

THE CHEAP EMBEDDING PRINCIPLE: DYNAMICAL UPPER BOUNDS FOR HOMOLOGY GROWTH

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ABSTRACT. We provide upper bounds for logarithmic torsion homology growth and Betti number growth of groups, phrased in the language of measured group theory.

1. INTRODUCTION

We provide new upper bounds for logarithmic torsion homology growth and Betti number gradients along systems of finite index normal subgroups via the following two basic principles:

- ① If A “embeds” into B and B is “small”, then also A is “small”.
- ② The asymptotic behaviour along finite index normal subgroups is encoded in the dynamical system given by the profinite completion.

We apply these two principles in the setting of chain complexes over crossed product rings and orbit equivalence relation rings associated with dynamical systems. Typically, “embeddings” refer to chain maps that admit homotopy retractions and “smallness” will be measured in terms of “dimensions” or “determinants” over various rings.

The principle ① was used to prove vanishing results for L^2 -Betti numbers and homology gradients in the presence of amenable covers with small multiplicity [Sau09, Sau16]. The principle ② was previously established for L^2 -Betti numbers [Gab02a], rank gradients of groups [AN12], and stable integral simplicial volume [LP16, Löh20b]. In these cases, the dynamical point of view was the key to proving novel types of inheritance results, leading to a deeper understanding of the invariants and concrete calculations and estimates [Gab02a, Gab00, FLPS16, FLMQ21].

Similarly, our approach provides new perspectives on calculations and estimates for logarithmic torsion homology growth and Betti number gradients over finite fields. We will now describe the setting and method in more detail:

1.1. Setup and dynamical sizes. Let Γ be a countable group. Let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a *standard Γ -action*, i.e., an essentially free probability measure preserving action on a standard Borel probability space. As coefficients, we consider Z to be \mathbb{Z} (with the usual norm) or a finite field (with the trivial norm). We write $L^\infty(\alpha, Z)$ for the $Z\Gamma$ -module of essentially bounded measurable functions $X \rightarrow Z$ up to equality μ -almost everywhere; i.e., elements of $L^\infty(\alpha, Z)$ are represented by finite Z -linear combinations of characteristic functions on measurable subsets of X . This leads to the crossed product ring $R := L^\infty(\alpha, Z) * \Gamma$ of the action α .

We assume that Γ is *of type FP_{n+1}* , i.e., the trivial $\mathbb{Z}\Gamma$ -module \mathbb{Z} admits a projective resolution that is finitely generated in degrees $\leq n + 1$.

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We fix a free $Z\Gamma$ -resolution $C_* \rightarrow Z$ of the trivial $Z\Gamma$ -module Z . By elementary homological algebra, the subsequent definitions will be independent of the choice of the resolution of the group. On the dynamical side, we consider marked projective augmented chain complexes over R (see Section 2 for precise definitions).

Definition 1.1 (α -embedding). In this situation, an α -embedding (up to degree n) is a pair that consists of a marked projective augmented R -chain complex $D_* \twoheadrightarrow L^\infty(\alpha, Z)$ and a $Z\Gamma$ -chain map $C_* \rightarrow D_*$ up to degree $n+1$ extending the inclusion $Z \rightarrow L^\infty(\alpha, Z)$ as constant functions. We write $A_n(\alpha)$ for the class of all augmented complexes arising in α -embeddings up to degree n .

For every $\mathbb{R}_{\geq 0}$ -valued isomorphism invariant Δ of marked projective augmented R -chain complexes, we may define

$$\Delta(\alpha) := \inf_{(D_* \twoheadrightarrow L^\infty(\alpha, Z)) \in A_n(\alpha)} \Delta(D_* \twoheadrightarrow L^\infty(\alpha, Z)).$$

For example, for $n \in \mathbb{N}$, we obtain the following invariants of $\Gamma \curvearrowright (X, \mu)$, still under the assumption that Γ is of type FP_{n+1} .

- The *measured embedding dimension* $\text{medim}_n^Z(\alpha)$ over Z in degree n : Here, we take

$$\Delta(D_* \twoheadrightarrow L^\infty(\alpha, Z)) := \dim_R(D_n).$$

- The *measured embedding volume* $\text{mevol}_n(\alpha)$ in degree n : Here, we take $Z = \mathbb{Z}$ and

$$\Delta(D_* \twoheadrightarrow L^\infty(\alpha, \mathbb{Z})) := \text{lognorm}(\partial_{n+1}^D).$$

The quantity lognorm is a crude approximation of the logarithmic determinant, introduced in Section 6.

In fact, this setting also extends to the equivalence relation ring $Z\mathcal{R}$ over the orbit relation $\mathcal{R} := \mathcal{R}_\alpha := \{(x, \gamma \cdot x) \mid x \in X, \gamma \in \Gamma\}$ of α . This leads to the same values of measured embedding dimension and of measured embedding volume (Corollary 17.1). However, it is not clear whether one may obtain suitable orbit equivalence invariants in this way because $Z\mathcal{R}$ in general might not be flat over $Z\Gamma$ and because $Z\mathcal{R} \otimes_{Z\Gamma} Z$ in general is not isomorphic to $L^\infty(\alpha, Z)$.

1.2. Upper bounds for gradient invariants. Let Γ be a countable residually finite group (satisfying suitable finiteness properties). If $\Gamma_* = (\Gamma_i)_{i \in I}$ is a directed system of finite index normal subgroups of Γ , we may study Γ through Γ_* via the residually finite point of view and the asymptotic behaviour of invariants of the subgroups in Γ_* : Let F be an $\mathbb{R}_{\geq 0}$ -valued isomorphism invariant of residually finite groups (satisfying suitable finiteness properties). Then we obtain an associated (upper) gradient invariant via

$$\widehat{F}(\Gamma, \Gamma_*) := \limsup_{i \in I} \frac{F(\Gamma_i)}{[\Gamma : \Gamma_i]}.$$

For example, for $n \in \mathbb{N}$ and Γ of type FP_{n+1} , we have the following invariants of Γ_* :

- The (upper) *Betti number gradient* over Z in degree n :

$$\widehat{b}_n(\Gamma, \Gamma_*; Z) := (\text{rk}_Z H_n(\cdot; Z))^\wedge = \limsup_{i \in I} \frac{\text{rk}_Z H_n(\Gamma_i; Z)}{[\Gamma : \Gamma_i]}.$$

- The (upper) *logarithmic torsion homology gradient* in degree n :

$$\widehat{t}_n(\Gamma, \Gamma_*) := (\log \# \text{tors } H_n(\cdot; \mathbb{Z}))^\wedge = \limsup_{i \in I} \frac{\log \# \text{tors } H_n(\Gamma_i; \mathbb{Z})}{[\Gamma : \Gamma_i]}.$$

On the dynamical side, we can consider the corresponding profinite completion $\widehat{\Gamma}_* := \varprojlim_{i \in I} \Gamma/\Gamma_i$. Then the left translation action $\Gamma \curvearrowright \widehat{\Gamma}_*$ is measure preserving with respect to the Haar measure. If this action is essentially free, we are in the dynamical setting of Section 1.1 and we can compare the gradient invariants with dynamical invariants.

Theorem 1.2 (dynamical upper bounds; Theorem 8.1). *Let $n \in \mathbb{N}$ and let Γ be a residually finite group of type FP_{n+1} . Let $(\Gamma_i)_{i \in I}$ be a directed system of finite index normal subgroups of Γ with $\bigcap_{i \in I} \Gamma_i = 1$ (e.g., a residual chain in Γ or the system of all finite index normal subgroups). Then:*

$$\begin{aligned} \widehat{b}_n(\Gamma, \Gamma_*; Z) &\leq \text{medim}_n^Z(\Gamma \curvearrowright \widehat{\Gamma}_*) \quad \text{if } Z \text{ is } \mathbb{Z} \text{ or a finite field} \\ \widehat{t}_n(\Gamma, \Gamma_*) &\leq \text{mevol}_n(\Gamma \curvearrowright \widehat{\Gamma}_*). \end{aligned}$$

The proof of Theorem 1.2 relies on the fundamental observation that the subalgebra of cylinder sets is dense in $\widehat{\Gamma}_*$ with respect to the Haar measure. Therefore, we can approximate augmented complexes over the crossed product ring $R := L^\infty(\widehat{\Gamma}_*, Z) * \Gamma$ with arbitrary precision by augmented complexes that only involve single, deep enough, subgroups Γ_i . We can then apply classical homological algebra to interpret the embeddings as chain homotopy retracts and use standard estimates for Betti numbers and logarithmic torsion.

On a technical level, the underlying approximation result is challenging and is obtained through a balanced sequence of deformations and strictifications of (almost) chain complexes and (almost) chain maps (Section 4 and Section 5). We thus develop a quantitative homological algebra over R . In particular, this includes various norms on marked projective modules over R , norms for homomorphisms between such modules, and the introduction of a Gromov–Hausdorff distance between marked projective chain complexes over R (Section 3). The proof of Theorem 1.2 is completed in Section 8.

The measured embedding dimension is also an upper bound for the L^2 -Betti numbers:

Theorem 1.3 (Theorem 8.6). *Let $n \in \mathbb{N}$, let Γ be a group of type FP_{n+1} , and let α be a standard Γ -action. Then:*

$$b_n^{(2)}(\Gamma) \leq \text{medim}_n^{\mathbb{Z}}(\alpha).$$

1.3. Examples. As in the case of L^2 -Betti numbers, we can use a Rokhlin lemma to show that amenable groups are homologically dynamically small (Section 11). In particular, we obtain:

Theorem 1.4 (amenable groups; Theorem 11.11). *Let $n \in \mathbb{N}$, let Γ be a countable infinite amenable group of type FP_{n+1} , and let α be a standard Γ -action. Then:*

$$\begin{aligned} \text{medim}_n^{\mathbb{Z}}(\alpha) &= 0 \quad \text{if } Z \text{ is } \mathbb{Z} \text{ or a finite field} \\ \text{mevol}_n(\alpha) &= 0. \end{aligned}$$

Theorem 1.2 and Theorem 1.4 refine the well-known results that amenable residually finite groups of type FP_∞ have vanishing Betti gradients over every field [CG86, LLS11] and vanishing logarithmic torsion growth [KKN17] in all degrees.

For free groups, we obtain (Section 12.2):

Proposition 1.5 (free groups; Proposition 12.3). *Let $d \in \mathbb{N}_{>0}$, let F_d be a free group of rank d , and let α be a standard action of F_d . Let Z be \mathbb{Z} or a finite field. Then:*

- (i) For all $n \in \mathbb{N}$, we have $\text{mevol}_n(\alpha) = 0$;

- (ii) For all $n \in \mathbb{N} \setminus \{1\}$, we have $\text{medim}_n^{\mathbb{Z}}(\alpha) = 0$;
- (iii) We have $\text{medim}_1^{\mathbb{Z}}(\alpha) \geq d - 1$;
- (iv) If α is the profinite completion with respect to a directed system $(\Gamma_i)_{i \in I}$ of finite index normal subgroups of F_d with $\bigcap_{i \in I} \Gamma_i = 1$, then

$$\text{medim}_1^{\mathbb{Z}}(\alpha) = d - 1.$$

We prove estimates for amalgamated products over \mathbb{Z} (Section 12.1). As a consequence of the calculation for free groups, we may then also treat surface groups (Section 12.3). Moreover, we obtain inheritance results for direct products with an amenable factor (Section 13) and for finite index subgroups (Section 14).

Remark 1.6 (relation with the cheap rebuilding property). The definition of the measured embedding dimension and the measured embedding volume as well as the corresponding dynamical upper bounds is inspired by the work of Abért, Bergeron, Frączyk, and Gaboriau [ABFG25] on the cheap rebuilding property. This property of groups implies the vanishing of both Betti number growth and logarithmic torsion homology growth. The present authors introduced an algebraic version of this property [LLM⁺]. We show that an equivariant algebraic version implies the vanishing of medim and mevol (Section 14.2).

Via the measured embedding volume, the logarithmic torsion growth estimates are decoupled from the homology gradient estimates. In particular, one may use the measured embedding volume to prove vanishing of the logarithmic torsion homology growth where previous methods are not applicable because of non-vanishing L^2 -Betti numbers.

In the future, we plan to combine the algebraic techniques developed in our previous paper [LLM⁺] with the dynamical approach in order to obtain bootstrapping theorems for measured embedding dimensions and volumes.

1.4. Dynamical inheritance properties. By design, the measured embedding dimension and the measured embedding volume are of dynamical nature. More precisely, we establish the following concrete instances of dynamical behaviour:

- monotonicity under weak containment;
- reduction to ergodic actions;
- invariance under weak bounded orbit equivalence;
- comparison with cost;
- comparison with integral foliated simplicial volume.

As in the fixed price problem for cost and integral foliated simplicial volume, in general, it is not clear how the measured embedding dimension/volume depend on the underlying dynamical system. Similarly to the case of cost and integral foliated simplicial volume, we show that one can always restrict to ergodic dynamical systems (Corollary 16.4) and monotonicity under weak containment.

Theorem 1.7 (weak containment; Theorem 15.30). *Let $n \in \mathbb{N}$, let Γ be a countable group of type FP_{n+1} , and let α, β be standard Γ -actions with $\alpha \prec \beta$. Then, we have*

$$\begin{aligned} \text{medim}_n^{\mathbb{Z}}(\beta) &\leq \text{medim}_n^{\mathbb{Z}}(\alpha) \quad \text{if } Z \text{ is } \mathbb{Z} \text{ or a finite field} \\ \text{mevol}_n(\beta) &\leq \text{mevol}_n(\alpha). \end{aligned}$$

In particular, for groups with property EMD^* (Definition 15.7), we obtain in combination with Theorem 1.2:

Corollary 1.8 (Corollary 16.5). *Let $n \in \mathbb{N}$, let Γ be a residually finite group of type FP_{n+1} that satisfies EMD^* , and let α be a standard Γ -action. Then*

$$\begin{aligned} \widehat{b}_n(\Gamma, \Gamma_*; Z) &\leq \text{medim}_n^{\mathbb{Z}}(\Gamma \curvearrowright \widehat{\Gamma}) \leq \text{medim}_n^{\mathbb{Z}}(\alpha) \quad \text{if } Z \text{ is } \mathbb{Z} \text{ or a finite field} \\ \widehat{t}_n(\Gamma, \Gamma_*) &\leq \text{mevol}_n(\Gamma \curvearrowright \widehat{\Gamma}) \leq \text{mevol}_n(\alpha). \end{aligned}$$

It is an open problem to determine how (vanishing of) homology gradients over finite fields or torsion homology growth behaves under orbit equivalence. As a step towards this problem, we show that measured embedding dimension and measured embedding volume are compatible with weak bounded orbit equivalence (Definition 18.14). In particular, these invariants provide upper bounds for homology growth over finite fields and for torsion homology growth that are multiplicative under weak bounded orbit equivalences.

Theorem 1.9 (weak bounded orbit equivalence; Theorem 18.2). *Let $n \in \mathbb{N}$, let Γ and Λ be groups of type FP_{n+1} , and let α and β be standard actions of Γ and Λ , respectively, that are weakly bounded orbit equivalent of index c . Then, we have*

$$\begin{aligned} \text{medim}_n^Z(\alpha) &= c \cdot \text{medim}_n^Z(\beta) \quad \text{if } Z \text{ is } \mathbb{Z} \text{ or a finite field} \\ \text{mevol}_n(\alpha) &= c \cdot \text{mevol}_n(\beta). \end{aligned}$$

We deduce a proportionality result for hyperbolic 3-manifolds:

Theorem 1.10 (Theorem 18.21). *Let M and N be oriented closed connected hyperbolic 3-manifolds and let $\Gamma := \pi_1(M)$, $\Lambda := \pi_1(N)$. Then*

$$\begin{aligned} \frac{\text{medim}_1^Z(\Gamma \curvearrowright \hat{\Gamma})}{\text{vol}(M)} &= \frac{\text{medim}_1^Z(\Lambda \curvearrowright \hat{\Lambda})}{\text{vol}(N)} \quad \text{if } Z \text{ is } \mathbb{Z} \text{ or a finite field} \\ \frac{\text{mevol}_1(\Gamma \curvearrowright \hat{\Gamma})}{\text{vol}(M)} &= \frac{\text{mevol}_1(\Lambda \curvearrowright \hat{\Lambda})}{\text{vol}(N)}. \end{aligned}$$

“Small” resolutions over the equivalence relation ring lead to upper bounds for the measured embedding dimension/volume, but at the moment, the case of general orbit equivalence is out of reach, because the equivalence relation rings do not exhibit the same level of exactness and finiteness properties as the crossed product rings.

Proposition 1.11 (small resolutions over the equivalence relation ring; Proposition 17.2). *Let Z denote \mathbb{Z} (with the standard norm) or a finite field (with the trivial norm). Let \mathcal{R} be a measured standard equivalence relation on a standard Borel probability space (X, μ) , let $n \in \mathbb{N}$, and let D_* be a marked projective $Z\mathcal{R}$ -resolution of $L^\infty(\alpha, Z)$ (up to degree $n+1$). Then: If Γ is a countable group of type FP_{n+1} and if α is standard probability action of Γ on (X, μ) that induces \mathcal{R} , then*

$$\begin{aligned} \text{medim}_n^Z(\alpha) &\leq \dim_{Z\mathcal{R}}(D_n) \\ \text{mevol}_n(\alpha) &\leq \text{lognorm}(\partial_{n+1}^D) \quad \text{if } Z = \mathbb{Z}. \end{aligned}$$

Similarly to the estimates for L^2 -Betti numbers via resolutions over dynamical rings, we obtain upper bounds for the measured embedding dimension/volume through cost and the integral foliated simplicial volume:

Theorem 1.12 (cost estimate; Theorem 19.1). *Let Γ be an infinite group of type FP_2 and let α be a standard action of Γ . Then*

$$\text{medim}_1^{\mathbb{Z}}(\alpha) \leq \text{cost}(\alpha) - 1.$$

Theorem 1.13 (integral foliated simplicial volume estimate; Theorem 20.1). *Let M be an oriented closed connected aspherical n -manifold with fundamental group Γ , let α be a standard Γ -action, and let $k \in \{0, \dots, n\}$. Then*

$$\begin{aligned} \text{medim}_k^{\mathbb{Z}}(\alpha) &\leq \binom{n+1}{k+1} \cdot |M|^\alpha \\ \text{mevol}_k(\alpha) &\leq \log(k+2) \cdot \binom{n+1}{k+1} \cdot |M|^\alpha. \end{aligned}$$

The integral foliated simplicial volumes is zero for aspherical manifolds admitting an amenable open cover of small multiplicity [LMS22]. By Theorem 1.13, in this case also the measured embedding dimensions and volumes are zero. This applies to many geometrically interesting situations including, e.g., aspherical manifolds with amenable fundamental group, aspherical graph 3-manifolds and smooth aspherical manifolds admitting a smooth circle action without fixed points (Example 20.6). Moreover, Theorem 1.13 also provides upper bounds for the measured embedding volumes of aspherical 3-manifolds and, more generally, for Riemannian manifolds in terms of Riemannian volumes (Example 20.4 and Example 20.5).

1.5. Open problems and further motivation. The following beautiful result is proved by Abért–Bergeron–Frączyk–Gaboriau. In our previous paper [LLM⁺] we revisit this result and put it into a larger homological context.

Theorem 1.14 ([ABFG25]). *Let $\Gamma = \mathrm{SL}_d(\mathbb{Z})$ with $d \geq 3$. Let p be a prime. Then $\widehat{t}_n(\Gamma, \Gamma_*) = 0$ and $\widehat{b}_n(\Gamma, \Gamma_*; \mathbb{F}_p) = 0$ for every $n \in \{0, \dots, d-2\}$ and every residual chain Γ_* of Γ .*

We also have $\widehat{b}_n(\Gamma, \Gamma_*; \mathbb{Z}) = 0$ in the same range but this follows easily from Lück’s approximation theorem and the computation of L^2 -Betti numbers.

Their result is more general than stated above and provides a vanishing result for arithmetic lattices in all degrees less than the \mathbb{Q} -rank. Conjecturally, one should be able to replace the \mathbb{Q} -rank by the \mathbb{R} -rank and allow for more general, Benjamini–Schramm convergent, sequences of lattices.

Conjecture 1.15. *Let G be a semisimple Lie group with finite center and without compact factors. Let r be the \mathbb{R} -rank of G , and assume that $r \geq 2$. Let $(\Gamma_i)_{i \in I}$ be a sequence of irreducible lattices in G whose covolumes tend to infinity. Then*

$$\lim_{i \in I} \frac{\mathrm{rk}_{\mathbb{F}_p} H_n(\Gamma_i; \mathbb{F}_p)}{\mathrm{vol}(G/\Gamma_i)} = 0 \quad \text{and} \quad \lim_{i \in I} \frac{\log \# \mathrm{tors} H_n(\Gamma_i; \mathbb{Z})}{\mathrm{vol}(G/\Gamma_i)} = 0$$

for every $n \in \{0, \dots, r-1\}$.

In particular, $\widehat{t}_n(\Gamma, \Gamma_*) = 0$ and $\widehat{b}_n(\Gamma, \Gamma_*; \mathbb{F}_p) = 0$ for every lattice Γ in G , every $n \in \{0, \dots, r-1\}$ and every residual chain Γ_* of Γ .

Some evidence comes from the following breakthrough result of Frączyk–Mellick–Wilkens [FMW, Theorem B] in degree 1.

Theorem 1.16 ([FMW]). *Let $(\Gamma_i)_{i \in I}$ and G and $r \geq 2$ be as in Conjecture 1.15. Then*

$$\lim_{i \in I} \frac{\mathrm{rk}_{\mathbb{F}_p} H_1(\Gamma_i; \mathbb{F}_p)}{\mathrm{vol}(G/\Gamma_i)} = 0.$$

Part of the motivation for writing this foundational paper is the authors’ programme to tackle the above conjecture by a dynamical-homological approach that hopefully allows to extend the result by Abért–Bergeron–Frączyk–Gaboriau to other lattices (of lower \mathbb{Q} -rank) via orbit equivalence techniques.

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Part 1. Dynamical upper bounds for homology growth

The main goal is to prove the dynamical upper bounds for Betti number gradients and logarithmic torsion homology gradients in terms of measured embedding dimension and measured embedding volume, respectively. We first develop a framework of quantitative homological algebra over rings associated with standard actions and prove strictification and deformation results (Sections 2–6). We then explain the passage to finite index subgroups (Section 7) and prove the dynamical upper bounds (Section 8).

2. BASIC NOTIONS

We recall basic notions on rings and modules associated with dynamical systems: crossed product rings and equivalence relation rings. As we are interested in quantitative aspects, we will also introduce corresponding norms, sizes, and dimensions.

Setup 2.1. Let Γ be a countable group. We consider a standard Γ -action $\alpha: \Gamma \curvearrowright (X, \mu)$, i.e., an essentially free measure preserving action of Γ on a standard Borel probability space (X, μ) .

Moreover, let S be an algebra of measurable sets of X with $\Gamma \cdot S \subset S$.

