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Geometric Group Theory

An Introduction

~~Monster Collection~~ Errata



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This is the collection of errata for the book *Geometric Group Theory: An Introduction* [1]. Despite of careful proofreading before publication, some monsters managed to hide in the book (GROAR!). Whenever one of these monsters gets caught, it will be put into this list. I am very grateful to all monster hunters!

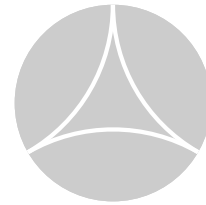
Notation. References of the form “Theorem 4.3.1” point to the corresponding items in the book [1].

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Errata



p. 48, Exercise 2.E.34



Location. p. 48, Exercise 2.E.34, caught by Johannes Witzig (03/2018)

Problem. There is a hypothesis missing in the first part of this exercise: If G is finitely generated and H is finite, then $\bigoplus_H G$ is finitely generated as well.

Fix. The hypothesis that the group H is infinite should be added.

p. 73, Definition 3.E.1 and Quick check 3.E.27



Location. p. 73, Definition 3.E.1 and Quick check 3.E.27, caught by Jan Fricke (04/2018)

Problem (Definition 3.E.1). Definition 3.E.1 only deals with finite chromatic numbers, but Definition 3.E.2 also talks about infinite chromatic numbers.

Fix. One can extend Definition 3.E.1 to also cover infinite chromatic numbers: A simple version would be to define the chromatic number to be infinite if no finite colouring exists; a more sophisticated version would be to take the minimal cardinal number such that a colouring of this cardinality exists (the class of cardinal numbers is well-ordered).

Problem (Quick check 3.E.27). In the first question of Quick check 3.E.27, the generating set is missing in the formula.

Fix. The formula should read $\text{ch}(\text{Cay}(S_6, S)) \geq 2017$.

p. 73, Exercise 3.E.29



Location. p. 73, Exercise 3.E.29, caught by Matthias Uschold (03/2019)

Problem (Exercise 3.E.29). In the first part, the claimed inequality, in general, does not hold if $N = G$.

Fix. In the first part, one should add the hypothesis that $N \neq G$.

p. 112, definition of the Grigorchuk group



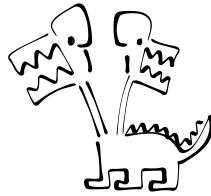
Location. p. 112, definition of the Grigorchuk group, caught by Philip Dowerk (06/2018)

Problem. In the definition of d , words of type $1w$ are mapped to $0b(w)$. This is a typo (in fact, this definition would not even result in an automorphism).

Fix. The definition of d should be corrected to:

$$\begin{aligned}d: \{0, 1\}^* &\longrightarrow \{0, 1\}^* \\ \varepsilon &\longmapsto \varepsilon \\ 0w &\longmapsto 0w \\ 1w &\longmapsto 1b(w).\end{aligned}$$

p. 145, first two lines



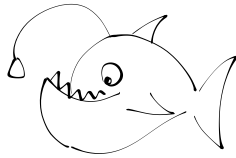
Location. p. 145, first two lines, caught by Daniel Kasprowski (11/2018)

Problem. In the first two lines, it says that “a metric space is locally compact if and only if it is proper.”

This is *not* true: Every proper metric space is locally compact. However, the converse is not true: If we equip an infinite set with the discrete metric, then this metric space is locally compact, but *not* proper.

Fix. The sentence should be replaced by: For example, every proper metric space is locally compact.

p. 217, proof of Theorem 7.2.11



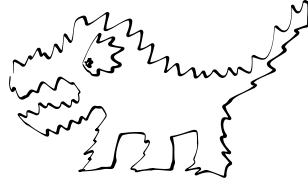
Location. p. 217, proof of Theorem 7.2.11, caught by Philip Dowerk (07/2018)

Problem. On p. 217, in the last paragraph, the numbers r' and s' are introduced. In these definitions, it is claimed that $r' \leq \Delta$ and $s' \geq L' - \Delta$.

