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07.01.2022

Recap: homology thy on Top^2 with values in ${}_R\mathrm{Mod}$:
$((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}})$. ES-axioms:
homotopy inv, LES of pairs, excision
(dim. axiom, additivity)

Goal: compute $h_k(S^n)$ using the suspension iso

$\boxed{3.2.2}$ HOMOLOGY OF SPHERES

Corollary (homology of spheres).

1. For all $n\in\mathbb{N}$, $k\in\mathbb{Z}$, there is a natural R-iso

$$h_k(S^n, \{e_1\}) \underset{R}{\cong} h_{k+1}(\underbrace{\Sigma S^n}_{\underset{\mathrm{Top}}{\cong} S^{n+1}}, \{[e_1, 0]\}) \underset{R}{\cong} h_{k+1}(S^{n+1}, \{e_1\}).$$

2. Inductively: for all $n\in\mathbb{N}$, $k\in\mathbb{Z}$:

$$h_k(S^n) \underset{R}{\overset{\text{Prop. 3.1.6}}{\cong}} h_k(S^n, \{e_1\}) \oplus h_k(\{e_1\})$$

$$\overset{\text{1. + ind.}}{\underset{R}{\cong}} h_{k-n}(S^0, \{e_1\}) \oplus h_k(\{e_1\})$$

$$\overset{\text{excision}}{\underset{R}{\cong}} h_{k-n}(\bullet) \oplus h_k(\bullet).$$

S^0: $\bullet \quad \bullet \; e_1$

3. In particular: If (h_*, ∂_*) is ordinary, then:

$$\forall_{k\in\mathbb{Z}} \quad h_k(S^0) \underset{R}{\cong} \begin{cases} h_0(\bullet)\oplus h_0(\bullet) & \text{if } k=0 \\ 0 & \text{if } k\neq 0 \end{cases}$$

$$\forall_{n\in\mathbb{N}_{>0}} \forall_{k\in\mathbb{Z}} \quad h_k(S^n) \underset{R}{\cong} \begin{cases} h_0(\bullet) & \text{if } k\in\{0,n\} \\ 0 & \text{if } k\notin\{0,n\} \end{cases}$$

$\square$

<u>Warning</u>: How <u>not</u> to apply excision:

Let $n\in\mathbb{N}_{>0}$, $S,N\in S^n$ be South/North pole.

Then

$$h_k(S^n,\{S\}) \underset{R}{\overset{\text{Prop. 3.14}}{\cong}} h_k(S^n, \underbrace{S^n\setminus\{N\}}_{=:A})$$

apply excision to $A\subset A\subset S^n$?

Not applicable!!

$$\overset{?}{\underset{R}{\cong}} h_k(S^n\setminus A, A\setminus A)$$

$$\underset{R}{\cong} h_k(S^n\setminus A) \underset{R}{\cong} h_k(\bullet).$$

In general: this will be wrong!

<u>Corollary</u> (invariance of dim II). If there ex. an ordinary homology thy (h_*, ∂_*) on Top^2 with values in ${}_{\mathbb{Z}}\mathsf{Mod}$ and coefficients (iso to) $\mathbb{Z}$, $\leftarrow h_0(\bullet)\underset{\mathbb{Z}}{\cong}\mathbb{Z}$

Thm: If $n, m \in \mathbb{N}$ and $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ are open and non-empty with $U \underset{Top}{\cong} V$, then $n = m$.

Proof idea: use excision to reduce the problem to $h_k(\text{spheres})$

Wlog, we assume $n, m > 0$.

Let $x \in U$, let $f : U \to V$ be a homeo, and $y := f(x)$.

Because U is open, then ex. $\varepsilon \in \mathbb{R}_{>0}$ s.t. $U_\varepsilon(x) \subset U$. For all $k \in \mathbb{N}_{>0}$, we have

$$\begin{aligned}
\boxed{h_k(U, U \setminus \{x\})} \quad (\ast) &\underset{\mathbb{Z}}{\overset{\text{exc.}}{\cong}} h_k(U \setminus W, (U \setminus \{x\}) \setminus W) \qquad W := U \setminus U_\varepsilon(x) \\
&\underset{\mathbb{Z}}{\cong} h_k(U_\varepsilon(x), U_\varepsilon(x) \setminus \{x\}) \\
\text{homeo inv.} \quad &\underset{\mathbb{Z}}{\cong} h_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\
\text{LES of triple} + \mathbb{R}^n \simeq \bullet \quad &\underset{\mathbb{Z}}{\cong} h_{k-1}(\mathbb{R}^n \setminus \{0\}, \{e_1\}) \\
&\underset{\mathbb{Z}}{\cong} \boxed{h_{k-1}(S^{n-1}, \{e_1\})}
\end{aligned}$$

and

$$h_k(V, V\setminus\{y\}) \underset{\mathbb{Z}}{\cong} h_{k-1}(S^{m-1}, \{e_1\}).$$