Let Z be the ring of integers \mathbb{Z} (with the usual absolute value) or a finite field with the trivial norm

$$x \mapsto \begin{cases} 0 & \text{if } x = 0; \\ 1 & \text{if } x \neq 0. \end{cases}$$

Remark 2.2. Most of our results can be generalised to the setting of principal ideal domains Z with a *discrete norm*, i.e., a function $|\cdot|: Z \rightarrow \mathbb{R}_{\geq 0}$ satisfying:

- $|0| = 0$ and $|x| \geq 1$ for all $x \in Z \setminus \{0\}$;
- $|x + y| \leq |x| + |y|$ for all $x, y \in Z$;
- $|x \cdot y| \leq |x| \cdot |y|$ for all $x, y \in Z$.

However, for simplicity, we state all results for the case $Z = \mathbb{Z}$ or Z being a finite field only.

Remark 2.3. When dealing with L^∞ -function spaces, we take the liberty of using pointwise notation. All corresponding notions such as equalities, defining equalities, suprema/infima, estimates, etc. are to be interpreted in the “almost every” sense.

2.1. Rings. We consider the following rings:

- The ring $L^\infty(\alpha) := L^\infty(X, \mu, Z)$ of essentially bounded measurable functions $X \rightarrow Z$ up to equality almost everywhere. The Γ -action on X induces a left Γ -action on $L^\infty(\alpha)$:

$$\forall \lambda \in L^\infty(\alpha) \quad \forall \gamma \in \Gamma \quad \gamma \cdot \lambda := (x \mapsto \lambda(\gamma^{-1} \cdot x)).$$

- The subring L of $L^\infty(\alpha)$ generated by S . (Every element of L is a finite Z -linear combination of characteristic functions over members of S).
- The *crossed product ring* $L * \Gamma \subset L^\infty(\alpha) * \Gamma$, i.e., the free L -module with basis Γ , endowed with the multiplication given by

$$(\lambda, \gamma) \cdot (\lambda', \gamma') := (\lambda \cdot (\gamma \cdot \lambda'), \gamma \cdot \gamma').$$

Sometimes we also write $\lambda \cdot \gamma$ instead of (λ, γ) for an element in the crossed product ring.

- Let $\mathcal{R} := \{(\gamma \cdot x, x) \mid x \in X, \gamma \in \Gamma\} \subset X \times X$ be the orbit relation of α . Let ν be the non-negative measure on the Borel σ -algebra of \mathcal{R} defined by

$$\nu(A) := \int_X \#(A \cap (\{x\} \times X)) \, d\mu(x).$$

The *equivalence relation ring* $Z\mathcal{R}$ is defined as

$$Z\mathcal{R} := \left\{ \lambda \in L^\infty(\mathcal{R}, \nu, Z) \mid \sup_{x \in X} \#\{y \mid \lambda(x, y) \neq 0\} < \infty, \right. \\ \left. \sup_{y \in X} \#\{x \mid \lambda(x, y) \neq 0\} < \infty \right\}$$

equipped with the convolution product

$$(\lambda \cdot \lambda')(x, y) := \sum_{w \in [x]_{\mathcal{R}}} \lambda(x, w) \cdot \lambda'(w, y).$$

We have a commutative diagram of canonical inclusions of rings (because the action of Γ on X is essentially free):

$$\begin{array}{ccccc} L^\infty(\alpha) & \longrightarrow & L^\infty(\alpha) * \Gamma & \longrightarrow & Z\mathcal{R} \\ \uparrow & & \uparrow & & \uparrow \\ L & \longrightarrow & L * \Gamma & \longleftarrow & Z\Gamma \end{array}$$

Under the ring inclusion $L^\infty(\alpha) * \Gamma \rightarrow Z\mathcal{R}$, the element $(\lambda, \gamma) \in L^\infty(\alpha) * \Gamma$ corresponds to the function

$$(\gamma' \cdot x, x) \mapsto \begin{cases} \lambda(\gamma \cdot x) & \text{if } \gamma' = \gamma; \\ 0 & \text{otherwise.} \end{cases}$$

A recurring theme will be that we need to control ℓ^1 -norms or supports.

Definition 2.4. Let $\lambda: \mathcal{R} \rightarrow Z$ be a function. Let

$$N_1(\lambda, \cdot): X \rightarrow \mathbb{N} \cup \{\infty\} \\ y \mapsto \#\{x \in X \mid \lambda(x, y) \neq 0\}$$

and $N_1(\lambda) := \sup_{y \in X} N_1(\lambda, y)$. Symmetrically, we define $N_2(\lambda, \cdot)$ and $N_2(\lambda)$.

If $\lambda \in Z\mathcal{R}$, then $N_1(\lambda) < \infty$ and $N_2(\lambda) < \infty$; per our convention in Remark 2.3, this is interpreted in the “almost everywhere” sense.

Lemma 2.5. *Let $\lambda, \lambda' \in Z\mathcal{R}$. Then*

$$|\lambda \cdot \lambda'|_1 \leq N_2(\lambda') \cdot |\lambda'|_\infty \cdot |\lambda|_1.$$

Proof. We have

$$\begin{aligned}
|\lambda \cdot \lambda'|_1 &= \int_{\mathcal{R}} |\lambda \cdot \lambda'| \, d\nu \\
&= \int_X \sum_{y \in [x]_{\mathcal{R}}} |\lambda \cdot \lambda'(x, y)| \, d\mu(x) \\
&= \int_X \sum_{y \in [x]_{\mathcal{R}}} \left| \sum_{w \in [x]_{\mathcal{R}}} \lambda(x, w) \cdot \lambda'(w, y) \right| \, d\mu(x) \\
&\leq \int_X \sum_{y \in [x]_{\mathcal{R}}} \sum_{w \in [x]_{\mathcal{R}}} |\lambda(x, w)| \cdot |\lambda'(w, y)| \, d\mu(x) \\
&= \int_X \sum_{w \in [x]_{\mathcal{R}}} |\lambda(x, w)| \cdot \sum_{y \in [x]_{\mathcal{R}}} |\lambda'(w, y)| \, d\mu(x) \\
&\leq \int_X \sum_{w \in [x]_{\mathcal{R}}} |\lambda(x, w)| \cdot N_2(\lambda', w) \cdot |\lambda'|_{\infty} \, d\mu(x) \\
&\leq N_2(\lambda') \cdot |\lambda'|_{\infty} \cdot \int_X \sum_{w \in [x]_{\mathcal{R}}} |\lambda(x, w)| \, d\mu(x) \\
&= N_2(\lambda') \cdot |\lambda'|_{\infty} \cdot \int_{\mathcal{R}} |\lambda| \, d\nu \\
&= N_2(\lambda') \cdot |\lambda'|_{\infty} \cdot |\lambda|_1.
\end{aligned}$$

□

Definition 2.6 (support). Let $\lambda \in Z\mathcal{R}$. We define (up to measure 0)

- the *support* of λ by

$$\text{supp}(\lambda) := \{(x, y) \in \mathcal{R} \mid \lambda(x, y) \neq 0\} \subset X \times X$$

- and the *1-support* of λ by

$$\text{supp}_1(\lambda) := \text{proj}_1(\text{supp}(\lambda)) \subset X,$$

where $\text{proj}_1: X \times X \rightarrow X$ denotes the projection onto the first factor.

Remark 2.7. Let $\lambda \in Z\mathcal{R}$. We have $\lambda(x, y) = \chi_{\text{supp}(\lambda)}(x, y) \cdot \lambda(x, y)$ for all $(x, y) \in \mathcal{R}$. Moreover, $\lambda = \chi_{\text{supp}_1(\lambda)} \cdot \lambda$ (with respect to the convolution product); indeed, under the inclusion $L^\infty(\alpha) \rightarrow Z\mathcal{R}$, the element $\chi_{\text{supp}_1(\lambda)} \in L^\infty(\alpha)$ corresponds to the function

$$(\gamma \cdot x, x) \mapsto \chi_{\text{supp}_1(\lambda)}(\gamma \cdot x, x) = \begin{cases} \chi_{\text{supp}_1(\lambda)}(x) & \text{if } \gamma = e; \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we have that $\mu(\text{supp}_1(\lambda)) \leq \nu(\text{supp}(\lambda)) \leq |\lambda|_1$.

Remark 2.8. For $(\lambda, \gamma) \in L^\infty(\alpha) * \Gamma \subset Z\mathcal{R}$, we have:

$$\begin{aligned}
N_1((\lambda, \gamma)) &\leq 1 \\
N_2((\lambda, \gamma)) &\leq 1 \\
\text{supp}_1((\lambda, \gamma)) &= \gamma^{-1} \cdot \text{supp}(\lambda).
\end{aligned}$$

Hence, in general we have $N_1(\sum_{j=1}^k (\lambda_j, \gamma_j)) \leq k$, $N_2(\sum_{j=1}^k (\lambda_j, \gamma_j)) \leq k$, and $\text{supp}_1(\sum_{j=1}^k (\lambda_j, \gamma_j)) \subset \bigcup_{j=1}^k \gamma_j^{-1} \cdot \text{supp}(\lambda)$.

2.2. Base changes. Using the inclusion relations between our basic rings, we can also consider the associated base change and induction functors.

Remark 2.9. We view $L^\infty(\alpha)$ as a $Z\mathcal{R}$ -module via the following scalar multiplication: Let $\iota: L^\infty(\alpha) \hookrightarrow L^\infty(\alpha) * \Gamma \hookrightarrow Z\mathcal{R}$ be the canonical inclusion. We define $\varepsilon: Z\mathcal{R} \rightarrow L^\infty(\alpha)$ by

$$\varepsilon(\lambda)(x) := \sum_{y \in [x]_\mathcal{R}} \lambda(x, y)$$

for all $\lambda \in Z\mathcal{R}$ and $x \in X$. For all $\lambda \in Z\mathcal{R}$ and $\lambda' \in L^\infty(\alpha)$, we set

$$\lambda \cdot \lambda' := \varepsilon(\lambda \cdot \iota(\lambda')) \in L^\infty(\alpha),$$

where the multiplication on the right hand side is the multiplication in $Z\mathcal{R}$.

More explicitly, if $A \subset X$ is a measurable subset, $\lambda \in L^\infty(\alpha)$, and $\gamma \in \Gamma$, then this action amounts to $(\lambda, \gamma) \cdot \chi_A = \lambda \cdot \chi_{\gamma \cdot A}$, where multiplication on the right hand side is the usual pointwise multiplication of L^∞ -functions.

Proposition 2.10. *The modules $L^\infty(\alpha) * \Gamma$ and $L * \Gamma$ are flat over $Z\Gamma$.*

Proof. For a $Z\Gamma$ -module M , there is a canonical isomorphism

$$(L^\infty(\alpha) * \Gamma) \otimes_{Z\Gamma} M \cong_Z L^\infty(\alpha) \otimes_Z M.$$

The claim follows because $L^\infty(\alpha)$ is flat over Z : If Z is a finite field, then this is clear. If Z is \mathbb{Z} , then it is also known that $L^\infty(\alpha)$ is free abelian [Ste85]. For L , the same argument applies. \square

Remark 2.11. In general, the module $Z\mathcal{R}$ might not be flat over the group ring $Z\Gamma$.

2.3. Modules. We will mainly be interested in a simple type of projective modules and the base ring $L^\infty(\alpha) * \Gamma$.

Setup 2.12. Let L be the subring of $L^\infty(\alpha)$ generated by S and let R be an L -subalgebra of $Z\mathcal{R}$, e.g., one of the rings $L^\infty(\alpha)$, $L^\infty(\alpha) * \Gamma$, or $Z\mathcal{R}$.

Definition 2.13 (marked projective module). A *marked projective R -module* is a triple $(M, (A_i)_{i \in I}, \varphi)$, consisting of

- an R -module M ,
- a finite family $(A_i)_{i \in I}$ of measurable subsets of X , and
- an R -isomorphism $\varphi: M \rightarrow \bigoplus_{i \in I} R \cdot \chi_{A_i}$.

In the following, we will abbreviate

$$\langle A_i \rangle := R \cdot \chi_{A_i}.$$

The *dimension* of the marked projective R -module $(M, (A_i)_{i \in I}, \varphi)$ is given by

$$\dim(M) := \sum_{i \in I} \mu(A_i) \in \mathbb{R}_{\geq 0}.$$

To simplify notation, we will leave φ implicit and call a description as above a *marked presentation*. Then, $\text{rk}(M) := \#I$ is the *rank* of this marked presentation. Finiteness is built into this definition of marked projective modules as this is the only case that we will consider.

Definition 2.14 (marked homomorphism). Let $f: M \rightarrow N$ be an R -homomorphism between marked projective R -modules and let $M = \bigoplus_{i \in I} \langle A_i \rangle$, $N = \bigoplus_{j \in J} \langle B_j \rangle$ be the marked presentations. We say that f is

- a *marked inclusion* if there exists an injective function $\sigma: I \rightarrow J$ such that $A_i \subset B_{\sigma(i)}$ and $f(\chi_{A_i} \cdot e_i) = \chi_{A_i} \cdot \chi_{B_{\sigma(i)}} \cdot e_{\sigma(i)}$;

- a *marked projection* if there exists an injective function $\tau: J \rightarrow I$ such that $B_j \subset A_{\tau(j)}$ and

$$f(\chi_{A_i} \cdot e_i) = \begin{cases} \chi_{A_i} \cdot \chi_{B_{\tau^{-1}(i)}} \cdot e_{\tau^{-1}(i)} & \text{if } i \in \text{im}(\tau); \\ 0 & \text{otherwise;} \end{cases}$$

- a *marked R -homomorphism* if f is a composition of marked inclusions and marked projections.

Remark 2.15 (canonical hull/complement, canonical inclusion, canonical projection). Marked projective modules are projective R -modules, as witnessed by the projections $R \rightarrow R$ of the form $\lambda \mapsto \lambda \cdot \chi_{A_i}$.

More precisely: Let $M = \bigoplus_{i \in I} \langle A_i \rangle$ be a marked projective R -module. Then the *canonical hull* of M is $F := \bigoplus_I R$. The marked inclusion $M \rightarrow F$ is the *canonical inclusion*. The R -linear marked projection $F \rightarrow M$ given on the standard basis $(e_i)_{i \in I}$ by

$$e_i \mapsto \chi_{A_i} \cdot e_i$$

is the *canonical projection to M* . We call

$$M' := \bigoplus_{i \in I} \langle X \setminus A_i \rangle$$

the *canonical complement of M* . By construction, the canonical inclusions and projections of M and M' combine into an isomorphism $M \oplus M' \cong_R F$.

Definition 2.16 (support). Let $M = \bigoplus_{i \in I} \langle A_i \rangle$ be a marked projective R -module. For $z = \sum_{i \in I} \lambda_i \cdot \chi_{A_i} \cdot e_i \in M$, we define the supports of z by

$$\text{supp}(z) := \bigcup_{i \in I} \text{supp}(\lambda_i \cdot \chi_{A_i}) \subset X \times X$$

$$\text{supp}_1(z) := \bigcup_{i \in I} \text{supp}_1(\lambda_i \cdot \chi_{A_i}) \subset X.$$

Remark 2.17. Let M be a marked projective R -module, let $z \in M$, and let $B := \text{supp}_1(z) \subset X$. Then

$$\chi_{X \setminus B} \cdot z = 0.$$

Indeed, $\chi_B \cdot z = z$ (which follows from Remark 2.7) and so $\chi_{X \setminus B} \cdot z = \chi_X \cdot z - \chi_B \cdot z = z - z = 0$.

Remark 2.18 (defining homomorphisms out of marked projective modules). Let $M = \bigoplus_{i \in I} \langle A_i \rangle$ be a marked projective R -module. Let $A \subset X$ be a measurable subset and let $z \in M$ with $\text{supp}_1(z) \subset A$. By Remark 2.7 we have $z = \chi_{\text{supp}_1(z)} \cdot z$. Hence,

$$\begin{aligned} f: \langle A \rangle &\rightarrow M \\ \lambda \cdot \chi_A &\mapsto \lambda \cdot z \end{aligned}$$

is a well-defined R -homomorphism: Indeed, $1 \mapsto z$ describes a well-defined R -homomorphism $\tilde{f}: R \rightarrow M$ and $\text{supp}_1(z) \subset A$ shows that

$$\tilde{f}(\lambda \cdot \chi_A) = \lambda \cdot \chi_A \cdot \tilde{f}(1) = \lambda \cdot \chi_A \cdot z = \lambda \cdot z$$

holds for all $\lambda \in R$. Therefore, f is obtained from \tilde{f} by composition with the canonical inclusion $\langle A \rangle \rightarrow R$.

If $A, B \subset X$ and $f: \langle A \rangle \rightarrow \langle B \rangle$ is an R -homomorphism, then f is given by right multiplication with $z \in R$ and evaluation at χ_A shows that we may choose z always in such a way that $\text{supp}(z) \subset A \times B$.

2.4. Chain complexes. The main objects will be chain complexes consisting of marked projective modules. We continue to work in Setup 2.12.

Definition 2.19 (marked projective chain complex). A *marked projective R -chain complex* is a pair (D_*, η) , consisting of

- an R -chain complex D_* of marked projective R -modules and
- a surjective R -homomorphism $\eta: D_0 \rightarrow L^\infty(\alpha)$, called *augmentation*.

We also write $\eta: D_* \twoheadrightarrow L^\infty(\alpha)$ for such a marked projective R -chain complex.

An *R -chain map* between marked projective R -chain complexes is an R -chain map between the underlying chain complexes that is compatible with the augmentations. A chain map *extends* $\text{id}_{L^\infty(\alpha)}$ if the map in degree -1 is $\text{id}_{L^\infty(\alpha)}$.

Remark 2.20. Marked projective R -chain complexes admit canonical inclusions and projections into/from free R -chain complexes.

Remark 2.21 (induction of resolutions). Let R contain $L * \Gamma$ and let (C_*, ζ) be a free $Z\Gamma$ -resolution of the trivial $Z\Gamma$ -module Z . Let $r \in \mathbb{N}$. Choosing a $Z\Gamma$ -basis of C_r , we can view $\text{Ind}_{Z\Gamma}^R C_r := R \otimes_{L*\Gamma} L \otimes_Z C_r$ as a marked projective R -module (possibly of infinite type). Hence, applying the functor $\text{Ind}_{Z\Gamma}^R := R \otimes_{L*\Gamma} L \otimes_Z \cdot$ to (C_*, ζ) leads to a marked projective R -chain complex (possibly of infinite type in each degree). In such situations, we will always consider this marked structure. In particular, also norms of elements and homomorphisms are interpreted in this way.

2.5. Norms. We use the ℓ^1 -norm to measure the size of elements in marked projective modules and consider the associated operator norm for homomorphisms between marked projective modules. We continue to work in Setup 2.12.

Definition 2.22. Let $M = \bigoplus_{i \in I} \langle A_i \rangle$ be a marked projective R -module. Then M carries the norm $\|\cdot\|_1$, inherited from the corresponding “norm”

$$\begin{aligned} \|\cdot\|_1: \bigoplus_{i \in I} R &\rightarrow \mathbb{R}_{\geq 0} \\ \sum_{i \in I} \lambda_i \cdot e_i &\mapsto \sum_{i \in I} |\lambda_i|_1 \end{aligned}$$

on the canonical hull of M .

Definition 2.23. Let $f: M \rightarrow N$ be an R -homomorphism between marked projective R -modules. Then the *norm* $\|f\|$ of f is the least real number c with

$$\forall z \in M \quad \|f(z)\|_1 \leq c \cdot \|z\|_1.$$

Remark 2.24. All R -homomorphisms $f: M \rightarrow N$ between marked projective R -modules have finite norm: Because the marked presentations are compatible with the ℓ^1 -norms, it suffices to show that maps of the form $f: R \rightarrow R$, $z \mapsto z \cdot \lambda$ for some $\lambda \in R$ have finite norm: Let $z \in R$. Then, by Lemma 2.5,

$$\|f(z)\|_1 = \|z \cdot f(1)\|_1 = \|z \cdot \lambda\|_1 \leq N_2(\lambda) \cdot |\lambda|_\infty \cdot \|z\|_1$$

and hence

$$\|f\| \leq N_2(\lambda) \cdot |\lambda|_\infty.$$

By definition of $Z\mathcal{R}$ (which contains R), we know that $|\lambda|_\infty$ and $N_2(\lambda)$ are indeed finite. However, one should note that in general this norm of f cannot be controlled directly in terms of $|\lambda|_1$. In particular, to compute norms of R -maps, it will in general not be sufficient to just compute the ℓ^1 -norms of the values on the canonical basis. This will cause some unpleasant detours later on. An alternative description of the operator norm is provided in Section 2.7.

In order to generalise the estimate from Remark 2.24 to homomorphisms between general marked projective modules, we introduce the following additional norm on homomorphisms. The said generalisation will be given in Lemma 2.31.

Definition 2.25. Let $f: M \rightarrow N$ be an R -homomorphism between marked projective R -modules $M = \bigoplus_{i \in I} \langle A_i \rangle$ and $N = \bigoplus_{j \in J} \langle B_j \rangle$. We set

$$\|f\|_\infty := \max_{(i,j) \in I \times J} |\lambda_{ij} \cdot \chi_{B_j}|_\infty,$$

where $(\lambda_{ij})_{(i,j) \in I \times J} \in M_{I \times J}(R)$ is the matrix that describes f (through right multiplication by this matrix).

Similarly, if $\eta: M \rightarrow L^\infty(\alpha)$ is an R -homomorphism, we set

$$\|\eta\|_\infty := \max_{i \in I} |\eta(\chi_{A_i} \cdot e_i)|_\infty.$$

2.6. Support and size estimates. In order to handle the deformation and strictification of elements and homomorphisms, we use 1-supports and 1-sizes. We continue to work in Setup 2.12.

Remark 2.26. If M is a marked projective R -module and $z \in M$, then (Remark 2.7)

$$\mu(\text{supp}_1(z)) \leq \nu(\text{supp}(z)) \leq \|z\|_1.$$

However, in general, such norm estimates will be too coarse. Therefore, we introduce the following size notions:

Definition 2.27 (size). Let $M = \bigoplus_{i \in I} \langle A_i \rangle$ be a marked projective R -module.

- If $z \in M$, we abbreviate

$$\text{size}_1(z) := \mu(\text{supp}_1(z)) \in [0, 1].$$

If $z = \sum_{i \in I} \lambda_i \cdot \chi_{A_i} \cdot e_i$, we write

$$\begin{aligned} N_1(z) &:= \sum_{i \in I} N_1(\lambda_i \cdot \chi_{A_i}) \in \mathbb{N} \\ |z|_\infty &:= \sum_{i \in I} |\lambda_i \cdot \chi_{A_i}|_\infty \in \mathbb{R}_{\geq 0}. \end{aligned}$$

- If N is a marked projective R -module and $f: M \rightarrow N$ is an R -homomorphism, we set

$$\begin{aligned} \text{size}_1 f &:= \sum_{i \in I} \text{size}_1(f(\chi_{A_i} \cdot e_i)) \in \mathbb{R}_{\geq 0} \\ N_1(f) &:= \sum_{i \in I} N_1(f(\chi_{A_i} \cdot e_i)) \in \mathbb{N} \\ \underline{N}_1(f) &:= \max_{i \in I} N_1(f(\chi_{A_i} \cdot e_i)) \in \mathbb{N}. \end{aligned}$$

Clearly, $\underline{N}_1(f) \leq N_1(f) \leq \text{rk}(M) \cdot \underline{N}_1(f)$. Similarly, we define $N_2(f)$ and $\underline{N}_2(f)$.