In general, this is *not* true: The geodesic and the quasi-geodesic could coincide except for a very small arc in the “middle”. Then Δ would be small, but r' and s' would close to $L'/2$.

Fix. These estimates in the definition of r' and s' should be removed (they are wrong and not used anywhere).

p. 276, Definition 8.3.1



Location. p. 276, Definition 8.3.1, caught by Juan C. Lanfranco (04/2019)

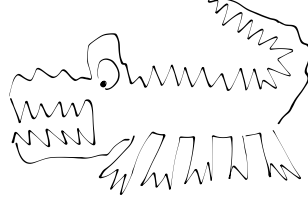
Problem. The definition of the topology on ∂X is misformulated: The quantifier over the limit ray γ is at the wrong position.

Fix. The correct definition of the topology on ∂X allows the limit representative γ to depend on the subsequence:

- We define a topology on ∂X through convergence of sequences in ∂X to a point in ∂X :

Let $(x_n)_{n \in \mathbb{N}} \subset \partial X$ and let $x \in \partial X$. We say that $(x_n)_{n \in \mathbb{N}}$ converges to x if there exist quasi-geodesic rays $(\gamma_n)_{n \in \mathbb{N}}$ representing the $(x_n)_{n \in \mathbb{N}}$ with the following property: Every subsequence of $(\gamma_n)_{n \in \mathbb{N}}$ contains a subsequence that converges (uniformly on compact subsets of $[0, \infty)$) to a quasi-geodesic ray that represents x .

p. 274, proof of Lemma 8.3.14



Location. p. 274, proof of Lemma 8.3.14, caught by Roman Sauer (01/2018)

Problem. At the beginning of the proof, it is claimed that $g^\infty = h^\infty$ implies via Theorem 8.3.4 that there is a constant $c \in \mathbb{R}_{>0}$ such that

$$\forall n \in \mathbb{N} \quad d_S(g^n, h^n) \leq c.$$

This is nonsense. We only know that the *quasi*-geodesics associated with g and h have finite Hausdorff distance.

For example, $g := 1$, $h := 2 \in \mathbb{Z}$ demonstrate this problem nicely: These elements lead to the same element in $\partial\mathbb{Z}$, but the distance between their powers grows linearly.

Fix. We can use the same argument as in the original proof of Lemma 8.3.14, using finite Hausdorff distance instead of uniformly bounded distance of the quasi-geodesic rays:

Proof. Let $S \subset G$ be a finite generating set of G . In view of Theorem 8.3.4, there exists a constant $c \in \mathbb{R}_{>0}$ such that

$$\forall m \in \mathbb{N} \quad \exists \tilde{m} \in \mathbb{N} \quad d_S(h^m, g^{\tilde{m}}) \leq c.$$

Moreover, we can assume without loss of generality that

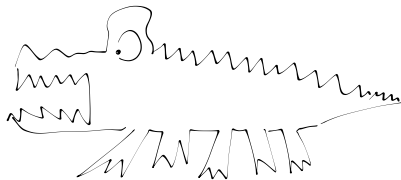
$$d_S(g, e) \leq c \quad \text{and} \quad d_S(e, h) \leq c.$$

We then obtain for all $m \in \mathbb{N}$ that

$$\begin{aligned} d_S(h^{-m} \cdot g \cdot h^m, g) &= d_S(g \cdot h^m, h^m \cdot g) \\ &\leq d_S(g \cdot h^m, g \cdot g^{\tilde{m}}) + d_S(g^{\tilde{m}+1}, g^{\tilde{m}}) + d_S(g^{\tilde{m}}, h^m) \\ &\quad + d_S(h^m, h^{m+1}) + d_S(h^{m+1}, h^m \cdot g) \\ &\leq d_S(h^m, g^{\tilde{m}}) + d_S(g, e) + c + d_S(e, h) + d_S(h, g) \\ &\leq c + c + c + c + d_S(h, e) + d_S(e, g) \\ &\leq 6 \cdot c, \end{aligned}$$