Therefore:

$$\begin{aligned}
\boxed{h_{n-1}(S^{n-1})} &\underset{\mathbb{Z}}{\cong} h_{n-1}(S^{n-1}, \{e_1\}) \oplus h_{n-1}(\bullet) \\
&\overset{(*)}{\underset{\mathbb{Z}}{\cong}} h_n(U, U\setminus\{x\}) \oplus h_{n-1}(\bullet) \\
&\underset{\mathbb{Z}}{\cong} h_n(V, V\setminus\{y\}) \oplus h_{n-1}(\bullet) \\
&\underset{\mathbb{Z}}{\cong} h_{n-1}(S^{m-1}, \{e_1\}) \oplus h_{n-1}(\bullet) \\
&\underset{\mathbb{Z}}{\cong} \boxed{h_{n-1}(S^{m-1})}.
\end{aligned}$$

f is a homeo

Because (h_*, ∂_*) is ordinary with $\mathbb{Z}$-coeffs, it follows that $n = m$. $\square$

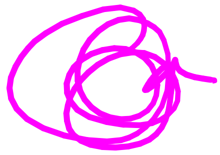
3.2.3 MAPPING DEGREES OF SELF-MAPS OF SPHERES

Corollary (mapping degrees on spheres). Let (h_*, ∂_*) be an ordinary homology theory.

1. If $n \in \mathbb{N}_{>0}$ and $j \in \{1, \dots, n+1\}$, then

$$h_n(r_j^{(n)}) = -\mathrm{id}_{h_n(S^n)}.$$

refl. at j-coord. (Thm 13.2.2)



2. For $d \in \mathbb{Z}$, let
$$f_d : S^1 \longrightarrow S^1$$
$$[t] \longmapsto [dt \bmod 1]$$

Then $h_1(f_d) = d \cdot \mathrm{id}_{h_1(S^1)}$.

3. For all $n \in \mathbb{N}_{>0}$ and all $d \in \mathbb{Z}$:

$$h_n(\underbrace{\Sigma^{n-1} f_d}) = d \cdot \mathrm{id}_{h_n(S^n)}.$$

viewed as a map $S^n \longrightarrow S^n$ using $\Sigma^{n-1} S^1 \cong_{\mathrm{Top}} S^n$ (Ex. 3.2.4).

In particular: If (h_*, ∂_*) is an ordinary homology thy with values in ${}_{\mathbb{Z}}\mathrm{Mod}$ and coeffs iso to $\mathbb{Z}$, then:

1. If $n \in \mathbb{N}_{>0}$, $j \in \{1, \dots, n+1\}$, then

$$\deg_{h_n} r_j^{(n)} = -1.$$

$\overbrace{F(X)}^{\cong \mathbb{Z}} \xrightarrow{F(f)} F(X)$

$F(f) = \deg_F(f) \cdot \mathrm{id}_{F(X)}$

2. For all $n \in \mathbb{N}_{>0}$, $d \in \mathbb{Z}$:

$$\deg_{h_n}(\Sigma^{n-1} f_d) = d.$$

($\leadsto$ $\deg_{Oh_n} : [S^n, S^n] \to \mathbb{Z}$ is surj.)

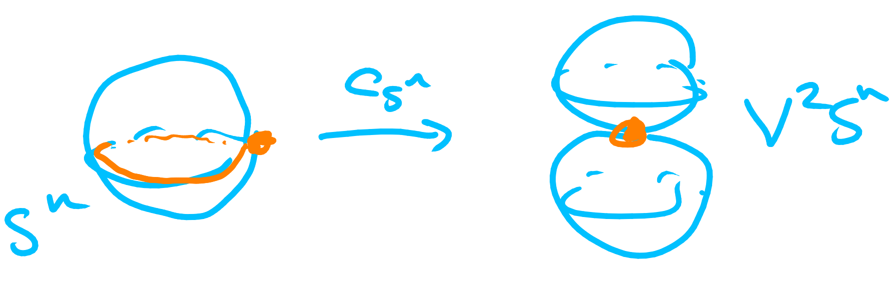
The proof is based on:

Lemma. (homology of "addition" of maps defined on spheres). Let (h_*, ∂_*) be an ordinary homology theory, let $n \in \mathbb{N}_{>0}$. For $d \in \mathbb{N}$, we write $V^d S^n$ for the top. space underlying $V^d(S^n, e_1)$.