Remark 2.28. Let $f: M \rightarrow N$ be a marked R -homomorphism (Definition 2.14) between marked projective R -modules and let $z \in M$. Then,

$$N_1(f(z)) \leq N_1(z), \quad N_2(f(z)) \leq N_2(z), \quad \text{and} \quad |f(z)|_\infty \leq |z|_\infty.$$

Lemma 2.29 (support estimates). Let $f: M \rightarrow N$ be an R -homomorphism between marked projective R -modules M and N and let $z \in M$.

- (i) For all $\lambda \in R$, we have $\text{size}_1(\lambda \cdot z) \leq N_1(\lambda) \cdot \text{size}_1(z)$;
- (ii) $\text{supp}_1(f(z)) \subset \text{supp}_1(z)$;
- (iii) $\text{size}_1(f(z)) \leq N_1(z) \cdot \text{size}_1(f)$;

- (iv) Let $g: L \rightarrow M$ be an R -homomorphism between marked projective R -modules. Then $\text{size}_1(f \circ g) \leq N_1(g) \cdot \text{size}_1(f)$;
- (v) Let $g: L \rightarrow M$ be a marked R -homomorphism between marked projective R -modules. Then $\text{size}_1(f \circ g) \leq \text{size}_1(f)$.

Proof. Let $M = \bigoplus_{i \in I} \langle A_i \rangle$ be the marked presentation of M and $z = \sum_{i \in I} \lambda_i \cdot \chi_{A_i} \cdot e_i$ with $\lambda_i \in R$.

(i) Let $A := \text{supp}(\lambda) \subset X \times X$ and $B := \text{supp}_1(z) \subset X$. If $x \in \text{supp}_1(\lambda \cdot z)$, then by definition of the convolution product there exists a $w \in [x]_{\mathcal{R}}$ with $(x, w) \in A$ and $w \in B$. Therefore,

$$\begin{aligned} \text{size}_1(\lambda \cdot z) &= \mu(\text{supp}_1(\lambda \cdot z)) \leq \mu(\{x \in X \mid A \cap \text{proj}_2^{-1}(B) \cap \{x\} \times X \neq \emptyset\}) \\ &\leq \int_X \#(A \cap \text{proj}_2^{-1}(B) \cap \{x\} \times X) \, d\mu(x) \\ &= \nu(A \cap \text{proj}_2^{-1}(B)). \end{aligned}$$

As ν can also be computed through proj_2 , we obtain

$$\begin{aligned} \text{size}_1(\lambda \cdot z) &\leq \nu(A \cap \text{proj}_2^{-1}(B)) \\ &= \int_X \#(A \cap \text{proj}_2^{-1}(B) \cap X \times \{y\}) \, d\mu(y) \\ &= \int_B \#(A \cap X \times \{y\}) \, d\mu(y) \\ &\leq N_1(\lambda) \cdot \mu(B) && (\text{because } A = \text{supp}(\lambda)) \\ &= N_1(\lambda) \cdot \text{size}_1(z). \end{aligned}$$

(ii) We compute

$$\text{supp}_1(f(z)) = \text{supp}_1(f(\chi_{\text{supp}_1 z} \cdot z)) = \text{supp}_1(\chi_{\text{supp}_1 z} \cdot f(z)) \subset \text{supp}_1(z).$$

(iii) We use part (i):

$$\begin{aligned} \text{size}_1(f(z)) &= \text{size}_1\left(\sum_{i \in I} \lambda_i \cdot f(\chi_{A_i} \cdot e_i)\right) \\ &= \mu\left(\text{supp}_1\left(\sum_{i \in I} \lambda_i \cdot f(\chi_{A_i} \cdot e_i)\right)\right) \\ &\leq \sum_{i \in I} \mu\left(\text{supp}_1(\lambda_i \cdot f(\chi_{A_i} \cdot e_i))\right) \\ &= \sum_{i \in I} \mu\left(\text{supp}_1(\lambda_i \cdot \chi_{A_i} \cdot f(\chi_{A_i} \cdot e_i))\right) && (\text{by Remark 2.18}) \\ &= \sum_{i \in I} \text{size}_1(\lambda_i \cdot \chi_{A_i} \cdot f(\chi_{A_i} \cdot e_i)) \\ &\leq \sum_{i \in I} N_1(\lambda_i \cdot \chi_{A_i}) \cdot \text{size}_1(f(\chi_{A_i} \cdot e_i)) && (\text{by part (i)}) \\ &\leq N_1(z) \cdot \text{size}_1(f). \end{aligned}$$

(iv) Let $L = \bigoplus_{j \in J} \langle B_j \rangle$ be the marked presentation of L . We have

$$\begin{aligned} \text{size}_1(f \circ g) &= \sum_{j \in J} \text{size}_1(f(g(\chi_{B_j} \cdot e_j))) \\ &\leq \sum_{j \in J} N_1(g(\chi_{B_j} \cdot e_j)) \cdot \text{size}_1(f) && (\text{by part (iii)}) \\ &= N_1(g) \cdot \text{size}_1(f). \end{aligned}$$

(v) By adding trivial summands to the marked presentations, we may assume that the marked R -homomorphism g is of the form

$$g: L = \bigoplus_{i \in I} \langle B_i \rangle \rightarrow \bigoplus_{i \in I} \langle A_i \rangle = M,$$

$$g(\chi_{B_i} \cdot e_i) = \chi_{B_i} \cdot \chi_{A_i} \cdot e_i.$$

We have

$$\begin{aligned} \text{size}_1(f \circ g) &= \sum_{i \in I} \text{size}_1(f(g(\chi_{B_i} \cdot e_i))) \\ &= \sum_{i \in I} \text{size}_1(f(\chi_{B_i} \cdot \chi_{A_i} \cdot e_i)) \\ &= \sum_{i \in I} \text{size}_1(\chi_{B_i} \cdot f(\chi_{A_i} \cdot e_i)) \\ &\leq \sum_{i \in I} N_1(\chi_{B_i}) \cdot \text{size}_1(f(\chi_{A_i} \cdot e_i)) \quad (\text{by part (i)}) \\ &\leq \sum_{i \in I} \text{size}_1(f(\chi_{A_i} \cdot e_i)) \\ &= \text{size}_1(f). \end{aligned}$$

This completes the proof. \square

Lemma 2.30. *Let $M = \bigoplus_{i \in I} \langle A_i \rangle$ be a marked projective R -module and let $\eta: M \rightarrow L^\infty(\alpha)$ be an R -homomorphism. Let $z \in M$. Then,*

$$|\eta(z)|_\infty \leq N_2(z) \cdot |z|_\infty \cdot \|\eta\|_\infty.$$

Proof. Without loss of generality, we may assume that $M = \langle A \rangle$. Since η is R -linear and by Remark 2.9, we have

$$\eta(z)(x) = (z \cdot \eta(\chi_A \cdot e))(x) = \sum_{y \in [x]_{\mathcal{R}}} z(x, y) \cdot \eta(\chi_A \cdot e)(y).$$

Thus,

$$\begin{aligned} |\eta(z)|_\infty &= \sup_{x \in X} \left| \sum_{y \in [x]_{\mathcal{R}}} z(x, y) \cdot \eta(\chi_A \cdot e)(y) \right| \\ &\leq \sup_{x \in X} \left| \sum_{y \in [x]_{\mathcal{R}}} z(x, y) \right| \cdot \|\eta\|_\infty \\ &\leq \sup_{x \in X} N_2(z, x) \cdot |z|_\infty \cdot \|\eta\|_\infty \\ &= N_2(z) \cdot |z|_\infty \cdot \|\eta\|_\infty. \end{aligned} \quad \square$$

Lemma 2.31 (norm estimate). *Let $f: M \rightarrow N$ be an R -homomorphism between marked projective R -modules. Then*

$$\|f\| \leq \underline{N}_2(f) \cdot \|f\|_\infty.$$

We abbreviate $K_f := \underline{N}_2(f) \cdot \|f\|_\infty$.

Proof. This is a straightforward calculation: Let $(\lambda_{ij})_{(i,j) \in I \times J}$ be the matrix describing f (through right multiplication by this matrix). Let $z \in M$, written

as $z = \sum_{i \in I} \lambda_i \cdot \chi_{A_i} \cdot e_i$. Then we obtain

$$\begin{aligned}
\|f(z)\|_1 &= \sum_{j \in J} \left| \sum_{i \in I} \lambda_i \cdot \chi_{A_i} \cdot \lambda_{ij} \cdot \chi_{B_j} \right|_1 \\
&\leq \sum_{j \in J} \sum_{i \in I} |\lambda_i \cdot \chi_{A_i} \cdot \lambda_{ij} \cdot \chi_{B_j}|_1 \\
&\leq \sum_{i \in I} \sum_{j \in J} N_2(\lambda_{ij} \cdot \chi_{B_j}) \cdot |\lambda_{ij} \cdot \chi_{B_j}|_\infty \cdot |\lambda_i \cdot \chi_{A_i}|_1 \quad (\text{Lemma 2.5}) \\
&\leq \|f\|_\infty \cdot \left(\sum_{i \in I} N_2 \left(\sum_{j \in J} \lambda_{ij} \cdot \chi_{B_j} \cdot e_j \right) \cdot |\lambda_i \cdot \chi_{A_i}|_1 \right) \\
&= \|f\|_\infty \cdot \left(\sum_{i \in I} N_2(f(\chi_{A_i} \cdot e_i)) \cdot |\lambda_i \cdot \chi_{A_i}|_1 \right) \\
&\leq \underline{N}_2(f) \cdot \|f\|_\infty \cdot \|z\|_1,
\end{aligned}$$

which shows that $\|f\| \leq \underline{N}_2(f) \cdot \|f\|_\infty$. \square

Remark 2.32. Let $A \subset X$ be measurable, let $\lambda \in R$ with $\text{supp}_1(\lambda) \subset A$, and let $f: \langle A \rangle \rightarrow R$ be the R -homomorphism given by right multiplication with λ . Then

$$\|f\|_\infty = |\lambda|_\infty \quad \text{and} \quad N_2(f) = \underline{N}_2(f) = N_2(\lambda).$$

Remark 2.33. Let $f: M \rightarrow N$ be a marked R -homomorphism between marked projective R -modules, i.e., a composition of marked inclusions and marked projections. Then

$$\underline{N}_1(f) \leq 1, \quad \underline{N}_2(f) \leq 1, \quad \|f\|_\infty \leq 1, \quad \|f\| \leq 1.$$

2.7. An explicit description of the operator norm. We provide an explicit description of the operator norm for homomorphisms between marked projective modules.

Setup 2.34. Let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a standard action of a countable group and let $R := L^\infty(\alpha) * \Gamma$ be the crossed product ring. Let $f: \bigoplus_{i \in I} \langle A_i \rangle \rightarrow \bigoplus_{j \in J} \langle B_j \rangle$ be an R -linear map. Then, f is given by right multiplication with a matrix $z := (z_{i,j})_{i,j}$ over the crossed product ring. There is a finite family $(U_k)_{k \in K}$ of pairwise disjoint measurable subsets of X and a finite subset $F \subset \Gamma$ such that for all $i \in I, j \in J$, we have

$$z_{i,j} = \sum_{(k,\gamma) \in K \times F} a_{i,j,k,\gamma} \cdot (\chi_{\gamma U_k}, \gamma),$$

where $a_{i,j,k,\gamma} \in \mathbb{Z}$.

Moreover, we call such a presentation *reduced* if for all $i \in I, j \in J, k \in K, \gamma \in F$ with $a_{i,j,k,\gamma} \neq 0$, the following hold:

- (1) $U_k \subset B_j$;
- (2) $\gamma U_k \subset A_i$.

Note that in particular, this implies that $\gamma^{-1} A_i \cap U_k = U_k \subset B_j$. It is straightforward to verify that we can always find a reduced presentation.

Proposition 2.35. *In the situation of Setup 2.34, let $f: M \rightarrow N$ be an R -homomorphism between marked projective R -modules. Let z be the matrix representing f as in Setup 2.34. Then,*

$$\|f\| = \max_{i \in I} \max \left\{ \sum_{j \in J, (k,\gamma) \in L} |a_{i,j,k,\gamma}| \mid L \subset K \times F \text{ with } \mu \left(\bigcap_{(k,\gamma) \in L} \gamma U_k \right) > 0 \right\}.$$

In particular, we have $\|f\| \in \mathbb{N}$.

As a first step, we show that we can restrict to modules of the form $L^\infty(A)$.

Lemma 2.36. *Let $A \subset X$ and $f: \langle A \rangle \rightarrow N$ be an R -homomorphism between marked projective R -modules. Then,*

$$\|f\| = \|f|_{L^\infty(A)}\|.$$

Here, $f|_{L^\infty(A)}$ denotes the precomposition of f with the canonical inclusion $L^\infty(A) \hookrightarrow R$.

Proof. For $\gamma \in \Gamma$, we define $f_\gamma: L^\infty(A) \rightarrow N$ by

$$f_\gamma(g) := f((\chi_X, \gamma) \cdot (g, 1)).$$

By definition of the norm as an ℓ^1 -norm over the $L^\infty(A)$ -summands, it is clear that

$$\|f\| = \sup_{\gamma \in \Gamma} \|f_\gamma\|.$$

It thus suffices to show that for $\gamma \in \Gamma$, we have $\|f_\gamma\| \leq \|f_1\|$. Indeed, because f is R -linear, for all $g \in L^\infty(A)$, we have

$$\begin{aligned} \|f_\gamma(g)\|_1 &= \|f((\chi_X, \gamma) \cdot (g, 1))\|_1 \\ &= \|(\chi_X, \gamma) \cdot f((g, 1))\|_1 \\ &= \|f((g, 1))\|_1 \\ &\leq \|f_1\| \cdot |g|_1, \end{aligned}$$

where in the penultimate step, we use that multiplication with (χ_X, γ) defines an isometry because the action of γ preserves the probability measure μ . \square

As a second step, we compute the value of the homomorphism f on specific small building blocks:

Lemma 2.37. *Let $f: \langle A \rangle \rightarrow \langle B \rangle$ be an R -homomorphism, given as in Setup 2.34 by right multiplication with*

$$z := \sum_{(k, \gamma) \in K \times F} a_{k, \gamma} \cdot (\chi_{\gamma U_k}, \gamma),$$

where $a_{k, \gamma} \in Z$, $F \subset \Gamma$ is a finite set, and $(U_k)_{k \in K}$ are pairwise disjoint measurable subsets of B with $\gamma U_k \subset A$ whenever $a_{k, \gamma} \neq 0$ (see Remark 2.18). For $L \subset K \times F$, we define

$$U(L) := \bigcap_{(k, \gamma) \in L} \gamma U_k \cap \bigcap_{(k, \gamma) \in (K \times F) \setminus L} A \setminus \gamma U_k.$$

Then, the $(U(L))_{L \subset K \times F}$ are pairwise disjoint subsets of A and

$$f((\chi_{U(L)}, 1)) = \sum_{\gamma \in \Gamma} a_{*, \gamma}^L \cdot (\chi_{U(L)}, \gamma),$$

where

$$a_{*, \gamma}^L := \begin{cases} 0 & \text{if } U(L) = \emptyset; \\ a_{k, \gamma} & \text{if there exists } k \in K \text{ with } (k, \gamma) \in L; \\ 0 & \text{otherwise.} \end{cases}$$

Because the U_k are pairwise disjoint, there is at most one $k \in K$ with $(k, \gamma) \in L$ unless $U(L) = \emptyset$.

Proof. By construction, the $U(L)$ are pairwise disjoint and for $(k, \gamma) \in K \times F$, we have

$$(2.1) \quad U(L) \cap \gamma U_k = \begin{cases} U(L) & \text{if } (k, \gamma) \in L; \\ \emptyset & \text{if } (k, \gamma) \notin L. \end{cases}$$

Thus, with Equation (2.1), we obtain

$$\begin{aligned}
f((\chi_{U(L)}, 1)) &= (\chi_{U(L)}, 1) \cdot \sum_{(k, \gamma) \in K \times F} a_{k, \gamma} \cdot (\chi_{\gamma U_k}, \gamma) \\
&= \sum_{(k, \gamma) \in K \times F} a_{k, \gamma} \cdot (\chi_{U(L) \cap \gamma U_k}, \gamma) \\
&= \sum_{(k, \gamma) \in L} a_{k, \gamma} \cdot (\chi_{U(L)}, \gamma) \\
&= \sum_{\gamma \in \Gamma} a_{*, \gamma}^L \cdot (\chi_{U(L)}, \gamma). \quad \square
\end{aligned}$$

Proof of Proposition 2.35. Because the operator norm is defined with respect to the ℓ^1 -norm on M , we may assume without loss of generality that $M = \langle A \rangle$. We will therefore drop the index i from the notation. Let m be the maximum on the right hand side of the claim. In order to prove that $\|f\| \geq m$, let $L_m \subset K \times F$ be a maximal subset realising the maximum, i.e.,

$$m = \sum_{j \in J, (k, \gamma) \in L_m} |a_{j, k, \gamma}| \quad \text{and} \quad \mu\left(\bigcap_{(k, \gamma) \in L_m} \gamma U_k\right) > 0.$$

Then, also $\mu(U(L_m)) > 0$, by maximality of L_m . We consider the characteristic function $x := (\chi_{U(L_m)}, 1)$ as a witness and compute

$$\begin{aligned}
|f(x)|_1 &= \sum_{j \in J} |f_j(\chi_{U(L_m)})|_1 \\
&= \sum_{j \in J} \left| \sum_{\gamma \in \Gamma} a_{j, *, \gamma}^{L_m} \cdot (\chi_{U(L_m)}, \gamma) \right|_1 \quad (\text{Lemma 2.37}) \\
&= \sum_{j \in J} \sum_{\gamma \in \Gamma} |a_{j, *, \gamma}^{L_m}| \cdot \mu(U(L_m)) \\
&= m \cdot |x|_1.
\end{aligned}$$

Since $|x|_1 = \mu(U(L_m)) > 0$, this proves that $\|f\| \geq m$.

To show the converse inequality, we use the canonical R -isomorphism

$$\langle A \rangle \cong \bigoplus_{L \subset K \times F} \langle U(L) \rangle,$$

where $U(L)$ is defined as in Lemma 2.37. Equipping the right hand side with the ℓ^1 -norms of the summands, the canonical isomorphism of R -modules is an isometry. Thus, it suffices to prove that $|f(x)|_1 \leq m \cdot |x|_1$ for $x \in \langle U(L) \rangle$. By Lemma 2.36, it suffices to consider $g \in L^\infty(U(L))$. We can write

$$g = \sum_{s \in S} g_s \cdot \chi_{V_s}$$

for a finite set S , $g_s \in Z$, and pairwise disjoint subsets V_s of $U(L)$. The calculation in Lemma 2.37 shows that

$$\begin{aligned}
|f(g)|_1 &= \sum_{j \in J} |f_j(g)|_1 \\
&= \sum_{j \in J} \left| \sum_{\gamma \in \Gamma} a_{j,*,\gamma}^L \cdot \sum_{s \in S} g_s \cdot (\chi_{V_s}, \gamma) \right|_1 \\
&\leq \left(\sum_{j \in J} \sum_{\gamma \in \Gamma} |a_{j,*,\gamma}^L| \right) \cdot |g|_1 \\
&\leq m \cdot |g|_1.
\end{aligned}$$

□

The notion of S -adapted modules and morphisms used below will be introduced later (Definition 5.2).

Corollary 2.38 (adaptation with the same norm). *Let $f: M \rightarrow N$ be an S -adapted morphism between marked projective R -modules. Let M' be a marked projective summand of M . Then, there exists a marked projective summand M'' of M that is S -adapted such that*

$$M' \subset M'' \quad \text{and} \quad \|f|_{M''}\| = \|f|_{M'}\|.$$

Proof. As it suffices to prove the claim componentwise in the domain, suppose that $M = \langle A \rangle$ and $M' = \langle A' \rangle$ with $A' \subset A$. Set

$$A'' := \bigcup_{\substack{L \subset K \times F \text{ s.t.} \\ \mu(A' \cap U(L)) > 0}} U(L),$$

where $U(L)$ is defined as in Lemma 2.37.

Then, since f is S -adapted, so is $M'' := \langle A'' \rangle$ and M'' contains $\langle A' \rangle$. Moreover, by the explicit description of the norm (Proposition 2.35), we have $\|f|_{M''}\| = \|f|_{M'}\|$ as the maximum ranges over the same sums of coefficients. □

3. ALMOST EQUALITY

We introduce quantitative notions of “almost equality” for homomorphisms between marked projective modules and marked projective chain complexes. Almost equality for homomorphisms requires that the homomorphisms are equal except on a marked summand of small dimension and that the norm on this exceptional summand is uniformly controlled. For the comparison of homomorphisms with different domains/targets, we introduce a controlled Gromov–Hausdorff distance. This admits a straightforward generalisation to chain complexes. In particular, we will be able to speak of marked projective chain complexes that are “almost equal”.

Setup 3.1. Let Γ be a countable group and let Z denote \mathbb{Z} (with the usual norm) or a finite field (with the trivial norm). We consider a standard Γ -action $\alpha: \Gamma \curvearrowright (X, \mu)$. Moreover, let \mathcal{R} be the associated orbit relation, and let $R \subset Z\mathcal{R}$ be a subring that contains $L^\infty(\alpha, Z) * \Gamma$.

3.1. Almost equality. We begin with a notion of almost equality that only requires the homomorphisms to be equal except on a marked summand of small dimension.

Definition 3.2 (marked decomposition, almost equality). In Setup 3.1, let $M = \bigoplus_{i \in I} \langle A_i \rangle$ be a marked projective R -module.

- A *marked decomposition* of M is the canonical R -isomorphism

$$M \cong_R \bigoplus_{i \in I} \langle A_i \setminus B_i \rangle \oplus \bigoplus_{i \in I} \langle B_i \rangle$$

induced by a family $(B_i)_{i \in I}$ of (possibly empty) measurable sets $B_i \subset A_i$.

- Let $\delta \in \mathbb{R}_{>0}$ and let $f, f': M \rightarrow N$ be R -homomorphisms between marked projective R -modules. We write $f =_\delta f'$ if there exists a marked decomposition $M \cong_R M_0 \oplus M_1$ with

$$f|_{M_0} = f'|_{M_0} \quad \text{and} \quad \dim(M_1) < \delta.$$

- If $z, z' \in R$, then we write $z =_\delta z'$ if $\mu(\text{supp}_1(z - z')) < \delta$.
- If $z, z' \in L^\infty(\alpha)$, then we write $z =_\delta z'$ if $\mu(\text{supp}(z - z')) < \delta$.

Example 3.3. Let $\gamma \in \Gamma$ and $U, V \subset X$ be measurable subsets. Let $f_U, f_V: R \rightarrow R$ be the R -linear maps given by right multiplication with $(\chi_{\gamma U}, \gamma)$ and $(\chi_{\gamma V}, \gamma)$, respectively. Set $\delta := \mu(U \triangle V)$, where \triangle denotes the symmetric difference. Then, $f_U =_\delta f_V$.

Remark 3.4. Let M be a marked projective R -module. The notion of almost equality can also be defined in the same way for R -linear maps $M \rightarrow L^\infty(\alpha)$. The following lemmas hold for such maps to $L^\infty(\alpha)$ in an analogous way.