and so $\{h^{-m} \cdot g \cdot h^m \mid m \in \mathbb{N}\} \subset B_{6,c}^{G,S}(g)$. Because this ball is finite, there exist $m, k \in \mathbb{N}$ with $m \neq k$ such that $h^{-m} \cdot g \cdot h^m = h^{-k} \cdot g \cdot h^k$. In other words, $h^{m-k} \cdot g = g \cdot h^{m-k}$. \square

p. 276, proof of Corollary 8.3.17



Location. p. 276, proof of Corollary 8.3.17, caught by Roman Sauer (01/2018)

Problem. The problem is the same one as in the proof of Lemma 8.3.14.

Fix. We can fix the proof of Corollary 8.3.17 basically in the same way as the fix for the proof of Lemma 8.3.14. To this end, we first provide a Hausdorff distance version of Lemma 7.5.14:

Lemma 1 (Close conjugates, Hausdorff distance version). *Let G be a hyperbolic group, let $g \in G$ be of infinite order, and let $S \subset G$ be a finite generating set of G . Then there is a constant $\Delta \in \mathbb{R}_{>0}$ with the following property: If $k \in \Gamma$ and $\varepsilon \in \{-1, 1\}$ satisfy*

$$\exists C \in \mathbb{R}_{>0} \quad \forall n \in \mathbb{Z} \quad \exists \tilde{n} \in \mathbb{Z} \quad d_S(k \cdot g^n \cdot k^{-1}, g^{\varepsilon \cdot \tilde{n}}) < C,$$

then

$$d_S(k, \langle g \rangle_G) \leq \Delta.$$

The proof is a straightforward adaption of the proof of Lemma 7.5.14:

Proof. We first need to make some of the constants explicit: By Theorem 7.5.9, there exists a constant $c \in \mathbb{R}_{\geq 1}$ such that the map

$$\begin{aligned} \mathbb{Z} &\longrightarrow G \\ n &\longmapsto g^n \end{aligned}$$

is a (c, c) -quasi-isometric embedding. Because G is hyperbolic, there exists a $\delta \in \mathbb{R}_{>0}$ such that G is (c, c, δ) -hyperbolic with respect to d_S (Exercise 7.E.13). We set

$$\Delta := 2 \cdot \delta$$

(however, one should note that c , whence δ , depends on g).

We now start with the actual proof: Let $k \in G$, let $\varepsilon \in \{-1, 1\}$ and let $C \in \mathbb{R}_{>0}$ with

$$\forall n \in \mathbb{Z} \quad \exists \tilde{n} \in \mathbb{Z} \quad d_S(k \cdot g^n \cdot k^{-1}, g^{\varepsilon \cdot \tilde{n}}) < C.$$

By Theorem 7.5.9, we can choose $n \in \mathbb{N}$ so big that

$$d_S(e, g^n) > C + 2 + 2 \cdot \delta + d_S(e, k).$$

We will abbreviate $m := -n$. We consider a quasi-geodesic quadrilateral with the vertices $k \cdot g^{-n}$, $k \cdot g^n$, $g^{\varepsilon \cdot \tilde{n}}$, $g^{\varepsilon \cdot \tilde{m}}$. To this end we pick $(1, 1)$ -quasi-geodesics γ from $g^{\varepsilon \cdot \tilde{m}}$ to $k \cdot g^n$, as well as γ_+ from $k \cdot g^n$ to $g^{\varepsilon \cdot \tilde{n}}$ and γ_- from $k \cdot g^{-n}$ to $g^{\varepsilon \cdot \tilde{m}}$. As “bottom” and “top” quasi-geodesics, we use the segments of $m \mapsto g^m$ and $m \mapsto k \cdot g^m$ (which by left-invariance of d_S is a (c, c) -quasi-geodesic embedding). We now argue similarly as in Lemma 7.5.5 and Lemma 7.4.11:

The conjugating element k lies on the “top” quasi-geodesic. Hence, by hyperbolicity, there is an x in $\text{im } \gamma$ or $\text{im } \gamma_-$ that is δ -close to k . We can rule out the case of $\text{im } \gamma_-$ as follows: For all $x \in \text{im } \gamma_-$ we have by the triangle inequality and the fact that γ_- is $(1, 1)$ -quasi-geodesic:

$$\begin{aligned} d_S(x, k) &\geq d_S(k \cdot g^{-n}, k) - d_S(x, k \cdot g^{-n}) \\ &\geq d_S(g^{-n}, e) - d_S(g^{\varepsilon \cdot \tilde{m}}, k \cdot g^{-n}) - 2. \end{aligned}$$

By the choice of n , we know $d_S(g^{-n}, e) = d_S(e, g^n) > C + 2 + 2 \cdot \delta + d_S(e, k)$. Moreover,

$$\begin{aligned} d_S(g^{\varepsilon \cdot \tilde{m}}, k \cdot g^{-n}) &\leq d_S(g^{\varepsilon \cdot \tilde{m}}, k \cdot g^{-n} \cdot k^{-1}) + d_S(k \cdot g^{-n} \cdot k^{-1}, k \cdot g^{-n}) \\ &\leq C + d_S(e, k). \end{aligned}$$

Putting these estimates together, we obtain $d_S(x, k) > 2 \cdot \delta \geq \delta$. Hence, there is an $x \in \text{im } \gamma$ with $d_S(k, x) \leq \delta$.

Analogously, we can use hyperbolicity in the “lower” quasi-geodesic triangle to show that there is a point $y \in \langle g \rangle_G$ with $d_S(x, y) \leq \delta$ (by ruling out $\text{im } \gamma_+$). Therefore,

$$d_S(k, \langle g \rangle_G) \leq d_S(k, y) \leq d_S(k, x) + d_S(x, y) \leq 2 \cdot \delta = \Delta,$$

as claimed. \square

Using this lemma, we can correct the proof of Corollary 8.3.17:

Proof. Clearly, the two alternatives exclude each other. We now consider the case when G is *not* virtually cyclic and we prove that then G has to contain a free group of rank 2. Because G is not virtually cyclic, G is infinite; in particular, G contains an element g of infinite order (Theorem 7.5.1). In view of Theorem 8.3.13, it suffices to find an element $h \in G$ of infinite order that is independent of g . Because G is not virtually cyclic, there exist elements $k \in G$ of arbitrarily large distance to $\langle g \rangle_G$. Therefore, Lemma 1 implies that there is a $k \in G$ such that the conjugate $h := k \cdot g \cdot k^{-1}$ satisfies for all $\varepsilon \in \{-1, 1\}$:

$$\forall C \in \mathbb{R}_{>0} \quad \exists n \in \mathbb{Z} \quad \forall \tilde{n} \in \mathbb{Z} \quad d_S(h^n, g^{\varepsilon \cdot \tilde{n}}) = \sup_{n \in \mathbb{Z}} d_S(k \cdot g^n k^{-1}, g^{\varepsilon \cdot \tilde{n}}) \geq C.$$

Since g , hence h , has infinite order, using Theorem 8.3.4 we can reformulate the previous expression as

$$\{h^\infty, h^{-\infty}\} \neq \{g^\infty, g^{-\infty}\}.$$

By the first part of Theorem 8.3.13, this already implies that g and h are independent; therefore, the second part of Theorem 8.3.13 can be applied. \square

Bibliography



- [1] Clara Löh. *Geometric Group Theory: An Introduction, Universitext*, Springer, 2018. Cited on page: ii