1. The inclusions $(i_j : S^n \to V^d S^n)_{j \in \{1,\dots,d\}}$ and collapse maps $(p_j : V^d S^n \to S^n)_{j \in \{1,\dots,d\}}$ induce for all $k \in \mathbb{Z} \setminus \{0\}$ an iso

$$h_k(V^d S^n) \underset{R}{\cong} \bigoplus_d h_k(S^n).$$

2. Let $c_{S^n} : S^n \to V^2 S^n$ be the pinching map:



For every top. space X and all $f,g \in \mathrm{map}(S^n, X)$ with $f(e_1)=g(e_1)$, we have

$$\underbrace{h_n\Big(\overbrace{(\underbrace{f\vee g}_{\vee^2 S^n \to X})\circ c_{S^n}}^{S^n \to S^n}\Big)}_{\text{geom. add.}} = \underbrace{h_n(f)+h_n(g)}_{\text{alg. add.}}.$$

Proof. 1. This follows inductively from excision ("similar" to Ex 2.2.14) (check!)

2. We have

$$h_n(\vee^2 S^n) \overset{1.}{\cong} \bigoplus^2 h_n(S^n)$$

$$\boxed{h_n((f\vee g)\circ c_{S^n})} = h_n(f\vee g)\circ h_n(c_{S^n})$$

$$= h_n(f\vee g)\circ h_n(i_1)\circ h_n(p_1)\circ h_n(c_{S^n})$$

$i_1 \circ p_2 \simeq \mathrm{const}$

$$0 = \Big\{ + \quad " \quad i_1 \quad p_2 \quad "$$

$h_n(i_1\circ p_2)$ factors over $h_n(\bullet) \cong 0$

$$+ \quad " \quad i_2 \quad p_2 \quad "$$

$$0 = \Big\{ + \quad " \quad i_2 \quad p_1 \quad "$$

$$= h_n(\underbrace{(f\vee g)\circ i_1}_{=f}) \circ h_n(\underbrace{p_1\circ c_{S^n}}_{\simeq \mathrm{id}_{S^n}})$$

$$+ h_n(\underbrace{(f\vee g)\circ i_2}_{=g}) \circ h_n(\underbrace{p_2\circ c_{S^n}}_{\simeq \mathrm{id}_{S^n}})$$

$$= \boxed{h_n(f) + h_n(g)}.$$

id on $h_n(S^n)$

$\square$

<u>Rmk.</u> (Hurewicz hom.). Let (h_*, ∂_*) be an ord. homlgy thy on Top^2 with values in $\mathbb{Z}\mathrm{Mod}$ and $\mathbb{Z}$-coeffs, let $n \in \mathbb{N}_{>0}$, and let $[S^n] \in h_n(S^n) \underset{\mathbb{Z}}{\cong} \mathbb{Z}$ be a generator.

If (X, x_0) is a ptd space, then there is a hom ← previous lemma

$$\pi_n(X, x_0) \longrightarrow h_n(X)$$

$$[f]_* \longmapsto h_n(f)([S^n])$$

$(S^n, e_1) \to (X, x_0)$

⇝ nat. trafo $\pi_n \Longrightarrow h_n \circ$ forget basept.

Proof of corollary:

1. Steps:

① Reduction to $r_2^{(n)}$: $r_j^{(n)}$ is obtained from $r_2^{(n)}$ through conj. with the homeo

$$g_j : S^n \longrightarrow S^n$$
$$x \longmapsto (x_1, x_j, x_3, \dots, x_{j-1}, x_2, x_{j+1}, \dots, x_{n+1}).$$

Namely:

$$r_j^{(n)} = g_j \circ r_2^{(n)} \circ g_j^{-1}.$$

Thus: if $h_n(r_2^{(n)}) = -id$, then also

$$h_n(r_j^{(n)}) = -id.$$