Lemma 3.5. *In the situation of Setup 3.1, let $f, f': M \rightarrow N$ be R -homomorphisms between marked projective R -modules and let $\delta \in \mathbb{R}_{>0}$. Then,*

$$f =_\delta f' \iff \text{size}_1(f - f') < \delta.$$

Proof. Let $M = \bigoplus_{i \in I} \langle A_i \rangle$ be the marked presentation of M .

We first assume that $f =_\delta f'$. Let $M \cong_R \bigoplus_{i \in I} \langle A_i \setminus B_i \rangle \oplus \bigoplus_{i \in I} \langle B_i \rangle$ be a corresponding marked decomposition. In particular, we have $(f - f')(\chi_{A_i} \cdot e_i) = (f - f')(\chi_{B_i} \cdot e_i)$ for all $i \in I$. Then, we obtain

$$\begin{aligned} \text{size}_1(f - f') &= \sum_{i \in I} \text{size}_1((f - f')(\chi_{B_i} \cdot e_i)) \\ &\leq \sum_{i \in I} \text{size}_1(\chi_{B_i} \cdot e_i) && \text{(Lemma 2.29 (ii))} \\ &= \sum_{i \in I} \mu(B_i) = \dim\left(\bigoplus_{i \in I} \langle B_i \rangle\right) < \delta. \end{aligned}$$

Conversely, let $\text{size}_1(f - f') < \delta$. For $i \in I$, we set $B_i := \text{supp}_1((f - f')(\chi_{A_i} \cdot e_i))$. We consider

$$M_0 := \bigoplus_{i \in I} \langle A_i \setminus B_i \rangle \quad \text{and} \quad M_1 := \bigoplus_{i \in I} \langle B_i \rangle.$$

Then $M_0 \oplus M_1$ is a marked decomposition of M . Because of $\text{size}_1(f - f') < \delta$ and the definition of B_i , we have

$$\dim M_1 = \sum_{i \in I} \mu(B_i) = \sum_{i \in I} \text{size}_1((f - f')(\chi_{A_i} \cdot e_i)) = \text{size}_1(f - f') < \delta.$$

Moreover, by construction,

$$f(\chi_{A_i \setminus B_i} \cdot e_i) = f'(\chi_{A_i \setminus B_i} \cdot e_i)$$

for all $i \in I$ (Remark 2.17). Hence, $f|_{M_0} = f'|_{M_0}$. \square

Lemma 3.6. *In the situation of Setup 3.1, let L, M, N be marked projective R -modules, let $f, f': M \rightarrow N$ be R -homomorphisms, and let $\delta, \delta' \in \mathbb{R}_{>0}$. We assume that $f =_\delta f'$.*

- (i) *If $f'': M \rightarrow N$ is an R -homomorphism with $f' =_{\delta'} f''$, then $f =_{\delta + \delta'} f''$.*

(ii) If $h: N \rightarrow L$ is an R -homomorphism, then

$$h \circ f =_{\delta} h \circ f'.$$

(iii) If $g: L \rightarrow M$ is an R -homomorphism, then

$$f \circ g =_{N_1(g) \cdot \delta} f' \circ g.$$

(iv) If $g: L \rightarrow M$ is a marked R -homomorphism, then

$$f \circ g =_{\delta} f' \circ g.$$

(v) If $g, g': L \rightarrow M$ are R -homomorphisms with $g =_{\delta'} g'$, then

$$f \circ g =_{N_1(g) \cdot \delta + \delta'} f' \circ g'.$$

(vi) If $g, g': M \rightarrow N$ are R -homomorphisms with $g =_{\delta'} g'$, then

$$f + g =_{\delta + \delta'} f' + g'$$

Proof. Parts (i), (ii) and (vi) are immediate from the definition. Parts (iii) and (iv) follow from the characterisation in Lemma 3.5 and the estimate in Lemma 2.29 (iv) and (v), respectively. Part (v) follows from parts (i), (ii), and (iii). \square

We obtain the following consequences.

Lemma 3.7. *Let $M = \bigoplus_{i \in I} M_i$ and $N = \bigoplus_{j \in J} N_j$ be marked projective R -modules. Let $f, g: M \rightarrow N$ be R -homomorphisms. For $i \in I$ and $j \in J$, let $f_{i,j}, g_{i,j}: M_i \rightarrow N_j$ denote the restrictions to the specified summands in the domain and codomain. Let $\delta \in \mathbb{R}_{>0}$ such that for all $i \in I, j \in J$, we have $f_{i,j} =_{\delta} g_{i,j}$. Then, we have $f =_{\#I \cdot \#J \cdot \delta} g$.*

Proof. This follows directly from Lemma 3.6. \square

Lemma 3.8. *Let $\delta, \delta' \in \mathbb{R}_{>0}$ and let the following be a diagram of projective R -modules:*

$$\begin{array}{ccccc} M & \xrightarrow{g_1} & M' & \xrightarrow{f_1} & M'' \\ \downarrow \partial & & \downarrow \partial' & & \downarrow \partial'' \\ N & \xrightarrow{g_0} & N' & \xrightarrow{f_0} & N'' \end{array}$$

If $\partial' \circ g_1 =_{\delta} g_0 \circ \partial$ and $\partial'' \circ f_1 =_{\delta'} f_0 \circ \partial'$, then

$$\partial'' \circ f_1 \circ g_1 =_{\delta + N_1(g_1) \cdot \delta'} f_0 \circ g_0 \circ \partial.$$

Proof. In view of Lemma 3.6, we have

$$\begin{aligned} \partial'' \circ f_1 \circ g_1 &=_{N_1(g_1) \cdot \delta'} f_0 \circ \partial' \circ g_1, \\ f_0 \circ \partial' \circ g_1 &=_{\delta} f_0 \circ g_0 \circ \partial, \end{aligned}$$

which combines to the claimed almost equality. \square

Remark 3.9. Let $A \subset X$ be measurable, let $\lambda, \lambda' \in R$ with $\text{supp}_1(\lambda) \subset A$, $\text{supp}_1(\lambda') \subset A$, and let $f, f': \langle A \rangle \rightarrow R$ be the R -homomorphisms given by right multiplication by λ and λ' , respectively. Then the following are equivalent:

- (i) $\lambda =_{\delta} \lambda'$;
- (ii) $\mu(\text{supp}_1(\lambda - \lambda')) < \delta$;
- (iii) $f =_{\delta} f'$.

Moreover, we recall that (Remark 2.26)

$$\mu(\text{supp}_1(\lambda - \lambda')) \leq \nu(\text{supp}(\lambda - \lambda')) \leq |\lambda - \lambda'|_1.$$

3.2. Controlled almost equality. Almost equality of maps is not robust with respect to norm estimates. We thus proceed to a controlled version of the notion of almost equality from Section 3.1:

Definition 3.10. In the situation of Setup 3.1, let M and N be marked projective R -modules, let $f, f': M \rightarrow N$ be R -homomorphisms, and let $\delta, K \in \mathbb{R}_{>0}$. We then say that f is (δ, K) -almost equal to f' if there exists a marked decomposition $M \cong_R M_0 \oplus M_1$ with

$$f|_{M_0} = f'|_{M_0} \quad \text{and} \quad \dim(M_1) < \delta \quad \text{and} \quad \|f|_{M_1} - f'|_{M_1}\| \leq K.$$

In this case, we write $f =_{\delta, K} f'$.

Proposition 3.11. In the situation of Setup 3.1, let M, N be marked projective R -modules, let $f, f', f'': M \rightarrow N$ be R -homomorphisms, and let $\delta, \delta', K, K' \in \mathbb{R}_{>0}$. Then the following hold:

- (i) We have $f =_{\delta, K} f$.
- (ii) We have $f =_{\delta, K} f'$ if and only if $f - f' =_{\delta, K} 0$.
- (iii) We have $f =_{\delta, K} f'$ if and only if $f =_{\delta} f'$ and $\|f - f'\| \leq K$.
- (iv) If $f =_{\delta, K} f'$ and $\delta \leq \delta', K \leq K'$, then $f =_{\delta', K'} f'$.
- (v) If $f =_{\delta, K} f'$ and $f' =_{\delta', K'} f''$, then $f =_{\delta+\delta', K+K'} f''$.
- (vi) Let $g, g': M \rightarrow N$ be R -homomorphisms with $g =_{\delta', K'} g'$. If $f =_{\delta, K} f'$, then $f + g =_{\delta+\delta', K+K'} f' + g'$.
- (vii) Let L and P be marked projective R -modules and let $h: N \rightarrow P, g: L \rightarrow M$ be R -homomorphisms. If $f =_{\delta, K} f'$, then

$$h \circ f =_{\delta, \|h\| \cdot K} h \circ f' \quad \text{and} \quad f \circ g =_{N_1(g) \cdot \delta, \|g\| \cdot K} f' \circ g.$$

Moreover, if $g: L \rightarrow M$ is a marked R -homomorphism, then $f \circ g =_{\delta, K} f' \circ g$.

- (viii) Let L and P be marked projective R -modules and let $g, g': L \rightarrow M$ be R -homomorphisms with $g =_{\delta', K'} g'$. If $f =_{\delta, K} f'$, then

$$f \circ g =_{N_1(g) \cdot \delta + \delta', \|g\| \cdot K + \|f'\| \cdot K'} f' \circ g'.$$

Proof. (i)–(vi) These properties are straightforward.

(vii) This follows from Lemma 3.6 and Remark 2.33.

(viii) This follows from parts (vii) and (v). □

3.3. A Gromov–Hausdorff distance for homomorphisms. We introduce a notion of Gromov–Hausdorff distance for homomorphisms between marked projective modules. As in the case of metric spaces, the Gromov–Hausdorff distance is defined by inclusions into joint ambient objects.

Definition 3.12 (marked symmetric difference). Let $M = \bigoplus_{i \in I} \langle A_i \rangle$ be a marked projective R -module and let $N = \bigoplus_{i \in I} \langle B_i \rangle, N' = \bigoplus_{i \in I} \langle B'_i \rangle$ be marked projective summands of M . Then we define the *marked symmetric difference of N and N'* by

$$N \oplus N' := \bigoplus_{i \in I} \langle B_i \triangle B'_i \rangle.$$

Definition 3.13 (Gromov–Hausdorff distance for homomorphisms). In the situation of Setup 3.1, let M, N, M', N' be marked projective R -modules, let $f: M \rightarrow N$ and $f': M' \rightarrow N'$ be R -homomorphisms, and let $\delta, K \in \mathbb{R}_{>0}$. We then say that $d_{\text{GH}}^K(f, f') < \delta$ if there exist marked projective R -modules L, P and marked inclusions $\varphi: M \rightarrow L, \varphi': M' \rightarrow L, \psi: N \rightarrow P, \psi': N' \rightarrow P$ with the following properties:

- $\dim(\varphi(M) \oplus \varphi'(M')) < \delta$
- $\dim(\psi(N) \oplus \psi'(N')) < \delta$

- $F =_{\delta, K} F'$, where $F := \psi \circ f \circ \pi_\varphi$, $F' := \psi' \circ f' \circ \pi_{\varphi'}$ and $\pi_\varphi, \pi_{\varphi'}$ are the marked projections associated with the marked inclusions φ and φ' , respectively.

$$\begin{array}{ccc}
 M & \xrightarrow{f} & N \\
 \varphi \updownarrow \pi_\varphi & & \psi \updownarrow \pi_\psi \\
 L & \overset{F}{\dashrightarrow} & P \\
 \varphi' \updownarrow \pi_{\varphi'} & \overset{F'}{\dashrightarrow} & \psi' \updownarrow \pi_{\psi'} \\
 M' & \xrightarrow{f'} & N'
 \end{array}$$

Proposition 3.14. *In the situation of Setup 3.1, let M, M', M'', N, N', N'' be marked projective R -modules, let $f: M \rightarrow N$, $f': M' \rightarrow N'$, $f'': M'' \rightarrow N''$ be R -homomorphisms, and let $\delta, \delta', K, K' \in \mathbb{R}_{>0}$. Then the following hold:*

- (i) *If $d_{\text{GH}}^K(f, f') < \delta$, then*

$$|\dim M - \dim M'| < \delta \quad \text{and} \quad |\dim N - \dim N'| < \delta.$$

- (ii) *If $M = M'$ and $N = N'$ and $f =_{\delta, K} f'$, then $d_{\text{GH}}^K(f, f') < \delta$.*
 (iii) *If $d_{\text{GH}}^K(f, f') < \delta$ and $\delta \leq \delta'$, $K \leq K'$, then $d_{\text{GH}}^{K'}(f, f') < \delta'$.*
 (iv) *If $d_{\text{GH}}^K(f, f') < \delta$, then there exist marked R -homomorphisms $\Phi: M \rightarrow M'$ and $\Psi: N \rightarrow N'$ with $\Psi \circ f =_{\delta, K} f' \circ \Phi$.*

$$\begin{array}{ccc}
 M & \xrightarrow{f} & N \\
 \Phi \downarrow & & \downarrow \Psi \\
 M' & \xrightarrow{f'} & N'
 \end{array}$$

- (v) *If $d_{\text{GH}}^K(f, f') < \delta$ and $d_{\text{GH}}^{K'}(f', f'') < \delta'$, then*

$$d_{\text{GH}}^{K+K'}(f, f'') < \delta + \delta'.$$

Proof. (i) This follows from

$$|\dim M - \dim M'| = |\dim \varphi(M) - \dim \varphi'(M')| \leq \dim(\varphi(M) \oplus \varphi'(M')) < \delta$$

and similarly for N and N' .

(ii) and (iii) These are clear.

(iv) Take $\Phi := \pi_{\varphi'} \circ \varphi$ and $\Psi := \pi_{\psi'} \circ \psi$. Then

$$\begin{aligned}
 \Psi \circ f &= \pi_{\psi'} \circ \psi \circ f \circ \pi_\varphi \circ \varphi = \pi_{\psi'} \circ F \circ \varphi \\
 &=_{\delta, K} \pi_{\psi'} \circ F' \circ \varphi && \text{(Proposition 3.11 (vii))} \\
 &= \pi_{\psi'} \circ \psi' \circ f' \circ \pi_{\varphi'} \circ \varphi = f' \circ \Phi.
 \end{aligned}$$

(v) Suppose $d_{\text{GH}}^K(f, f') < \delta$ and $d_{\text{GH}}^{K'}(f', f'') < \delta'$ witnessed by

$$\begin{array}{ccc}
 M & \xrightarrow{f} & N \\
 \varphi \updownarrow \pi_\varphi & & \psi \updownarrow \pi_\psi \\
 L & \xrightarrow[F']{F} & P \\
 \varphi' \updownarrow \pi_{\varphi'} & & \psi' \updownarrow \pi_{\psi'} \\
 M' & \xrightarrow{f'} & N' \\
 \rho \updownarrow \pi_\rho & & \theta \updownarrow \pi_\theta \\
 L' & \xrightarrow[G']{G} & P' \\
 \rho' \updownarrow \pi_{\rho'} & & \theta' \updownarrow \pi_{\theta'} \\
 M'' & \xrightarrow{f''} & N''
 \end{array}$$

with $F =_{\delta, K} F'$ and $G =_{\delta', K'} G'$. We may assume that $L = \bigoplus_{i \in I} \langle A_i \rangle$, $L' = \bigoplus_{i \in I} \langle A'_i \rangle$, and $M' = \bigoplus_{i \in I} \langle B'_i \rangle$ with $B'_i \subset A_i$ and $B'_i \subset A'_i$. We define

$$L'' := \bigoplus_{i \in I} \langle A_i \cup A'_i \rangle$$

and similarly, we define P'' . We have

$$\begin{array}{ccc}
 L & \xrightarrow[F']{F} & P \\
 \uparrow & & \downarrow \\
 L'' & & P'' \\
 \downarrow & & \uparrow \\
 L' & \xrightarrow[G']{G} & P'
 \end{array}$$

We denote the compositions $L'' \rightarrow P''$ by $\widetilde{F}, \widetilde{F}', \widetilde{G}, \widetilde{G}'$, respectively. Then we have $\widetilde{F} =_{\delta, K} \widetilde{F}'$ and $\widetilde{G} =_{\delta', K'} \widetilde{G}'$ by Proposition 3.11 (vii), and $\widetilde{F}' = \widetilde{G}$. By Proposition 3.11 (vi), we conclude $\widetilde{F} =_{\delta + \delta', K + K'} \widetilde{G}'$, witnessing that $d_{\text{GH}}^{K+K'}(f, f'') < \delta + \delta'$. \square

3.4. A Gromov–Hausdorff distance for chain complexes. We extend the notion of Gromov–Hausdorff distance to chain complexes and, more generally, to sequences of homomorphisms.

Definition 3.15 (marked projective sequence). In the situation of Setup 3.1, let $n \in \mathbb{N}$. A *marked projective n -sequence* (over R) is a sequence (D_*, η) of the form

$$D_{n+1} \xrightarrow{\partial_{n+1}} D_n \longrightarrow \cdots \longrightarrow D_0 \xrightarrow{\partial_0 = \eta} L^\infty(\alpha),$$

consisting of marked projective R -modules D_0, \dots, D_{n+1} and R -homomorphisms $\partial_0 := \eta, \partial_1, \dots, \partial_{n+1}$.

Clearly, marked projective R -chain complexes (up to degree $n+1$) are marked projective n -sequences.

For the Gromov–Hausdorff distance between sequences, we require the inclusions into a common ambient module to exist simultaneously for all degrees in the given range:

Definition 3.16 (Gromov–Hausdorff distance for sequences). In the situation of Setup 3.1, let $n \in \mathbb{N}$, let $(D_*, \eta), (D'_*, \eta')$ be marked projective n -sequences over R , and let $\delta, K \in \mathbb{R}_{>0}$. We then say that $d_{\text{GH}}^K(D_*, D'_*, n) < \delta$ if there exist marked

projective R -modules P_0, \dots, P_{n+1} and marked inclusions $\varphi_r: D_r \rightarrow P_r$, $\varphi'_r: D'_r \rightarrow P_r$ for all $r \in \{0, \dots, n+1\}$ with the following properties:

- For all $r \in \{0, \dots, n+1\}$, we have

$$\dim(\varphi_r(D_r) \otimes \varphi'_r(D'_r)) < \delta.$$

- For all $r \in \{0, \dots, n+1\}$, we have

$$F_r =_{\delta, K} F'_r,$$

where $F_r := \varphi_{r-1} \circ \partial_r \circ \pi_{\varphi_r}$ and $F'_r := \varphi'_{r-1} \circ \partial'_r \circ \pi_{\varphi'_r}$. Here, $P_{-1} := L^\infty(\alpha)$ and $\varphi_{-1} := \text{id}_{L^\infty(\alpha)} =: \varphi'_{-1}$.

$$\begin{array}{ccc} D_{n+1} & \xrightarrow{\partial_{n+1}} & D_n & \dots & D_0 & \xrightarrow{\partial_0} & L^\infty(\alpha) \\ \varphi_{n+1} \downarrow \uparrow \pi_{\varphi_{n+1}} & & \varphi_n \downarrow \uparrow \pi_{\varphi_n} & & \varphi_0 \downarrow \uparrow \pi_{\varphi_0} & & \parallel \\ P_{n+1} & \xrightarrow{F_{n+1}} & P_n & \dots & P_0 & \xrightarrow{F_0} & L^\infty(\alpha) \\ \varphi'_{n+1} \downarrow \uparrow \pi_{\varphi'_{n+1}} & & \varphi'_n \downarrow \uparrow \pi_{\varphi'_n} & & \varphi'_0 \downarrow \uparrow \pi_{\varphi'_0} & & \parallel \\ D'_{n+1} & \xrightarrow{\partial'_{n+1}} & D'_n & \dots & D'_0 & \xrightarrow{\partial'_0} & L^\infty(\alpha) \end{array}$$

Proposition 3.17. *In the situation of Setup 3.1, let $n \in \mathbb{N}$. Let (D_*, η) , (D'_*, η') , (D''_*, η'') be marked projective n -sequences and let $\delta, \delta', K, K' \in \mathbb{R}_{>0}$. If we have $d_{\text{GH}}^K(D_*, D'_*, n) < \delta$ and $d_{\text{GH}}^{K'}(D'_*, D''_*, n) < \delta'$, then*

$$d_{\text{GH}}^{K+K'}(D_*, D''_*, n) < \delta + \delta'.$$

Proof. The proof is similar to that of Proposition 3.14 (v). \square

4. STRICTIFICATION

We prove two strictification results:

- Theorem 4.8: Every marked projective “almost” chain complex is “close” (in the Gromov–Hausdorff sense) to an actual chain complex.
- Theorem 4.15: Every “almost” chain map is “close” to an actual chain map to a target complex that is “close” to the original target complex.

The control on the constants is delicate, in particular, in degree 0.

Setup 4.1. Let Γ be a countable group and let Z denote \mathbb{Z} (with the usual norm) or a finite field (with the trivial norm). We consider a standard Γ -action $\alpha: \Gamma \curvearrowright (X, \mu)$. Moreover, let \mathcal{R} be the associated orbit relation and let $R \subset Z\mathcal{R}$ be a subring that contains $L^\infty(\alpha, Z) * \Gamma$.

4.1. Almost chain complexes and almost chain maps. Almost chain complexes are sequences that “almost” satisfy the chain complex equations. Almost chain maps are sequences of homomorphisms between almost chain complexes that “almost” satisfy the chain map equations.

Definition 4.2 (almost chain complex). In the situation of Setup 4.1, let $n \in \mathbb{N}$ and $\delta \in \mathbb{R}_{>0}$. A *marked projective δ -almost n -chain complex (over R)* is a marked projective n -sequence (D_*, η) over R (Definition 3.15) such that

$$\forall_{r \in \{0, \dots, n\}} \quad \partial_r \circ \partial_{r+1} =_\delta 0$$

and such that η is δ -surjective, i.e., there exists $z \in D_0$ with $\eta(z) =_\delta 1$.

Definition 4.3 (almost chain map). Let $\delta, \varepsilon \in \mathbb{R}_{>0}$, let $n \in \mathbb{N}$, and let (C_*, ζ) and (D_*, η) be marked projective δ -almost n -chain complexes. An ε -almost n -chain map $C_* \rightarrow D_*$ extending $\text{id}_{L^\infty(\alpha)}$ is a sequence $(f_r: C_r \rightarrow D_r)_{r \in \{0, \dots, n+1\}}$ of R -homomorphisms with

$$\eta \circ f_0 =_\varepsilon \zeta \quad \text{and} \quad \forall_{r \in \{1, \dots, n+1\}} \quad \partial_r^D \circ f_r =_\varepsilon f_{r-1} \circ \partial_r^C.$$

We say that f_* is *marked* if every f_r is a marked R -homomorphism.

Lemma 4.4 (compositions of almost chain maps). Let $\delta, \varepsilon, \varepsilon' \in \mathbb{R}_{>0}$ and let $f_*: C_* \rightarrow D_*$ be an ε -almost n -chain map and $g_*: D_* \rightarrow E_*$ be an ε' -almost n -chain map between marked projective δ -almost n -chain complexes extending the identity. Then, $g_* \circ f_*$ is an $(\varepsilon + N\varepsilon')$ -almost n -chain map extending the identity, where $N := \max_{r \in \{0, \dots, n+1\}} N_1(f_r)$.

Proof. This follows from Lemma 3.8. \square

Proposition 4.5. In the situation of Setup 4.1, let $n \in \mathbb{N}$, and let (D_*, η) and (D'_*, η') be marked projective R -chain complexes (up to degree $n+1$) satisfying $d_{\text{GH}}^K(D_*, D'_*, n) < \delta$. Then there exist marked δ -almost n -chain maps $\Phi_*: D_* \rightarrow D'_*$ and $\Phi'_*: D'_* \rightarrow D_*$ extending $\text{id}_{L^\infty(\alpha)}$ with

$$\Phi'_r \circ \Phi_r =_\delta \text{id}_{D_r}, \quad \Phi_r \circ \Phi'_r =_\delta \text{id}_{D'_r}$$

for all $r \in \{0, \dots, n+1\}$.

Proof. Let $P_*, \varphi_*, \varphi'_*$ be witnesses for $d_{\text{GH}}^K(D_*, D'_*, n) < \delta$ as in Definition 3.16. Set $\Phi_r := \pi_{\varphi'_r} \circ \varphi_r$ and $\Phi'_r := \pi_{\varphi_r} \circ \varphi'_r$. Then Φ_* and Φ'_* are marked δ -almost n -chain maps, since we have

$$\begin{aligned} \Phi_r \circ \partial_{r+1} &= \pi_{\varphi'_r} \circ \varphi_r \circ \partial_{r+1} \\ &= \pi_{\varphi'_r} \circ \varphi_r \circ \partial_{r+1} \circ \pi_{\varphi_{r+1}} \circ \varphi_{r+1} \\ &= \pi_{\varphi'_r} \circ F_{r+1} \circ \varphi_{r+1} \\ &=_\delta \pi_{\varphi'_r} \circ F'_{r+1} \circ \varphi_{r+1} && \text{(Lemma 3.6)} \\ &= \pi_{\varphi'_r} \circ \varphi'_r \circ \partial'_{r+1} \circ \pi_{\varphi'_{r+1}} \circ \varphi_{r+1} \\ &= \partial'_{r+1} \circ \Phi_{r+1} \end{aligned}$$

and similarly for Φ'_* . We may assume that $P_r = \bigoplus_{i \in I} \langle A_i \rangle$, $D_r = \bigoplus_{i \in I} \langle B_i \rangle$, $D'_r = \bigoplus_{i \in I} \langle B'_i \rangle$ with $B_i, B'_i \subset A_i$ and that φ_r, φ'_r are the obvious marked inclusions. Then $\Phi'_r \circ \Phi_r: D_r \rightarrow D_r$ is the marked R -homomorphism given coordinate-wise by the projection $\langle B_i \rangle \rightarrow \langle B_i \cap B'_i \rangle$. Hence

$$\begin{aligned} \text{size}_1(\Phi'_r \circ \Phi_r - \text{id}_{D_r}) &\leq \sum_{i \in I} \mu(B_i \setminus (B_i \cap B'_i)) \\ &\leq \sum_{i \in I} \mu(B_i \triangle B'_i) = \dim(\varphi_r(D_r) \ominus \varphi'_r(D'_r)) \leq \delta \end{aligned}$$

and similarly for $\Phi_r \circ \Phi'_r$. \square

Lemma 4.6. In the situation of Setup 4.1, let $n \in \mathbb{N}$ and let $\delta, K, \varepsilon \in \mathbb{R}_{>0}$. Let (D_*, η) be a marked projective δ -almost n -chain complex, let $z \in D_0$ with $\eta(z) =_\delta 1$, and let $(\widehat{D}_*, \widehat{\eta})$ be a marked projective n -sequence. If $d_{\text{GH}}^K(\widehat{D}_*, D_*, n) < \varepsilon$, then $(\widehat{D}_*, \widehat{\eta})$ is a marked projective $\widehat{\delta}$ -almost n -chain complex, where

$$\begin{aligned} \widehat{\delta} &:= \max\{\delta + (1 + \nu_n(D_*)) \cdot \varepsilon, \delta + N_1(z) \cdot \varepsilon\} \\ \nu_n(D_*) &:= \max\{\|\eta\|_\infty, N_1(\partial_1^D), \dots, N_1(\partial_{n+1}^D)\}. \end{aligned}$$

Moreover, there exists a $\widehat{z} \in \widehat{D}_0$ with $\widehat{\eta}(\widehat{z}) =_{\widehat{\delta}} 1$, $N_1(\widehat{z}) \leq N_1(z)$, $N_2(\widehat{z}) \leq N_2(z)$, and $|\widehat{z}|_\infty \leq |z|_\infty$.

Proof. Suppose that $d_{\text{GH}}^K(\widehat{D}_*, D_*, n) < \varepsilon$ is witnessed by

$$\begin{array}{ccccc}
 D_{r+1} & \xrightarrow{\partial_{r+1}} & D_r & \xrightarrow{\partial_r} & D_{r-1} \\
 \varphi_{r+1} \downarrow \uparrow \pi_{\varphi_{r+1}} & & \varphi_r \downarrow \uparrow \pi_{\varphi_r} & & \varphi_{r-1} \downarrow \uparrow \pi_{\varphi_{r-1}} \\
 P_{r+1} & \xrightarrow{F_{r+1}} & P_r & \xrightarrow{F_r} & P_{r-1} \\
 \widehat{\varphi}_{r+1} \downarrow \uparrow \pi_{\widehat{\varphi}_{r+1}} & & \widehat{\varphi}_r \downarrow \uparrow \pi_{\widehat{\varphi}_r} & & \widehat{\varphi}_{r-1} \downarrow \uparrow \pi_{\widehat{\varphi}_{r-1}} \\
 \widehat{D}_{r+1} & \xrightarrow{\widehat{\partial}_{r+1}} & \widehat{D}_r & \xrightarrow{\widehat{\partial}_r} & \widehat{D}_{r-1}
 \end{array}$$

By Lemma 3.6, we have

$$\widehat{F}_r \circ \widehat{F}_{r+1} =_{\varepsilon + N_1(F_{r+1}) \cdot \varepsilon} F_r \circ F_{r+1}$$

and

$$F_r \circ F_{r+1} = \varphi_{r-1} \circ \partial_r \circ \partial_{r+1} \circ \pi_{\varphi_{r+1}} =_{\delta} 0,$$

since $\pi_{\varphi_{r+1}}$ is a marked R -homomorphism. Moreover, since $N_1(F_{r+1}) = N_1(\partial_{r+1})$, the previous equalities together imply $\widehat{F}_r \circ \widehat{F}_{r+1} =_{\varepsilon + N_1(\partial_{r+1}) \cdot \varepsilon + \delta} 0$ (Lemma 3.6 (i)). We conclude that

$$\widehat{\partial}_r \circ \widehat{\partial}_{r+1} = \pi_{\widehat{\varphi}_{r-1}} \circ \widehat{F}_r \circ \widehat{F}_{r+1} \circ \widehat{\varphi}_{r+1} =_{\varepsilon + N_1(\partial_{r+1}) \cdot \varepsilon + \delta} 0.$$

In degree 0, we consider the diagram

$$\begin{array}{ccc}
 R & \xrightarrow{f_1} & L^\infty(\alpha) \\
 f_z \downarrow & & \parallel \\
 D_0 & \xrightarrow{\eta} & L^\infty(\alpha) \\
 \pi_{\widehat{\varphi}_0} \circ \varphi_0 \downarrow & & \parallel \\
 \widehat{D}_0 & \xrightarrow{\widehat{\eta}} & L^\infty(\alpha)
 \end{array}$$

where the R -homomorphisms f_z, f_1 are given by $z \in D_0$ and $1 \in L^\infty(\alpha)$, respectively. Since $\eta(z) =_{\delta} 1$ in $L^\infty(\alpha)$, we have $\eta \circ f_z =_{\delta} f_1$. By Proposition 4.5 and its proof, we have $\eta =_{\varepsilon} \widehat{\eta} \circ \pi_{\widehat{\varphi}_0} \circ \varphi_0$. Then Lemma 3.8 yields

$$f_1 =_{\delta + N_1(z) \cdot \varepsilon} \widehat{\eta} \circ \pi_{\widehat{\varphi}_0} \circ \varphi_0 \circ f_z.$$

Thus, the element $\widehat{z} := \pi_{\widehat{\varphi}_0} \circ \varphi_0(z) \in \widehat{D}_0$ is as desired by Remark 2.28. \square

In particular, every sequence “close” to a chain complex is an almost chain complex. Conversely, also every almost complex is “close” to a strict chain complex; this is the content of the strictification theorem (Theorem 4.8). Similarly, we establish strictification for almost chain maps (Theorem 4.15).

4.2. Strictification of almost chain complexes. To formulate and prove the strictification theorems in the appropriate uniformity, we bound the complexity of the input data as follows:

Definition 4.7. In the situation of Setup 4.1, let $\delta \in \mathbb{R}_{>0}$, let $n \in \mathbb{N}$, and let (D_*, η) be a marked projective δ -almost n -chain complex. We set

$$\begin{aligned}
 \kappa_n(D_*) &:= \max\{\|\eta\|, \|\partial_1^D\|, \dots, \|\partial_{n+1}^D\|\} \\
 \nu_n(D_*) &:= \max\{\|\eta\|_\infty, \underline{N}_1(\partial_1^D), \dots, \underline{N}_1(\partial_{n+1}^D)\} \\
 \nu_n(D_*) &:= \max\{\|\eta\|_\infty, N_1(\partial_1^D), \dots, N_1(\partial_{n+1}^D)\}.
 \end{aligned}$$

Moreover, if $\kappa \in \mathbb{R}_{>0}$, then we say that $\overline{\kappa}_n(D_*) < \kappa$ if

$$\max\{\text{rk}(D_1), \dots, \text{rk}(D_{n+1}), \kappa_n(D_*), \nu_n(D_*)\} < \kappa$$

and there exists a $z \in D_0$ with

$$\eta(z) =_\delta 1, \quad N_1(z) < \kappa, \quad N_2(z) < \kappa, \quad |z|_\infty < \kappa.$$

Theorem 4.8. *In the situation of Setup 4.1, let $n \in \mathbb{N}$ and let $\kappa \in \mathbb{R}_{>0}$. Then, there exists a $K \in \mathbb{R}_{>0}$ such that: For every $\delta \in \mathbb{R}_{>0}$ and every marked projective δ -almost n -chain complex (D_*, η) with $\bar{\kappa}_n(D_*) < \kappa$, there exists a marked projective R -chain complex $(\hat{D}_*, \hat{\eta})$ (up to degree $n+1$) with*

$$d_{\text{GH}}^K(\hat{D}_*, D_*, n) \leq K \cdot \delta.$$

Moreover, \hat{D}_* can be chosen such that D_* is a subcomplex of \hat{D}_* and such that the inclusion map $D_* \hookrightarrow \hat{D}_*$ is a $(K \cdot \delta)$ -almost n -chain map.

Remark 4.9. Moreover, if (D_*, η) is S -adapted (Definition 5.2) and if there is an S -adapted $z \in D_0$ with $\eta(z) =_\delta 1$, $N_1(z) < \kappa$, $N_2(z) < \kappa$, $|z|_\infty < \kappa$, we can choose $(\hat{D}_*, \hat{\eta})$ to be S -adapted.

Before giving the proof of Theorem 4.8, we discuss the case of degree 0 separately:

Lemma 4.10. *In the situation of Setup 4.1, let $n \in \mathbb{N}$, let $\delta \in \mathbb{R}_{>0}$, and let (D_*, η) be a marked projective δ -almost n -chain complex. Moreover, let $z \in D_0$ with $\eta(z) =_\delta 1$ and let $K := |1 - \eta(z)|_\infty$. Then, there exists a marked projective δ -almost n -chain complex $(\hat{D}_*, \hat{\eta})$ with*

$$d_{\text{GH}}^K(\hat{D}_*, D_*, n) < \delta$$

and the following additional control:

- The R -homomorphism $\hat{\eta}: \hat{D}_0 \rightarrow L^\infty(\alpha)$ is surjective. More precisely, there exists a $\hat{z} \in \hat{D}_0$ with $\hat{\eta}(\hat{z}) = 1$ and $N_1(\hat{z}) \leq N_1(z) + 1$, $N_2(\hat{z}) \leq N_2(z) + 1$, and $|\hat{z}|_\infty \leq |z|_\infty + 1$.
- We have $\bar{\kappa}_n(\hat{D}_*) \leq \bar{\kappa}_n(D_*) + K + 1$.

Remark 4.11. By Lemma 2.30, we have $K = |1 - \eta(z)|_\infty \leq (\bar{\kappa}_n(D_*))^3 + 1$. We may assume $K = 0$ if η is already surjective.

Moreover, if (D_*, η) and z are S -adapted (Definition 5.2), we may choose $(\hat{D}_*, \hat{\eta})$ and \hat{z} also to be S -adapted.

Proof of Lemma 4.10. We consider the error term $B := \text{supp}(\eta(z) - 1) \subset X$ and set

$$\hat{D}_0 := D_0 \oplus \langle B \rangle \quad \text{and} \quad \forall_{r \in \{1, \dots, n+1\}} \quad \hat{D}_r := D_r.$$

Furthermore, we define $\hat{\partial}_r := \partial_r$ for all $r \in \{1, \dots, n+1\}$ and

$$\hat{\eta}: \hat{D}_0 = D_0 \oplus \langle B \rangle \rightarrow L^\infty(\alpha)$$

$$D_0 \ni x \mapsto \eta(x)$$

$$\langle B \rangle \ni \chi_B \cdot e \mapsto 1 - \eta(z).$$

Then, $\hat{\eta}$ is a well-defined R -homomorphism. By construction, $\hat{\eta}$ is surjective; indeed, for $\hat{z} := z + \chi_B \cdot e$, we have $\hat{\eta}(\hat{z}) = 1$, $N_1(\hat{z}) \leq N_1(z) + 1$, $N_2(\hat{z}) \leq N_2(z) + 1$, and $|\hat{z}|_\infty \leq |z|_\infty + 1$. Moreover, by hypotheses $\dim \langle B \rangle = \mu(B) < \delta$ and $\|\hat{\eta}|_{\langle B \rangle}\| \leq K$ (by definition of K). Hence, $\hat{\eta} =_{\delta, K} \eta$.

In addition, we have $\hat{\eta} \circ (D_0 \hookrightarrow \hat{D}_0) = \eta$ and $\hat{\eta} \circ \hat{\partial}_1 = \eta \circ \partial_1$. Thus, $(\hat{D}_*, \hat{\eta})$ also is a δ -almost n -chain complex and $d_{\text{GH}}^K(\hat{D}_*, D_*, n) < \delta$.

We are left to show that $\bar{\kappa}_n(\hat{D}_*) \leq \bar{\kappa}_n(D_*) + K + 1$. To this end, it is sufficient to estimate the quantities associated to $\hat{\eta}$ and to the module \hat{D}_0 . We have

$$\|\hat{\eta}\|_\infty = \max \{ \|\eta\|_\infty, |1 - \eta(z)|_\infty \} \leq \bar{\kappa}_n(D_*) + K$$

as well as

$$\|\widehat{\eta}\| \leq \|\eta\| + |1 - \eta(z)|_1 \leq \|\eta\| + |1 - \eta(z)|_\infty \leq \kappa_n(D_*) + K.$$

Moreover, by definition, $\text{rk}(\widehat{D}_0) \leq \text{rk}(D_0) + 1$. Therefore, we conclude that $\bar{\kappa}_n(\widehat{D}_*) \leq \bar{\kappa}_n(D_*) + K + 1$. \square

Remark 4.12. In the upcoming proof and in the proof of Theorem 4.15 below, we will take the liberty of writing “ const_κ ” for constants that depend only on κ (and n). For instance, $\kappa^2 + 1$ could be subsumed in “ const_κ ”, but $\dim D_1$ cannot.

Proof of Theorem 4.8. In view of the preparation in Lemma 4.10 (and Remark 4.11, Proposition 3.17), we may assume without loss of generality that $\eta: D_0 \rightarrow L^\infty(\alpha)$ is surjective, which will simplify the notation in the proof below.

Let $\widehat{D}_{-1} := L^\infty(\alpha)$, $\widehat{\partial}_{-1} := 0$. It suffices to show the following: For all $r \in \{0, \dots, n+1\}$, there exist marked projective R -modules $\widehat{D}_r = D_r \oplus E_r$ with $\dim E_r < \text{const}_\kappa \cdot \delta$ and an R -homomorphism $\widehat{\partial}_r: \widehat{D}_r \rightarrow \widehat{D}_{r-1}$ with the following properties:

$$\widehat{\partial}_{r-1} \circ \widehat{\partial}_r = 0, \quad \widehat{\partial}_0|_{D_0} = \eta, \quad \widehat{\partial}_r|_{D_r} =_{\text{const}_\kappa \cdot \delta, 1} \partial_r, \quad \|\widehat{\partial}_r|_{E_r}\| \leq \text{const}_\kappa.$$

We proceed by induction over the degree, modifying the chain modules and the boundary operator (twice) in each degree:

For convenience, we set $\widetilde{\partial}_0 := \eta$ and $E_{-1} := 0$, $\partial_{-1} := 0$.

For the induction step, let $r \in \{0, \dots, n\}$ and suppose that we already constructed $\widehat{D}_0, \dots, \widehat{D}_{r-1}$ and R -homomorphisms $\widetilde{\partial}_r: D_r \rightarrow \widehat{D}_{r-1}$ as well as $\widehat{\partial}_j: \widehat{D}_j \rightarrow \widehat{D}_{j-1}$ for all $j \in \{0, \dots, r-1\}$ subject to the following conditions:

$$\widehat{\partial}_{r-1} \circ \widetilde{\partial}_r = 0 \quad \text{and} \quad \forall_{j \in \{0, \dots, r-1\}} \quad \widehat{\partial}_{j-1} \circ \widehat{\partial}_j = 0.$$

Moreover, we assume that D_{r-1} is a marked projective summand in \widehat{D}_{r-1} of codimension $< \text{const}_\kappa \cdot \delta$, that $\widetilde{\partial}_r =_{\text{const}_\kappa \cdot \delta, 1} \partial_r$, that $\widetilde{\partial}_r$ and $\widehat{\partial}_{r-1}$ satisfy the claimed norm bounds, and that $\widehat{\partial}_{r-1}|_{D_{r-1}} =_{\text{const}_\kappa \cdot \delta, 1} \partial_{r-1}$. We have:

$$\begin{array}{ccccccc} D_{r+1} & \xrightarrow{\partial_{r+1}} & D_r & \xrightarrow{\partial_r} & D_{r-1} & \xrightarrow{\partial_{r-1}} & \dots \longrightarrow D_0 \\ & & \searrow \widetilde{\partial}_r & & \downarrow & & \downarrow \\ & & & & \widehat{D}_{r-1} & \xrightarrow{\widehat{\partial}_{r-1}} & \dots \longrightarrow \widehat{D}_0 \end{array}$$

Let $D_{r+1} = \bigoplus_{i \in I} \langle A_i \rangle$ be the marked presentation of D_{r+1} . For $i \in I$, we consider the error term

$$B_i := \text{supp}_1(\widetilde{\partial}_r \circ \partial_{r+1}(\chi_{A_i} \cdot e_i)) \subset X.$$

Then $B_i \subset A_i$ and from $\partial_r \circ \partial_{r+1} =_\delta 0$ and $\widetilde{\partial}_r =_{\text{const}_\kappa \cdot \delta} \partial_r$, we obtain

$$\begin{aligned} \sum_{i \in I} \mu(B_i) &\leq \sum_{i \in I} \text{size}_1(\partial_r \circ \partial_{r+1}(\chi_{A_i} \cdot e_i)) + \sum_{i \in I} \text{size}_1((\widetilde{\partial}_r - \partial_r) \circ \partial_{r+1}(\chi_{A_i} \cdot e_i)) \\ &< \delta + \#I \cdot \text{size}_1(\widetilde{\partial}_r - \partial_r) \cdot \underline{N}_1(\partial_{r+1}) \quad (\text{Lemmas 3.5 and 2.29}) \\ &\leq \delta + \text{rk}(D_{r+1}) \cdot \text{size}_1(\widetilde{\partial}_r - \partial_r) \cdot \underline{N}_1(\partial_{r+1}) \\ &\leq \delta + \text{const}_\kappa \cdot \delta \leq \text{const}_\kappa \cdot \delta. \quad (\text{Lemma 3.5}) \end{aligned}$$

We set

$$E_r := \bigoplus_{i \in I} \langle B_i \rangle \quad \text{and} \quad \widehat{D}_r := D_r \oplus E_r.$$

In particular, $\dim(E_r) < \text{const}_\kappa \cdot \delta$. Moreover, we define

$$\begin{aligned} \tilde{\partial}_{r+1}: D_{r+1} &\rightarrow \hat{D}_r \\ \chi_{A_i} \cdot e_i &\mapsto (\partial_{r+1}(\chi_{A_i} \cdot e_i), -\chi_{B_i} \cdot e_i) \end{aligned}$$

and

$$\begin{aligned} \hat{\partial}_r: \hat{D}_r = D_r \oplus E_r &\rightarrow \hat{D}_{r-1} \\ D_r \ni x &\mapsto \tilde{\partial}_r(x) \\ E_r \ni \chi_{B_i} \cdot e_i &\mapsto \tilde{\partial}_r \circ \partial_{r+1}(\chi_{A_i} \cdot e_i); \end{aligned}$$

both $\tilde{\partial}_{r+1}$ and $\hat{\partial}_r$ are well-defined R -homomorphisms. The marked decomposition $D_{r+1} \cong_R \bigoplus_{i \in I} \langle A_i \setminus B_i \rangle \oplus \bigoplus_{i \in I} \langle B_i \rangle$ shows that $\tilde{\partial}_{r+1} =_{\text{const}_\kappa \cdot \delta} \partial_{r+1}$. By construction, we have

$$\hat{\partial}_r \circ \tilde{\partial}_{r+1} = 0 \quad \text{and} \quad \hat{\partial}_{r-1} \circ \hat{\partial}_r = 0,$$

To see the latter, we observe that $\hat{\partial}_{r-1} \circ \hat{\partial}_r|_{D_r} = \hat{\partial}_{r-1} \circ \tilde{\partial}_r = 0$ and for all $i \in I$

$$\hat{\partial}_{r-1} \circ \hat{\partial}_r(\chi_{B_i} \cdot e_i) = \hat{\partial}_{r-1} \circ \tilde{\partial}_r \circ \partial_{r+1}(\chi_{A_i} \cdot e_i) = 0,$$

where we used the inductive property $\hat{\partial}_{r-1} \circ \tilde{\partial}_r = 0$. Furthermore, by construction, we have $\|\tilde{\partial}_{r+1}\| \leq \|\partial_{r+1}\| + 1$ and $\hat{\partial}_r|_{D_r} = \tilde{\partial}_r =_{\text{const}_\kappa \cdot \delta} \partial_r$. Also, by induction, we have the estimate $\|\hat{\partial}_r\| \leq \|\tilde{\partial}_r\| \cdot (\|\partial_{r+1}\| + 1) \leq (\|\partial_r\| + 1) \cdot (\|\partial_{r+1}\| + 1) \leq \text{const}_\kappa$.

Finally, we set $\hat{D}_{n+1} := D_{n+1}$ and $\hat{\partial}_{n+1} := \tilde{\partial}_{n+1}$. This concludes the construction. In particular, we obtain the claimed norm estimate for $\hat{\partial}_{n+1} = \tilde{\partial}_{n+1}$. \square

Remark 4.13. Clearly, the inductive construction in the above proof preserves S -adaptedness (Definition 5.2) throughout if we start with an S -adapted $z \in D_0$ with $\eta(z) =_\delta 1$.

4.3. Strictification of almost chain maps. To formulate and prove strictification of chain maps, we bound the complexity of the original chain map.

Definition 4.14. In the situation of Setup 4.1, let $\delta \in \mathbb{R}_{>0}$, let $n \in \mathbb{N}$, and let $f_*: C_* \rightarrow D_*$ be a δ -almost n -chain map between marked projective (almost) R -chain complexes. Then, we set

$$\kappa_n(f_*) := \max\{\|f_0\|, \dots, \|f_{n+1}\|\}.$$

Theorem 4.15. In the situation of Setup 4.1, let $n \in \mathbb{N}$ and $\kappa \in \mathbb{R}_{>0}$. Then, there exists a $K \in \mathbb{R}_{>0}$ such that: For all $\delta \in \mathbb{R}_{>0}$, if (C_*, ζ) and (D_*, η) are marked projective R -chain complexes (up to degree $n+1$) with $\max\{\kappa_n(C_*), \nu_n(C_*)\} < \kappa$, $\kappa_n(D_*) < \kappa$ and if $f_*: C_* \rightarrow D_*$ is a δ -almost n -chain map extending $\text{id}_{L^\infty(\alpha)}$ with $\kappa_n(f_*) \leq \kappa$, then there exists a marked projective R -chain complex $(\hat{D}_*, \hat{\eta})$ (up to degree $n+1$) and an R -chain map $\hat{f}_*: C_* \rightarrow \hat{D}_*$ with the following properties:

- We have $d_{\text{GH}}^K(\hat{D}_*, D_*, n) < K \cdot \delta$.
- For all $r \in \{0, \dots, n+1\}$, we have $d_{\text{GH}}^K(\hat{f}_r, f_r) < K \cdot \delta$.

Remark 4.16. Moreover, if (C_*, ζ) and (D_*, η) as well as f_* are S -adapted (Definition 5.2), we can choose $(\hat{D}_*, \hat{\eta})$ and \hat{f}_* to be S -adapted.

Proof. Let $\hat{D}_{-1} := L^\infty(\alpha)$, $\hat{\partial}_{-1} := 0$, and $\hat{f}_{-1} := \text{id}_{L^\infty(\alpha)}$. It suffices to show the following: For all $r \in \{0, \dots, n+1\}$, there exist marked projective R -modules $\hat{D}_r = D_r \oplus E_r$ with $\dim(E_r) < \text{const}_\kappa \cdot \delta$ and R -homomorphisms $\hat{\partial}_r: \hat{D}_r \rightarrow \hat{D}_{r-1}$, $\hat{f}_r: C_r \rightarrow \hat{D}_r$ with the following properties:

$$\hat{\partial}_r \circ \hat{\partial}_{r-1} = 0, \quad \hat{\partial}_r|_{D_r} = \partial_r^D, \quad \|\hat{\partial}_r|_{E_r}\| \leq \text{const}_\kappa$$

and

$$\widehat{f}_r =_{\text{const}_\kappa \cdot \delta, 1} f_r, \quad \|\widehat{f}_r\| \leq \|f_r\| + 1.$$

We proceed by induction over the degree. Let $r \in \{-1, \dots, n\}$ and let us suppose that \widehat{f}_* and \widehat{D}_* are already constructed up to degree r with the claimed properties. We extend the construction to degree $r+1$: To this end, we consider the error function

$$\Delta := \partial_{r+1}^D \circ f_{r+1} - \widehat{f}_r \circ \partial_{r+1}^C : C_{r+1} \rightarrow \widehat{D}_r.$$

Let $C_{r+1} = \bigoplus_{i \in I} \langle A_i \rangle$ be the marked presentation of C_{r+1} ; for $i \in I$, we set

$$B_i := \text{supp}_1(\Delta(\chi_{A_i} \cdot e_i))$$

and

$$E_{r+1} := \bigoplus_{i \in I} \langle B_i \rangle \quad \text{and} \quad \widehat{D}_{r+1} := D_{r+1} \oplus E_{r+1}.$$

We then consider the well-defined R -homomorphisms

$$\begin{aligned} \widehat{f}_{r+1} : C_{r+1} &\rightarrow \widehat{D}_{r+1} \\ \chi_{A_i} \cdot e_i &\rightarrow (f_{r+1}(\chi_{A_i} \cdot e_i), -\chi_{B_i} \cdot e_i) \end{aligned}$$

and

$$\begin{aligned} \widehat{\partial}_{r+1} : \widehat{D}_{r+1} &\rightarrow \widehat{D}_r \\ D_{r+1} \ni x &\mapsto \partial_{r+1}^D(x) \\ \chi_{B_i} \cdot e_i &\mapsto \Delta(\chi_{A_i} \cdot e_i). \end{aligned}$$

In particular, $\widehat{\partial}_{r+1}|_{D_{r+1}} = \partial_{r+1}^D$ and $\widehat{f}_{r+1} =_{\dim(E_{r+1}), 1} f_{r+1}$. It remains to show that this construction has also all the other claimed, inductive, properties:

Dimensions. By construction, we have

$$\Delta = (\partial_{r+1}^D \circ f_{r+1} - f_r \circ \partial_{r+1}^C) - (\widehat{f}_r - f_r) \circ \partial_{r+1}^C.$$

The first difference is δ -almost equal to 0, because f_* is a δ -almost n -chain map. Moreover, we know $(\widehat{f}_r - f_r) \circ \partial_{r+1}^C =_{\text{const}_\kappa \cdot N_1(\partial_{r+1}^C) \cdot \delta} 0$ from the inductive property $\widehat{f}_r =_{\text{const}_\kappa \cdot \delta} f_r$ and Lemma 3.6 (iii). Unifying all constants, we conclude that $\Delta =_{\text{const}_\kappa \cdot \delta} 0$ (Lemma 3.6 (i)). In particular, with Lemma 3.5 we obtain

$$\dim(E_{r+1}) = \sum_{i \in I} \mu(B_i) = \text{size}_1(\Delta) < \text{const}_\kappa \cdot \delta.$$

Chain complex property. On the one hand, for all $x \in D_{r+1}$, we have $\widehat{\partial}_{r+1}(x) = \partial_{r+1}^D(x) \in D_r$ and so with the strict chain complex property of (D_*, η) we calculate

$$\widehat{\partial}_r \circ \widehat{\partial}_{r+1}(x) = \widehat{\partial}_r \circ \partial_{r+1}^D(x) = \partial_r^D(\partial_{r+1}^D(x)) = 0.$$

On the other hand, for all $i \in I$, by construction, we have

$$\widehat{\partial}_r \circ \widehat{\partial}_{r+1}(\chi_{B_i} \cdot e_i) = \widehat{\partial}_r \circ \partial_{r+1}^D \circ f_{r+1}(\chi_{A_i} \cdot e_i) - \widehat{\partial}_r \circ \widehat{f}_r \circ \partial_{r+1}^C(\chi_{A_i} \cdot e_i).$$

Because $\partial_{r+1}^D \circ f_{r+1}(\chi_{A_i} \cdot e_i)$ lies in D_r , the first term equals $\partial_r^D \circ \partial_{r+1}^D \circ f_{r+1}(\chi_{A_i} \cdot e_i)$, which is zero. For the second term, by induction, we have

$$\widehat{\partial}_r \circ \widehat{f}_r \circ \partial_{r+1}^C(\chi_{A_i} \cdot e_i) = \widehat{f}_{r-1} \circ \partial_r^C \circ \partial_{r+1}^C(\chi_{A_i} \cdot e_i),$$

which is also zero. Therefore, $\widehat{\partial}_r \circ \widehat{\partial}_{r+1} = 0$.

Chain map property. The fact that $\widehat{\partial}_{r+1} \circ \widehat{f}_{r+1} - \widehat{f}_r \circ \partial_{r+1}^C = 0$ is immediate from the construction.

Norm estimates. By construction, $\|\widehat{f}_{r+1}\| \leq \|f_{r+1}\| + 1$ and $\|\widehat{\partial}_{r+1}|_{E_{r+1}}\| \leq \|\Delta\|$. Moreover, $\|\Delta\|$ can be subsumed in const_κ . \square

5. DEFORMATION

We explain how to adapt modules, maps, chain complexes, and chain maps to a dense subalgebra of the measurable sets. Our key example is the algebra of cylinder sets in profinite completions along directed systems of finite index normal subgroups.

We prove two deformation results:

- Theorem 5.8: Every marked projective chain complex is “close” (in the Gromov–Hausdorff sense) to an “adapted” chain complex.
- Theorem 5.10: Every chain map is “close” to an “adapted” chain map to a target complex that is “close” to the original target complex.

Setup 5.1. Let Γ be a countable group and let Z denote \mathbb{Z} (with the usual norm) or a finite field (with the trivial norm). We consider a standard Γ -action $\alpha: \Gamma \curvearrowright (X, \mu)$. Moreover, let \mathcal{R} be the associated orbit relation and let $R \subset Z\mathcal{R}$ be a subring that contains $L^\infty(\alpha, Z) * \Gamma$.

Let S be a subalgebra of all measurable sets of X that is μ -dense and that satisfies $\Gamma \cdot S \subset S$. We write L for the subring of $L^\infty(\alpha)$ generated by S .

5.1. Adapted objects/morphisms. Elements, homomorphisms, or modules are adapted to the algebra S if they involve only measurable subsets in S .

Definition 5.2 (adapted). In the situation of Setup 5.1, we say that:

- A marked projective R -module $M = \bigoplus_{i \in I} \langle A_i \rangle$ is *adapted to S* if $A_i \in S$ holds for all $i \in I$.
- An element in $Z\mathcal{R}$ is *adapted to S* if it lies in $L * \Gamma$.
- An R -homomorphism between marked projective R -modules is *adapted* if it is defined over $L * \Gamma$. These notions admit obvious extensions to the case where the target R -module is $L^\infty(\alpha)$.
- Marked projective chain complexes are *adapted to S* if the chain modules and the boundary operators are adapted to S . Chain maps between marked projective chain complexes are *adapted to S* if they consist of adapted homomorphisms.

5.2. Adapting module homomorphisms. Density of the subalgebra leads to a basic deformation observation, which will be the foundation for all other deformation results:

Lemma 5.3. *In the situation of Setup 5.1, let $A, B \in S$ and let $\lambda \in R$ with $\text{supp}(\lambda) \subset A \times B$. Then, for each $\delta \in \mathbb{R}_{>0}$, there exists an S -adapted element $\hat{\lambda} \in L * \Gamma \subset R$ with $\text{supp}(\hat{\lambda}) \subset A \times B$ and:*

- (i) $\hat{\lambda} =_\delta \lambda$;
- (ii) $|\hat{\lambda}|_\infty \leq |\lambda|_\infty$;
- (iii) $N_1(\hat{\lambda}) \leq N_1(\lambda)$ and $N_2(\hat{\lambda}) \leq N_2(\lambda)$;
- (iv) $|\hat{\lambda} - \lambda|_1 < \delta$.

Proof. We use that $L^\infty(\alpha) * \Gamma$ is L^1 -dense in $R \subset Z\mathcal{R}$ and that $L * \Gamma$ is L^1 -dense in $L^\infty(\alpha) * \Gamma$ (because S is μ -dense). We can write λ in the form

$$\lambda = \sum_{k \in \mathbb{N}} \lambda_k \cdot \chi_{\Delta(A_k, \gamma_k)},$$

with $\lambda_k \in Z$, $A_k \subset X$ measurable, $\gamma_k \in \Gamma$, and $\Delta(A_k, \gamma_k) := \{(\gamma_k \cdot x, x) \mid x \in A_k\}$. Moreover, we may assume without loss of generality that this decomposition is reduced in the sense that the sets $\Delta(A_k, \gamma_k)$ are pairwise disjoint.

We now make the following deformations: We truncate λ to a finite sum, approximate the A_k by elements in S , and finally adjust the resulting functions to satisfy the upper bounds for N_1 and N_2 .

Truncation. Because $|\lambda|_1 < \infty$, we can find a finite (non-empty) subset $K \subset \mathbb{N}$ such that

$$\lambda_K := \sum_{k \in K} \lambda_k \cdot \chi_{\Delta(A_k, \gamma_k)} \in L^\infty(\alpha) * \Gamma$$

satisfies $|\lambda_K - \lambda|_1 < \delta$.

Approximation. Because S is μ -dense and $A, B \in S$, for each $k \in K$, we find $\tilde{A}_k \in S$ with $\gamma_k \cdot \tilde{A}_k \times \tilde{A}_k \subset A \times B$ and

$$\mu(\tilde{A}_k \triangle A_k) < \frac{\delta}{\#K \cdot |\lambda|_\infty + 1}.$$

In addition, by inductively refining the choice of the \tilde{A}_k , we may assume that the sets $\Delta(\tilde{A}_k, \gamma_k)$ are pairwise disjoint. Then $\tilde{\lambda} := \sum_{k \in K} \lambda_k \cdot \chi_{\Delta(\tilde{A}_k, \gamma_k)}$ lies in $L * \Gamma$ and since each $\lambda_k \in Z$ it satisfies $|\tilde{\lambda}|_\infty \leq |\lambda|_\infty$ as well as

$$\begin{aligned} \nu(\text{supp}(\tilde{\lambda} - \lambda)) &\leq |\tilde{\lambda} - \lambda|_1 \\ &\leq |\tilde{\lambda} - \lambda_K|_1 + |\lambda_K - \lambda|_1 \\ &< \left| \sum_{k \in K} \lambda_k \cdot \chi_{\Delta(A_k, \gamma_k) \triangle \Delta(\tilde{A}_k, \gamma_k)} \right|_1 + \delta \\ &\leq |\lambda|_\infty \cdot \sum_{k \in K} \mu(A_k \triangle \tilde{A}_k) + \delta \\ &< 2 \cdot \delta. \end{aligned}$$

Controlling N_1 and N_2 . We consider the violating subset

$$E := \{(x, y) \in \text{supp}(\tilde{\lambda}) \mid N_1(\tilde{\lambda}, y) > N_1(\lambda) \text{ or } N_2(\tilde{\lambda}, x) > N_2(\lambda)\}.$$

By definition, E lies in the subalgebra $S \otimes S$ and $E \subset \text{supp}(\tilde{\lambda}) \triangle \text{supp}(\lambda)$. In particular,

$$\nu(E) \leq \nu(\text{supp}(\tilde{\lambda} - \lambda)) < 2 \cdot \delta.$$

We finally consider the modified function

$$\hat{\lambda} := \chi_{\mathcal{R} \setminus E} \cdot \tilde{\lambda}.$$

By construction, $\hat{\lambda} \in L * \Gamma$ and $\hat{\lambda}$ satisfies the following estimates:

- $N_1(\hat{\lambda}) \leq N_1(\lambda)$ and $N_2(\hat{\lambda}) \leq N_2(\lambda)$ (by construction of E);
- $|\hat{\lambda}|_\infty \leq |\tilde{\lambda}|_\infty \leq |\lambda|_\infty$;
- $\mu(\text{supp}_1(\hat{\lambda} - \lambda)) \leq \nu(\text{supp}(\hat{\lambda} - \lambda)) \leq \nu(E) + \nu(\text{supp}(\tilde{\lambda} - \lambda)) \leq 2 \cdot \delta + 2 \cdot \delta = 4 \cdot \delta$;
- $|\hat{\lambda} - \lambda|_1 \leq |\hat{\lambda} - \lambda|_\infty \cdot \nu(\text{supp}(\hat{\lambda} - \lambda)) \leq (|\hat{\lambda}|_\infty + |\lambda|_\infty) \cdot 4 \cdot \delta = |\lambda|_\infty \cdot 8 \cdot \delta$.

In particular, $\hat{\lambda} =_{4 \cdot \delta} \lambda$ (Remark 3.9). Rescaling δ by the factor $1/(4 + |\lambda|_\infty \cdot 8)$, which depends only on λ but not on δ , finishes the proof. \square

Lemma 5.4. *In the situation of Setup 5.1, let $f: M \rightarrow N$ be an R -homomorphism between S -adapted marked projective R -modules. Then, for each $\delta \in \mathbb{R}_{>0}$, there exists an S -adapted R -homomorphism $\hat{f}: M \rightarrow N$ such that:*

- (i) $\hat{f} =_\delta f$;
- (ii) $\|\hat{f}\|_\infty \leq \|f\|_\infty$;
- (iii) $\underline{N}_1(\hat{f}) \leq \underline{N}_1(f)$ and $\underline{N}_2(\hat{f}) \leq \underline{N}_2(f)$.

In particular, with $K_f := \underline{N}_2(f) \cdot \|f\|_\infty$, we obtain $\hat{f} =_{\delta, 2 \cdot K_f} f$.

Proof. Because of Lemma 2.31, it suffices to prove the first three claims. Indeed,

$$\|\widehat{f} - f\| \leq \|\widehat{f}\| + \|f\| \leq \underline{N}_2(\widehat{f}) \cdot \|\widehat{f}\|_\infty + \underline{N}_2(f) \cdot \|f\|_\infty \leq 2 \cdot K_f.$$

By definition of the involved norm and size invariants, it suffices to consider the case that M and N have rank 1; we thus consider the case that $M = \langle A \rangle$, $N = \langle B \rangle$ with $A, B \in S$ and that $f: M \rightarrow N$ is given by $\chi_A \mapsto \lambda \cdot \chi_B$ with $\lambda \in Z\mathcal{R}$ and $\text{supp}(\lambda) \subset A \times B$ (Remark 2.18). Applying the previous approximation result (Lemma 5.3), we find $\lambda \in L * \Gamma$ that satisfies

$$\widehat{\lambda} =_\delta \lambda, \quad |\widehat{\lambda}|_\infty \leq |\lambda|_\infty, \quad N_1(\widehat{\lambda}) \leq N_1(\lambda), \quad N_2(\widehat{\lambda}) \leq N_2(\lambda).$$

Therefore, the R -linear map $\widehat{f}: \langle A \rangle \rightarrow \langle B \rangle$ given by $\chi_A \mapsto \widehat{\lambda} \cdot \chi_B$ is well-defined and has the following properties (Remark 3.9, Remark 2.32):

- (i) $\widehat{f} =_\delta f$;
- (ii) $\|\widehat{f}\|_\infty = \|\widehat{f}(\chi_A)\|_\infty = |\widehat{\lambda}|_\infty \leq |\lambda|_\infty = \|f\|_\infty$;
- (iii) $\underline{N}_1(\widehat{f}) = N_1(\widehat{\lambda}) \leq N_1(\lambda) = \underline{N}_1(f)$ and $\underline{N}_2(\widehat{f}) = N_2(\widehat{\lambda}) \leq N_2(\lambda) = \underline{N}_2(f)$. \square

5.3. Adapting almost chain maps. As above, given an R -homomorphism f between marked projective R -modules, we set $K_f := \underline{N}_2(f) \cdot \|f\|_\infty$.

Proposition 5.5. *In the situation of Setup 5.1, let $n \in \mathbb{N}$, let $\delta \in \mathbb{R}_{>0}$, and let $f_*: (C_*, \zeta) \rightarrow (D_*, \eta)$ be a δ -almost n -chain map between marked projective S -adapted R -chain complexes. Then, there exists an S -adapted $(2 \cdot \delta)$ -almost n -chain map $\widehat{f}_*: (C_*, \zeta) \rightarrow (D_*, \eta)$ extending $\text{id}_{L^\infty(\alpha)}$ with the following properties: For all $r \in \{0, \dots, n+1\}$, we have*

- (i) $\widehat{f}_r =_\delta f_r$;
- (ii) $\|\widehat{f}_r\|_\infty \leq \|f_r\|_\infty$;
- (iii) $\underline{N}_1(\widehat{f}_r) \leq \underline{N}_1(f_r)$ and $\underline{N}_2(\widehat{f}_r) \leq \underline{N}_2(f_r)$.

In particular, $\widehat{f}_r =_{\delta, 2 \cdot K_{f_r}} f_r$.

Proof. For $\varepsilon \in \mathbb{R}_{>0}$, we apply Lemma 5.4 to f_0, \dots, f_{n+1} and to the parameter ε to obtain corresponding S -adapted $\widehat{f}_0, \dots, \widehat{f}_{n+1}$ with norm and multiplicity control. A straightforward computation (using the basic estimates from Lemma 3.6) shows that \widehat{f}_* is a $(\delta + (1 + \nu_n(C_*)) \cdot \varepsilon)$ -almost n -chain map extending $\text{id}_{L^\infty(\alpha)}$. We then choose our initial ε small enough. \square

5.4. Adapting chain complexes (almost). We can approximate chain complexes by adapted almost chain complexes by approximating the chain modules and boundary operators by adapted modules/homomorphisms:

Proposition 5.6. *In the situation of Setup 5.1, let $n \in \mathbb{N}$, let (D_*, η) be a marked projective R -chain complex (up to degree $n+1$), and let $z \in D_0$ with $\eta(z) = 1$. Then, there exists a $K \in \mathbb{R}_{>0}$ such that: For every $\delta \in \mathbb{R}_{>0}$, there exists a marked projective S -adapted δ -almost n -chain complex $(\widehat{D}_*, \widehat{\eta})$ with*

$$d_{\text{GH}}^K(\widehat{D}_*, D_*, n) < \delta$$

and the following additional control:

- For all $r \in \{0, \dots, n+1\}$, we have $\text{rk}(\widehat{D}_r) \leq \text{rk}(D_r)$.
- For all $r \in \{0, \dots, n+1\}$, we have $\underline{N}_1(\partial_r^{\widehat{D}}) \leq \underline{N}_1(\partial_r^D)$.
- There exists an S -adapted $\widehat{z} \in \widehat{D}_0$ with $\widehat{\eta}(\widehat{z}) =_\delta 1$, $N_1(\widehat{z}) \leq N_1(z)$, $N_2(\widehat{z}) \leq N_2(z)$, and $|\widehat{z}|_\infty \leq |z|_\infty$.

In particular, $\overline{\kappa}_n(\widehat{D}_*) \leq \overline{\kappa}_n(D_*) + K$.

As a preparation, we first adapt the chain modules:

Lemma 5.7. *In the situation of Setup 5.1, let $n \in \mathbb{N}$, let (D_*, η) be a marked projective R -chain complex (up to degree $n+1$), and let $z \in D_0$ with $\eta(z) = 1$. Then, there exists a $K \in \mathbb{R}_{>0}$ such that: For every $\delta \in \mathbb{R}_{>0}$, there exists a marked projective δ -almost n -chain complex $(\widehat{D}_*, \widehat{\eta})$ consisting of S -adapted chain modules (but not necessarily S -adapted boundary operators) with*

$$d_{\text{GH}}^K(\widehat{D}_*, D_*, n) < \delta$$

and the following additional control:

- For all $r \in \{0, \dots, n+1\}$, we have $\text{rk}(\widehat{D}_r) \leq \text{rk}(D_r)$.
- For all $r \in \{0, \dots, n+1\}$, we have $\underline{N}_1(\partial_r^{\widehat{D}}) \leq \underline{N}_1(\partial_r^D)$.
- There exists a $\widehat{z} \in \widehat{D}_0$ with $\widehat{\eta}(\widehat{z}) =_\delta 1$, $N_1(\widehat{z}) \leq N_1(z)$, $N_2(\widehat{z}) \leq N_2(z)$, and $|\widehat{z}|_\infty \leq |z|_\infty$.

Proof. Let $\delta \in \mathbb{R}_{>0}$. Because S is μ -dense, we can efficiently adapt the chain modules: For $r \in \mathbb{N}$ and the marked presentation $D_r = \bigoplus_{i \in I} \langle A_i \rangle$, we choose $\widehat{A}_i \in S$ in such a way that $\sum_{i \in I} \mu(\widehat{A}_i \triangle A_i) < \delta$. We then consider the S -adapted “sibling”

$$\widehat{D}_r := \bigoplus_{i \in I} \langle \widehat{A}_i \rangle.$$

In particular, $\text{rk}(\widehat{D}_r) = \text{rk}(D_r)$ and $|\dim \widehat{D}_r - \dim D_r| < \delta$.

We write $\Phi_r: D_r \rightarrow \widehat{D}_r$ for the composition of the canonical inclusion/projection to/from the joint canonical hull $P_r := \bigoplus_{i \in I} \langle A_i \cup \widehat{A}_i \rangle$ of D_r and \widehat{D}_r ; in particular, $\underline{N}_1(\Phi_r) \leq 1$. Similarly, in the other direction, we write $\Psi_r: \widehat{D}_r \rightarrow D_r$ for the canonical R -homomorphism. Moreover, we set $\Phi_{-1} := \text{id}_{L^\infty(\alpha)}$ and $\Psi_{-1} := \text{id}_{L^\infty(\alpha)}$.

Regarding the boundary operators, we consider the compositions

$$\widehat{\partial}_r := \Phi_{r-1} \circ \partial_r \circ \Psi_r: \widehat{D}_r \rightarrow \widehat{D}_{r-1}$$

for $r \in \{0, \dots, n+1\}$ and set $\widehat{\eta} := \widehat{\partial}_0$. Hence we have the commutative diagram:

$$\begin{array}{ccc} D_r & \xrightarrow{\partial_r} & D_{r-1} \\ \Psi_r \uparrow & & \downarrow \Phi_{r-1} \\ \widehat{D}_r & \xrightarrow[\widehat{\partial}_r]{} & \widehat{D}_{r-1}. \end{array}$$

By construction, $\underline{N}_1(\widehat{\partial}_r) \leq \underline{N}_1(\partial_r)$ and we claim that

$$d_{\text{GH}}^{2, \kappa_n(D_*)}(\widehat{D}_*, D_*, n) < (1 + \nu_n(D_*)) \cdot \delta.$$

Indeed, this is witnessed by the following diagram:

$$\begin{array}{ccc} D_r & \xrightarrow{\partial_r} & D_{r-1} \\ \varphi_r \updownarrow \pi_{\varphi_r} & & \varphi_{r-1} \updownarrow \pi_{\varphi_{r-1}} \\ P_r & \xrightarrow[\widehat{F}_r]{F_r} & P_{r-1} \\ \widehat{\varphi}_r \updownarrow \pi_{\widehat{\varphi}_r} & & \widehat{\varphi}_{r-1} \updownarrow \pi_{\widehat{\varphi}_{r-1}} \\ \widehat{D}_r & \xrightarrow[\widehat{\partial}_r]{} & \widehat{D}_{r-1} \end{array}$$

By construction, we have

$$\begin{aligned}\widehat{F}_r &= \widehat{\varphi}_{r-1} \circ \pi_{\widehat{\varphi}_{r-1}} \circ F_r \circ \widehat{\varphi}_r \circ \pi_{\widehat{\varphi}_r} \\ \text{id}_{P_r} &=_{\delta,1} \widehat{\varphi}_r \circ \pi_{\widehat{\varphi}_r} \\ \text{id}_{P_{r-1}} &=_{\delta,1} \widehat{\varphi}_{r-1} \circ \pi_{\widehat{\varphi}_{r-1}}.\end{aligned}$$

Using Proposition 3.11, we conclude

$$\begin{aligned}\widehat{F}_r &=_{\delta, \|F_r\|} \widehat{\varphi}_{r-1} \circ \pi_{\widehat{\varphi}_{r-1}} \circ F_r \\ \widehat{\varphi}_{r-1} \circ \pi_{\widehat{\varphi}_{r-1}} \circ F_r &=_{N_1(F_r) \cdot \delta, \|F_r\|} F_r\end{aligned}$$

and together, using $N_1(F_r) \leq N_1(\partial_r)$ and $\|F_r\| \leq \|\partial_r\|$,

$$\widehat{F}_r =_{(1+N_1(\partial_r)) \cdot \delta, 2 \cdot \|\partial_r\|} F_r.$$

This proves the claim. Since all the estimates depend only on D_* , we can now rescale our initial δ appropriately and apply Lemma 4.6. \square

Proof of Proposition 5.6. We set $K := 2 \cdot \max\{K_{\partial_0}, \dots, K_{\partial_{n+1}}\}$. Let $\delta \in \mathbb{R}_{>0}$. In view of Lemma 5.7 and the triangle inequality of the Gromov–Hausdorff distance between (almost) chain complexes (Proposition 3.17), we may assume without loss of generality that (D_*, η) is a marked projective δ -almost n -chain complex with S -adapted chain modules of the same ranks; however, $z \in D_0$ might not map to 1, but only satisfy the almost equality $\eta(z) =_\delta 1$ (while keeping control on N_1 , N_2 , and $|\cdot|_\infty$).

We then set $\widehat{D}_r := D_r$ for all $r \in \{0, \dots, n+1\}$ and apply the basic approximation lemma (Lemma 5.4) with accuracy δ to $\partial_0 = \eta, \partial_1, \dots, \partial_{n+1}$ to obtain S -adapted R -homomorphisms $\widehat{\partial}_r: \widehat{D}_r \rightarrow \widehat{D}_{r-1}$ for all $r \in \{0, \dots, n+1\}$ with

$$\widehat{\partial}_r =_{\delta, 2 \cdot K_{\partial_r}} \partial_r \quad \text{and} \quad \underline{N}_1(\widehat{\partial}_r) \leq \underline{N}_1(\partial_r).$$

Therefore, $d_{\text{GH}}^K(\widehat{D}_*, D_*, n) < \delta$.

Finally, approximating the carrier sets appearing in z well enough through elements of S , we find an S -adapted $\widehat{z} \in \widehat{D}_0$ with $\widehat{\eta}(\widehat{z}) =_{2 \cdot \delta} 1$, $N_1(\widehat{z}) \leq N_1(z)$, $N_2(\widehat{z}) \leq N_2(z)$, and $|\widehat{z}|_\infty \leq |z|_\infty$ (using Lemma 5.3 in each coordinate). Applying Lemma 4.6 and rescaling the initial parameter δ beforehand completes the proof. \square

5.5. Deformation of chain complexes. We first approximate the chain complex by adapted almost chain complexes (Proposition 5.6) and then strictify these almost chain complexes (Theorem 4.8):

Theorem 5.8. *In the situation of Setup 5.1, let $n \in \mathbb{N}$ and let (D_*, η) be a marked projective R -chain complex (up to degree $n+1$). Then there exists a $K \in \mathbb{R}_{>0}$ such that: For every $\delta \in \mathbb{R}_{>0}$, there exists an S -adapted marked projective chain complex \widehat{D}_* (up to degree $n+1$) with*

$$d_{\text{GH}}^K(\widehat{D}_*, D_*, n) < \delta.$$

Proof. First, we fix some constants: Let $z \in D_0$ with $\eta(z) = 1$ suitable for $\bar{\kappa}_n(D_*)$. We apply Proposition 5.6 to (D_*, η) , z , and $n \in \mathbb{N}$ and thus obtain a constant $K \in \mathbb{R}_{>0}$ with the properties in Proposition 5.6, controlling S -adapted almost chain complexes approximating (D_*, η) . We set

$$\kappa := \bar{\kappa}_n(D_*) + K + 1.$$

Let $K' \in \mathbb{R}_{>0}$ be a constant as provided by Theorem 4.8 when applied to the parameters n and κ .

Now, let $\delta \in \mathbb{R}_{>0}$. Let (D'_*, η'_*) be an S -adapted δ -almost n -chain complex as obtained from Proposition 5.6 for δ . In particular,

$$\bar{\kappa}_n(D'_*) \leq \bar{\kappa}_n(D_*) + K < \kappa \quad \text{and} \quad d_{\text{GH}}^K(D'_*, D_*, n) < \delta.$$

Thus, by the strictification theorem (Theorem 4.8), there exists an S -adapted marked projective R -chain complex $(\widehat{D}_*, \widehat{\eta})$ (up to degree $n+1$) with

$$d_{\text{GH}}^{K'}(\widehat{D}_*, D'_*, n) < K' \cdot \delta.$$

In total, we obtain (Proposition 3.17)

$$d_{\text{GH}}^{K+K'}(\widehat{D}_*, D_*, n) < \delta + K' \cdot \delta = (1 + K') \cdot \delta.$$

Unifying the constants and rescaling δ beforehand, gives the desired result. \square

5.6. Deformation of chain maps. We apply the previously established deformation theorem for chain complexes (Theorem 5.8), the approximation of almost chain maps (Proposition 5.5), and the strictification of chain maps (Theorem 4.15) to prove the following:

Theorem 5.9. *In the situation of Setup 5.1, let $n \in \mathbb{N}$. Let $(\widehat{C}_*, \widehat{\zeta})$ and (D_*, η) be marked projective R -chain complexes (up to degree $n+1$) with $(\widehat{C}_*, \widehat{\zeta})$ being S -adapted. Let $F_*: \widehat{C}_* \rightarrow D_*$ be an R -chain map extending $\text{id}_{L^\infty(\alpha)}$. Then there exists a $K \in \mathbb{R}_{>0}$ such that: For every $\delta \in \mathbb{R}_{>0}$, there exists an S -adapted R -chain map $\widehat{F}_*: \widehat{C}_* \rightarrow \widehat{D}_*$ with*

$$d_{\text{GH}}^K(\widehat{D}_*, D_*, n) < \delta \quad \text{and} \quad \forall_{r \in \{0, \dots, n+1\}} \quad d_{\text{GH}}^K(\widehat{F}_r, F_r) < \delta.$$

Proof. We first fix our set of constants: Let $K \in \mathbb{R}_{>0}$ be a constant as provided by Theorem 5.8 when applied to $n \in \mathbb{N}$ and the target complex (D_*, η) .

We set $N := \max\{N_1(F_0), \dots, N_1(F_{n+1}), 1\}$ and

$$\kappa := \max\{\kappa_n(\widehat{C}_*), \nu_n(\widehat{C}_*), \kappa_n(D_*) + K, N_2(F_0) \cdot \|F_0\|_\infty, \dots, N_2(F_{n+1}) \cdot \|F_{n+1}\|_\infty\} + 1.$$

Let $K' \in \mathbb{R}_{>0}$ be a constant as provided by Theorem 4.15 when applied to $n \in \mathbb{N}$ and κ .

Now, let $\delta \in \mathbb{R}_{>0}$. We proceed in the following steps:

$$\begin{aligned} F_*: \widehat{C}_* &\rightarrow D_* && (D_* \text{ chain complex, } F_* \text{ chain map}) \\ F'_*: \widehat{C}_* &\rightarrow D'_* && (D'_* \text{ adapted chain complex, } F'_* \text{ almost chain map}) \\ F''_*: \widehat{C}_* &\rightarrow D'_* && (D'_* \text{ adapted chain complex, } F''_* \text{ adapted almost chain map}) \\ \widehat{F}_*: \widehat{C}_* &\rightarrow \widehat{D}_* && (\widehat{D}_* \text{ adapted chain complex, } \widehat{F}_* \text{ adapted chain map}) \end{aligned}$$

Adapting the target complex. Theorem 5.8 yields an S -adapted marked projective R -chain complex (D'_*, η') (up to degree $n+1$) with

$$d_{\text{GH}}^K(D'_*, D_*, n) < \delta.$$

In particular, $\kappa_n(D'_*) \leq \kappa_n(D_*) + K < \kappa$. By Proposition 4.5, there exists a marked δ -almost n -chain map $\Phi_*: D_* \rightarrow D'_*$ extending $\text{id}_{L^\infty(\alpha)}$ with

$$\forall_{r \in \{0, \dots, n+1\}} \quad \|\Phi_r\|_\infty \leq 1 \quad \text{and} \quad \underline{N}_1(\Phi_r) \leq 1 \quad (\text{Remark 2.33}).$$

We consider the composition

$$F'_* := \Phi_* \circ F_*: \widehat{C}_* \rightarrow D'_*.$$

By construction, F'_* is an $(N \cdot \delta)$ -almost n -chain map extending $\text{id}_{L^\infty(\alpha)}$ (Lemma 4.4) and

$$\forall_{r \in \{0, \dots, n+1\}} \quad \underline{N}_2(F'_r) \leq \underline{N}_2(F_r), \quad \|F'_r\|_\infty \leq \|F_r\|_\infty, \quad d_{\text{GH}}^{\|F_r\|}(F'_r, F_r) < N \cdot \delta.$$

Adapting the chain map (almost). We apply Proposition 5.5 to obtain an S -adapted $(2 \cdot N \cdot \delta)$ -almost n -chain map $F''_*: \widehat{C}_* \rightarrow D'_*$ that satisfies

$$\begin{aligned} \kappa_n(F''_*) &\leq \max_{r \in \{0, \dots, n+1\}} (\underline{N}_2(F''_r) \cdot \|F''_r\|_\infty) && \text{(Lemma 2.31)} \\ &\leq \max_{r \in \{0, \dots, n+1\}} (\underline{N}_2(F'_r) \cdot \|F'_r\|_\infty) && \text{(Proposition 5.5)} \\ &\leq \max_{r \in \{0, \dots, n+1\}} (\underline{N}_2(F_r) \cdot \|F_r\|_\infty) && \text{(construction of } F'_*) \\ &\leq \kappa \end{aligned}$$

and $d_{\text{GH}}^{2 \cdot K_{F_r}}(F''_r, F'_r) < N \cdot \delta$.

Strictifying the chain map. Finally, we apply the strictification of chain maps (Theorem 4.15 and Remark 4.16) to $F''_*: \widehat{C}_* \rightarrow D'_*$ and the accuracy $2 \cdot N \cdot \delta$. This is possible because $\max\{\kappa_n(\widehat{C}_*), \nu_n(\widehat{C}_*)\} < \kappa$, $\kappa_n(D'_*) < \kappa$, and $\kappa_n(F''_*) \leq \kappa$. Hence, we obtain an S -adapted marked projective chain complex $(\widehat{D}_*, \widehat{\eta})$ (up to degree $n+1$) and an S -adapted R -chain map $\widehat{F}_*: \widehat{C}_* \rightarrow \widehat{D}_*$ (up to degree $n+1$) extending $\text{id}_{L^\infty(\alpha)}$ with

$$\begin{aligned} d_{\text{GH}}^{K'}(\widehat{D}_*, D'_*, n) &< K' \cdot 2 \cdot N \cdot \delta \\ \forall_{r \in \{0, \dots, n+1\}} \quad d_{\text{GH}}^{K'}(\widehat{F}_r, F''_r) &< K' \cdot 2 \cdot N \cdot \delta. \end{aligned}$$

Moreover, we have (Proposition 3.17 and Proposition 3.14)

$$\begin{aligned} d_{\text{GH}}^{K+K'}(\widehat{D}_*, D_*, n) &< \delta + K' \cdot 2 \cdot N \cdot \delta = (1 + K' \cdot 2 \cdot N) \cdot \delta \\ \forall_{r \in \{0, \dots, n+1\}} \quad d_{\text{GH}}^{\|F_r\| + 2 \cdot K_{F_r} + K'}(\widehat{F}_r, F_r) &< (N + N + K' \cdot 2 \cdot N) \cdot \delta. \end{aligned}$$

Unifying the constants and rescaling δ beforehand, gives the claimed result. \square

Theorem 5.10. *In the situation of Setup 5.1, let C_* be a free ZT-resolution of Z that has finite rank in degrees $\leq n+1$ and let $f_*: C_* \rightarrow D_*$ be an α -embedding (Definition 1.1). Then there exists a $K \in \mathbb{R}_{>0}$ such that: For every $\delta \in \mathbb{R}_{>0}$, there exists an S -adapted α -embedding $\widehat{f}_*: C_* \rightarrow \widehat{D}_*$ with*

$$d_{\text{GH}}^K(\widehat{D}_*, D_*, n) < \delta \quad \text{and} \quad \forall_{r \in \{0, \dots, n+1\}} \quad d_{\text{GH}}^K(\text{Ind}_{\text{ZT}}^R(\widehat{f}_r), \text{Ind}_{\text{ZT}}^R(f_r)) < \delta.$$

Proof. Let $(\widehat{C}_*, \widehat{\zeta})$ be the result of applying the functor Ind_Γ^R (Remark 2.21) to the given resolution (C_*, ζ) and let $F_*: \widehat{C}_* \rightarrow D_*$ be the R -chain map (up to degree $n+1$) induced by f_* . Theorem 5.9 yields an S -adapted R -chain map $\widehat{F}_*: \widehat{C}_* \rightarrow \widehat{D}_*$. In particular, the restriction $\widehat{f}_*: C_* \rightarrow \widehat{D}_*$ of \widehat{F}_* to the “ZT-subcomplex” C_* of \widehat{C}_* is an S -adapted α -embedding. \square

Our main application will be in Section 7 to approximate embeddings over profinite actions by chain complexes related to individual finite index subgroups.

Moreover, for invariants with controlled behaviour with respect to the Gromov–Hausdorff distance, there is no difference between considering embeddings with target complexes over the equivalence relation ring or the crossed product ring. In particular, this applies to the measured embedding dimension and the measured embedding volume (Section 17).

6. A LOGARITHMIC NORM FOR MORPHISMS

We introduce a refinement of the quantity $\dim(N) \cdot \log_+ \|f\|$ for homomorphisms $f: M \rightarrow N$ between marked projective modules called $\text{lognorm}(f)$. The key properties of this invariant are that it satisfies dimension estimates, subadditivity over marked decompositions of the domain, compatibility with almost equality (Section 6.2), compatibility with adaptedness (Proposition 6.9), and that it provides an

upper bound for the logarithmic torsion of cokernels (Theorem 7.7). The ad-hoc construction is given in Section 6.1.

6.1. Construction. We refine the expression “ $\dim \cdot \log_+ \|\cdot\|$ ” by allowing for marked decompositions of the domain and for taking marked ranks of images.

Setup 6.1. Let Γ be a countable group and let Z denote \mathbb{Z} (with the usual norm) or a finite field (with the trivial norm). We consider a standard Γ -action $\alpha: \Gamma \curvearrowright (X, \mu)$. Moreover, let \mathcal{R} be the associated orbit relation and let $R \subset Z\mathcal{R}$ be a subring that contains $L^\infty(\alpha, Z) * \Gamma$.

Definition 6.2 (lognorm). In the situation of Setup 6.1, let $f: M \rightarrow N$ be an R -homomorphism between marked projective R -modules.

- The *marked rank* of f is defined as

$$\text{rk}(f) := \inf \{ \dim(N') \mid N' \subset N \text{ is a marked direct summand with } f(M) \subset N' \} \\ \in [0, \dim(N)].$$

- We set

$$\text{lognorm}'(f) := \min \{ \dim(M) \cdot \log_+ \|f\|, \text{rk}(f) \cdot \log_+ \|f\| \} \in \mathbb{R}_{\geq 0}.$$

- Let $D(M)$ denote the “set” of all finite marked decompositions of M . For $(M_i)_{i \in I} \in D(M)$, we set

$$\text{lognorm}'(f, (M_i)_{i \in I}) := \sum_{i \in I} \text{lognorm}'(f|_{M_i}: M_i \rightarrow N) \in \mathbb{R}_{\geq 0}.$$

- Finally, we let

$$\text{lognorm}(f) := \inf_{M_* \in D(M)} \text{lognorm}'(f, M_*) \in \mathbb{R}_{\geq 0}.$$

The value $\text{lognorm}(f)$ depends on the marked structure. We can adapt the definition to subalgebras as follows.

Remark 6.3 (adapted lognorm). Let $S \subset R$ be a subalgebra and $f: M \rightarrow N$ be an S -adapted homomorphism between S -adapted marked projective R -modules. We define $D_S(M)$ as the “set” of all finite marked S -adapted decompositions of M and

$$\text{lognorm}_S(f) := \inf_{M_* \in D_S(M)} \text{lognorm}'(f, M_*) \in \mathbb{R}_{\geq 0}.$$

6.2. Basic properties. We collect some basic properties of lognorm.

Proposition 6.4 (lognorm properties). *In the situation of Setup 6.1, let $f: M \rightarrow N$ be an R -homomorphism between marked projective R -modules. Then the following hold:*

- (i) Dimension estimates. *We have*

$$\text{lognorm}(f) \leq \dim(M) \cdot \log_+ \|f\| \quad \text{and} \quad \text{lognorm}(f) \leq \dim(N) \cdot \log_+ \|f\|.$$

- (ii) Subadditivity. *If $M \cong_R M_0 \oplus M_1$ is a marked decomposition, then*

$$\text{lognorm}(f) \leq \text{lognorm}(f|_{M_0}) + \text{lognorm}(f|_{M_1}).$$

- (iii) Marked inclusion estimates. *If $i: M' \hookrightarrow M$ and $j: N \hookrightarrow N'$ are marked inclusions of marked projective R -modules, then*

$$\text{lognorm}(j \circ f) = \text{lognorm}(f) \quad \text{and} \quad \text{lognorm}(f \circ i) \leq \text{lognorm}(f).$$

- (iv) Almost equality. *If $\delta \in \mathbb{R}_{>0}$, $K \in \mathbb{R}_{\geq 0}$, and $g: M \rightarrow N$ is an R -homomorphism with $f =_{\delta, K} g$, then*

$$\text{lognorm}(f) \leq \text{lognorm}(g) + \delta \cdot \log_+ K.$$

- (v) Gromov–Hausdorff distance. If $\delta, K \in \mathbb{R}_{>0}$, and $f': M' \rightarrow N'$ is an R -homomorphism of marked projective R -modules with $d_{\text{GH}}^K(f, f') < \delta$, then

$$\text{lognorm}(f) \leq \text{lognorm}(f') + \delta \cdot \log_+ K.$$

Proof. (i) The trivial decomposition of M into the single summand M gives both dimension estimates.

(ii) Combining marked decompositions of M_0 and M_1 results in marked decompositions of M . This leads to subadditivity.

(iii) For every marked decomposition $M_* \in D(M)$, we see easily that

$$\text{lognorm}'(j \circ f, M_*) = \text{lognorm}'(f, M_*),$$

because j is a marked inclusion. Taking the infimum over all M_* in $D(M)$ thus shows that $\text{lognorm}(j \circ f) = \text{lognorm}(f)$.

Every marked decomposition $(M_k)_{k \in I} \in D(M)$ induces a marked decomposition $(M'_k)_{k \in I} \in D(M')$ such that for every $k \in I$, the restriction $i|_{M'_k}: M'_k \rightarrow M_k$ is a marked inclusion; we have

$$\text{lognorm}'(f \circ i|_{M'_k}) \leq \text{lognorm}'(f|_{M_k}).$$

Therefore, $\text{lognorm}'(f \circ i, M'_*) \leq \text{lognorm}'(f, M_*)$. Taking the infimum over $D(M)$ shows that $\text{lognorm}(f \circ i) \leq \text{lognorm}(f)$.

(iv) This is a direct consequence of parts (i)–(iii).

(v) By part (iv) and the definition of Gromov–Hausdorff distance, it suffices to show that $\text{lognorm}(\psi \circ f \circ \pi_\varphi) = \text{lognorm}(f)$, if $\varphi: M \rightarrow L$, $\psi: N \rightarrow P$ are marked inclusions and $\pi_\varphi: L \rightarrow M$ is the marked projection associated to φ . By part (iii), we are left to show that $\text{lognorm}(f \circ \pi_\varphi) = \text{lognorm}(f)$. Since $\pi_\varphi \circ \varphi = \text{id}_M$, part (iii) yields $\text{lognorm}(f) \leq \text{lognorm}(f \circ \pi_\varphi)$. Conversely, part (ii) yields

$$\text{lognorm}(f \circ \pi_\varphi) \leq \text{lognorm}(f \circ \pi_\varphi|_{\varphi(M)}) = \text{lognorm } f.$$

This finishes the proof. \square

For a subgroup Λ of Γ , we denote the restricted action by $\alpha|_\Lambda: \Lambda \curvearrowright (X, \mu)$. The inclusion of crossed product rings $L^\infty(\alpha|_\Lambda) * \Lambda \hookrightarrow L^\infty(\alpha) * \Gamma$ induces induction and restriction functors $\text{Ind}_{L^\infty(\alpha|_\Lambda) * \Lambda}^{L^\infty(\alpha) * \Gamma}$ and $\text{Res}_{L^\infty(\alpha|_\Lambda) * \Lambda}^{L^\infty(\alpha) * \Gamma}$ between module categories.

Lemma 6.5. *In the situation of Setup 6.1, let Λ be a subgroup of Γ .*

- (i) *Let g be a map between marked projective $L^\infty(\alpha|_\Lambda) * \Lambda$ -modules. Then*

$$\text{lognorm}(\text{Ind}_{L^\infty(\alpha|_\Lambda) * \Lambda}^{L^\infty(\alpha) * \Gamma} g) \leq \text{lognorm}(g).$$

- (ii) *Suppose that Λ has finite index in Γ . Let f be a map between marked projective $L^\infty(\alpha) * \Gamma$ -modules. Then*

$$\text{lognorm}(\text{Res}_{L^\infty(\alpha|_\Lambda) * \Lambda}^{L^\infty(\alpha) * \Gamma} f) \leq [\Gamma : \Lambda] \cdot \text{lognorm}(f).$$

Proof. (ii) Let $\varphi: M \rightarrow N$ be a map between marked projective $L^\infty(\alpha) * \Gamma$ -modules. Then

$$\begin{aligned} \dim_{L^\infty(\alpha|_\Lambda) * \Lambda}(\text{Res}_{L^\infty(\alpha|_\Lambda) * \Lambda}^{L^\infty(\alpha) * \Gamma} M) &\leq [\Gamma : \Lambda] \cdot \dim_{L^\infty(\alpha) * \Gamma}(M) \\ \text{rk}_{L^\infty(\alpha|_\Lambda) * \Lambda}(\text{Res}_{L^\infty(\alpha|_\Lambda) * \Lambda}^{L^\infty(\alpha) * \Gamma} \varphi) &\leq [\Gamma : \Lambda] \cdot \text{rk}_{L^\infty(\alpha) * \Gamma}(\varphi) \\ \|\text{Res}_{L^\infty(\alpha|_\Lambda) * \Lambda}^{L^\infty(\alpha) * \Gamma} \varphi\| &\leq \|\varphi\|. \end{aligned}$$

The claim follows, since a marked decomposition of M induces a marked decomposition of $\text{Res}_{L^\infty(\alpha|_\Lambda) * \Lambda}^{L^\infty(\alpha) * \Gamma} M$.

Part (i) is proved similarly. \square

6.3. Explicit description of the marked rank. There is the following explicit description of the marked rank.

Lemma 6.6. *In the situation of Setup 6.1, let $R = L^\infty(\alpha) * \Gamma$. Let $f: M \rightarrow N$ be an $(L^\infty(\alpha) * \Gamma)$ -homomorphism between marked projective $(L^\infty(\alpha) * \Gamma)$ -modules. Fix a presentation of f as in Setup 2.34. Then, we have*

$$\text{rk}(f) = \sum_{j \in J} \mu \left(\bigcup_{\substack{(i,k,\gamma) \in I \times K \times F, \\ a_{i,j,k,\gamma} \neq 0}} \gamma^{-1} A_i \cap U_k \right).$$

Proof. Since the marked rank is defined via marked decompositions, we can assume without loss of generality that $N = \langle B \rangle$. We will thus drop $j \in J$ from the notation. Let

$$B' := \bigcup_{\substack{(i,k,\gamma) \in I \times K \times F, \\ a_{i,k,\gamma} \neq 0}} \gamma^{-1} A_i \cap U_k.$$

It suffices to show that $\langle B' \rangle$ is the smallest marked projective summand of N containing $f(M)$.

We first show that the image is contained in this summand. We denote by $\pi_{B \setminus B'}$ the canonical projection to the marked summand $\langle B \setminus B' \rangle$. For all $i \in I$, we have that $\pi_{B \setminus B'} \circ f|_{\langle A_i \rangle}$ is given by right multiplication with the element

$$\begin{aligned} & (\chi_{A_i}, 1) \cdot \sum_{\substack{(k,\gamma) \in K \times F, \\ a_{i,k,\gamma} \neq 0}} a_{i,k,\gamma} \cdot (\chi_{\gamma U_k}, \gamma) \cdot (\chi_{B \setminus B'}, 1) \\ &= \sum_{\substack{(k,\gamma) \in K \times F, \\ a_{i,k,\gamma} \neq 0}} a_{i,k,\gamma} \cdot (\chi_{A_i \cap \gamma U_k \cap \gamma B \setminus \gamma B'}, \gamma) \\ &= 0, \end{aligned}$$

because

$$\begin{aligned} A_i \cap \gamma U_k \cap \gamma B \setminus \gamma B' &\subset \gamma(\gamma^{-1} A_i \cap U_k \setminus B') \\ &\subset \gamma(\gamma^{-1} A_i \cap U_k \setminus (\gamma^{-1} A_i \cap U_k)) = \emptyset. \end{aligned}$$

This shows that $f(M) \subset \langle B' \rangle$.

Conversely, let $(i_0, k_0, \gamma_0) \in I \times K \times F$ such that $a_{i_0, k_0, \gamma_0} \neq 0$. It suffices to show that $\langle \gamma_0^{-1} A_{i_0} \cap U_{k_0} \rangle$ is contained in every marked projective summand containing $f(M)$. Indeed, let $\pi_1: L^\infty(\alpha) * \Gamma \rightarrow L^\infty(\alpha)$ be the $L^\infty(\alpha)$ -linear projection to the summand indexed by $1 \in \Gamma$. We have

$$\begin{aligned} & \pi_1 \circ f((\chi_{U_{k_0}}, \gamma_0^{-1}) \cdot (\chi_{A_{i_0}}, 1)) \\ &= \pi_1 \circ f(\chi_{U_{k_0} \cap \gamma_0^{-1} A_{i_0}}, \gamma_0^{-1}) \\ &= \pi_1 \left((\chi_{U_{k_0} \cap \gamma_0^{-1} A_{i_0}}, \gamma_0^{-1}) \cdot \sum_{(k,\gamma) \in K \times F} a_{i_0, k, \gamma} \cdot (\chi_{\gamma U_k}, \gamma) \right) \\ &= \pi_1 \left(\sum_{(k,\gamma) \in K \times F} a_{i_0, k, \gamma} \cdot (\chi_{U_{k_0} \cap \gamma_0^{-1} A_{i_0} \cap \gamma_0^{-1} \gamma U_k}, \gamma_0^{-1} \gamma) \right) \\ &= \sum_{k \in K} a_{i_0, k, \gamma_0} \cdot \chi_{U_{k_0} \cap \gamma_0^{-1} A_{i_0} \cap U_k} \quad (\text{projection}) \\ &= a_{i_0, k_0, \gamma_0} \cdot \chi_{U_{k_0} \cap \gamma_0^{-1} A_{i_0}} \quad ((U_k)_k \text{ pairwise disjoint}) \end{aligned}$$

Since $a_{i_0, k_0, \gamma_0} \neq 0$, we obtain from Lemma 6.7 below that $\langle U_{k_0} \cap \gamma_0^{-1} A_{i_0} \rangle$ is contained in the marked direct summand generated by $f(M)$. \square

Lemma 6.7 (recognising marked summands). *Let $M = \langle B \rangle$ be a marked summand of $L^\infty(\alpha) * \Gamma$ and let $x \in M$. Let $\pi_1: L^\infty(\alpha) * \Gamma \rightarrow L^\infty(\alpha)$ be the $L^\infty(\alpha)$ -linear projection to the summand corresponding to the identity element $1 \in \Gamma$. Assume that $a \in Z, a \neq 0$ and $A \subset X$ such that $\pi_1(x) = a \cdot \chi_A$. Then, $\langle A \rangle \subset M$.*

Proof. It suffices to show that $A \setminus B$ is a null-set. Because $x \in M = (L^\infty(\alpha) * \Gamma) \cdot (\chi_B, 1)$, we can fix a reduced presentation (see Setup 2.34) for x , i.e., $x = \lambda \cdot (\chi_B, 1)$ for some

$$\lambda = \sum_{(k, \gamma) \in K \times F} a_{(k, \gamma)} \cdot (\chi_{\gamma U_k}, \gamma),$$

with K and $F \subset \Gamma$ finite sets, $a_{(k, \gamma)} \in Z$, and $U_k \subset X$. Then, we have

$$\begin{aligned} a \cdot \chi_{A \setminus B} &= \chi_{A \setminus B} \cdot a \cdot \chi_A \\ &= \chi_{A \setminus B} \cdot \pi_1(x) \\ &= \pi_1(\chi_{A \setminus B} \cdot x) \\ &= \pi_1 \left((\chi_{A \setminus B}, 1) \cdot \sum_{(k, \gamma) \in K \times F} a_{(k, \gamma)} \cdot (\chi_{\gamma U_k}, \gamma) \cdot (\chi_B, 1) \right) \\ &= \sum_{k \in K} a_{(k, 1)} \cdot \chi_{A \setminus B} \cdot \chi_{U_k} \cdot \chi_B \\ &= 0. \end{aligned}$$

Thus, we have $a \cdot \chi_{A \setminus B} = 0$ almost everywhere with $a \neq 0$, which is only possible if $\mu(A \setminus B) = 0$. \square

Lemma 6.8. *In the situation of Setup 6.1, let $R = L^\infty(\alpha) * \Gamma$. Let $f: M \rightarrow N$ be an $(L^\infty(\alpha) * \Gamma)$ -homomorphism between marked projective $(L^\infty(\alpha) * \Gamma)$ -modules. Fix a presentation of f as in Setup 2.34. Then, in the notation of Setup 2.34, we have*

$$\text{rk}(f) \leq \dim(M) \cdot \#I \cdot \#J \cdot \#K \cdot \#F.$$

Proof. Because $\text{rk}(f)$ is subadditive, we can assume without loss of generality that $f: \langle A \rangle \rightarrow \langle B \rangle$ is given by right multiplication with

$$z := a \cdot (\chi_{\gamma U}, \gamma),$$

where $a \in Z, \gamma \in \Gamma$, and $U \subset X$. We have

$$(\chi_A, 1) \cdot z = a \cdot (\chi_{\gamma(\gamma^{-1}A \cap U)}, \gamma) \in \langle \gamma^{-1}A \cap U \rangle$$

and thus,

$$\text{rk}(f) \leq \mu(\gamma^{-1}A \cap U) \leq \mu(A) = \dim \langle A \rangle. \quad \square$$

6.4. The logarithmic norm of adapted homomorphisms. The logarithmic norm of adapted homomorphism can be computed through adapted decompositions:

Proposition 6.9 (lognorm, adapted homomorphisms). *In the situation of Setup 6.1, let $R = L^\infty(\alpha) * \Gamma$. Let S be a dense subalgebra of the set of all measurable subsets of X . Let $f: M \rightarrow N$ be an S -adapted homomorphism between S -adapted marked projective $(L^\infty(\alpha) * \Gamma)$ -modules. Then*

$$\text{lognorm}(f) = \text{lognorm}_S(f).$$

Proof. First, we record the following observation: Fix a presentation of f as in Setup 2.34 with $c \in \mathbb{N}$ summands. For every marked summand W of M , we obtain a presentation of $f|_W$ with the same number of summands. Thus, Lemma 6.8 yields that

$$(6.1) \quad \text{rk}(f|_W) \leq c \cdot \dim(W).$$

Clearly, $\text{lognorm}(f) \leq \text{lognorm}_S(f)$. For the converse estimate, let $M = \bigoplus_{i \in I} M_i$ be a marked decomposition of M and let $\varepsilon > 0$. Without loss of generality, we may assume $0 \notin I$. It suffices to show that there exists an S -adapted marked decomposition $(W_i)_{i \in I \cup \{0\}}$ such that

$$\text{lognorm}'(f, (W_i)_{i \in I \cup \{0\}}) \leq \text{lognorm}'(f, (M_i)_{i \in I}) + \varepsilon.$$

Because S is dense, for every $i \in I$, there exists an S -adapted marked summand U_i such that $\mu(M_i \oplus U_i) < \varepsilon$. Without loss of generality, we can assume that the $(U_i)_{i \in I}$ intersect pairwise trivially.

By Corollary 2.38, for every $i \in I$, there is an S -adapted marked summand V_i containing M_i such that

$$\|f|_{M_i}\| = \|f|_{V_i}\|.$$

For $i \in I$, we set

$$\begin{aligned} W_i &:= U_i \cap V_i \\ W_0 &:= M \ominus \bigoplus_{i \in I} W_i, \end{aligned}$$

where \ominus denotes the “marked complement”. By construction, the $(W_i)_{i \in I \cup \{0\}}$ form an S -adapted marked decomposition of M . For all $i \in I$, we have

$$\begin{aligned} \dim(W_i) &\leq \dim(M_i) + \varepsilon \\ \dim(W_0) &\leq \#I \cdot \varepsilon \\ \|f|_{W_i}\| &\leq \|f|_{M_i}\| \\ \|f|_{W_0}\| &\leq \|f\| \\ \text{rk}(f|_{W_i}) &\leq \text{rk}(f|_{M_i}) + \text{rk}(f|_{W_i \ominus M_i}) \\ &\leq \text{rk}(f|_{M_i}) + c \cdot \dim(W_i \ominus M_i) \quad (\text{Estimate (6.1)}) \\ &\leq \text{rk}(f|_{M_i}) + c \cdot \varepsilon. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} &\text{lognorm}'(f, (W_i)_{i \in I \cup \{0\}}) \\ &= \sum_{i \in I} \log_+ \|f|_{W_i}\| \cdot \min\{\dim W_i, \text{rk}(f|_{W_i})\} + \log_+ \|f|_{W_0}\| \cdot \min\{\dim W_0, \text{rk}(f|_{W_0})\} \\ &\leq \sum_{i \in I} \log_+ \|f|_{M_i}\| \cdot \min\{\dim M_i + \varepsilon, \text{rk}(f|_{M_i}) + c \cdot \varepsilon\} + \log_+ \|f\| \cdot \#I \cdot \varepsilon \\ &\leq \sum_{i \in I} \log_+ \|f|_{M_i}\| \cdot \min\{\dim M_i, \text{rk}(f|_{M_i})\} + \#I \cdot \log_+ \|f\| \cdot \varepsilon \cdot (c + 1) \\ &= \text{lognorm}(f, (M_i)_{i \in I}) + \varepsilon \cdot (1 + \#I \cdot \log_+ \|f\| \cdot (c + 1)). \end{aligned}$$

Rescaling ε appropriately proves the claim. \square

7. PASSING TO FINITE INDEX SUBGROUPS

We explain the passage from the dynamical view to finite index subgroups. More precisely, for dynamical systems $\Gamma \curvearrowright \widehat{\Gamma}_*$ coming from systems Γ_* of finite index normal subgroups, we reinterpret adapted modules and morphisms over the crossed product ring as $Z[\Gamma/\Gamma_i] * \Gamma$ -modules for large enough i (where the multiplication on $Z[\Gamma/\Gamma_i]$ is pointwise multiplication of functions $\Gamma/\Gamma_i \rightarrow Z$).

Adapting dynamical embeddings then leads to homotopy retracts at the level of finite index subgroups and thus eventually to the homological gradient bounds.

Setup 7.1. Let $n \in \mathbb{N}$, let Γ be a residually finite group of type FP_{n+1} , let $(\Gamma_i)_{i \in I}$ be a directed system of finite index normal subgroups of Γ with $\bigcap_{i \in I} \Gamma_i = 1$, let $\alpha: \Gamma \curvearrowright \widehat{\Gamma}_*$ be the associated dynamical system, and let Z be \mathbb{Z} (with the usual norm) or a finite field (with the discrete norm). We write (X, μ) for the probability space $\widehat{\Gamma}_*$.

Setup 7.2. In the situation of Setup 7.1, for each $i \in I$, we denote by

$$S_i := \{\pi_i^{-1}(A) \mid A \subset \Gamma/\Gamma_i\}$$

the *cylindrical sets* in the σ -algebra of $X = \widehat{\Gamma}_*$ corresponding to the canonical projection map $\pi_i: X \rightarrow \Gamma/\Gamma_i$. Let S be the union of all the S_i , namely the subalgebra given by

$$S := \bigcup_{i \in I} S_i = \{\pi_i^{-1}(A) \mid i \in I, A \subset \Gamma/\Gamma_i\}.$$

We will use “ Γ_* -adapted” as a synonym for “ S -adapted” to emphasise the origin of the subalgebra S . We write L_i for the subring of $L^\infty(\alpha, Z)$ generated by S_i .

Furthermore, we abbreviate $R := L^\infty(\alpha, Z) * \Gamma$ and $R_i := L_i * \Gamma$.

Then S is μ -dense in the set of all measurable subsets of X and $\Gamma \cdot S \subset S$ [Löh20b, Lemma 6.4.2]. Therefore, the deformation arguments from the previous sections apply.

7.1. Discretisation. Let $i \in I$. We first establish a correspondence between L_i and $Z[\Gamma/\Gamma_i]$. The projection $\pi_i: X \rightarrow \Gamma/\Gamma_i$ induces mutually inverse $Z\Gamma$ -isomorphisms

$$\begin{aligned} \pi_i^*: Z[\Gamma/\Gamma_i] &\rightarrow L_i, \quad \text{given by } \gamma \cdot \Gamma_i \mapsto \chi_{\pi_i^{-1}(\gamma \cdot \Gamma_i)} \\ \pi_{i,*}: L_i &\rightarrow Z[\Gamma/\Gamma_i], \quad \text{given by } \chi_A \mapsto \chi_{\pi_i(A)} \text{ for } A \in S_i. \end{aligned}$$

In this sense, we may view objects and morphisms over L_i (and whence those over R_i) as “discrete”. Under the above $Z\Gamma$ -isomorphism, the multiplication on L_i translates into pointwise multiplication on $Z[\Gamma/\Gamma_i]$ of functions $\Gamma/\Gamma_i \rightarrow Z$. Moreover, we will use the following “pre-induction” construction:

Definition 7.3. In the situation of Setup 7.2, let $i \in I$.

- Let $M = \bigoplus_{j \in J} \langle A_j \rangle$ be a marked projective R -module that is adapted to S_i . Then, we set

$$M(i) := \bigoplus_{j \in J} \langle A_j \rangle_{R_i} := \bigoplus_{j \in J} R_i \cdot (\chi_{A_j}, 1),$$

which is a marked projective R_i -module (and $R \otimes_{R_i} M(i) \cong_R M$).

- If $f: M \rightarrow N$ is an S_i -adapted R -homomorphism between S_i -adapted marked projective R -modules, then we write $f(i): M(i) \rightarrow N(i)$ for the corresponding R_i -homomorphism (defined by the same matrix).

If f and g are S_i -adapted and composable, then $(g \circ f)(i) = g(i) \circ f(i)$. Consequently, if (D_*, η) is a marked projective R -chain complex that is adapted to S_i , we naturally obtain a corresponding R_i -chain complex $(D_*(i), \eta(i))$, which augments to L_i . Adapted chain maps translate into corresponding R_i -chain maps.

7.2. Dimensions and norms. This discretisation is compatible with taking norms and dimensions:

Remark 7.4 (compatibility of dimensions). In the situation of Setup 7.2, let $i \in I$ and let M be a marked projective R -module that is S_i -adapted. Then the module $M(i)_\Gamma$ of coinvariants is a free Z -module and

$$\mathrm{rk}_Z(M(i)_\Gamma) = [\Gamma : \Gamma_i] \cdot \dim(M).$$

Indeed, using compatibility with marked decompositions, it suffices to consider the case that $M = \langle A \rangle$ with $A \in S_i$, say $A = \pi_i^{-1}(A_i)$ with $A_i \subset \Gamma/\Gamma_i$. Then, the map

$$e_{\gamma\Gamma_i} \mapsto 1 \otimes (\chi_{\pi_i^{-1}(\gamma\Gamma_i)}, 1) \cdot (\chi_A, 1)$$

induces a Z -isomorphism

$$\bigoplus_{A_i} Z \cong_Z L_i \cdot \chi_A \cong_Z Z \otimes_{Z\Gamma} (L_i * \Gamma) \cdot (\chi_A, 1) = M(i)_\Gamma.$$

Therefore,

$$\mathrm{rk}_Z(M(i)_\Gamma) = \#A_i = [\Gamma : \Gamma_i] \cdot \frac{1}{[\Gamma : \Gamma_i]} \cdot \#A_i = [\Gamma : \Gamma_i] \cdot \mu(A) = [\Gamma : \Gamma_i] \cdot \dim(M).$$

Remark 7.5 (compatibility of norms). In the situation of Setup 7.2, let $i \in I$ and let $f: M \rightarrow N$ be an S_i -adapted R -homomorphism between S_i -adapted marked projective R -modules. Then

$$\|f(i)_\Gamma\| \leq \|f\|,$$

where the operator norm on the left hand side is taken with respect to the ℓ^1 -norms induced by the canonical Z -bases on $M(i)_\Gamma$ and $N(i)_\Gamma$ (Remark 7.4). Indeed, for notational simplicity let us consider the case of $f: \langle A \rangle \rightarrow \langle B \rangle$ with $A, B \in S_i$, given by right multiplication by

$$z = \sum_{\lambda \in \Gamma} \sum_{\gamma\Gamma_i \in \Gamma/\Gamma_i} a_{\gamma\Gamma_i, \lambda} \cdot (\chi_{\pi_i^{-1}(\gamma\Gamma_i)}, \lambda).$$

Without loss of generality, we may assume that $a_{\gamma\Gamma_i, \lambda} = 0$ whenever $\lambda^{-1}\gamma\Gamma_i \notin \pi_i(B)$. Using the canonical Z -isomorphisms $M(i)_\Gamma \cong_Z \bigoplus_{\pi_i(A)} Z$ and $N(i)_\Gamma \cong_Z \bigoplus_{\pi_i(B)} Z$ from Remark 7.4, we obtain for all $\gamma\Gamma_i \in \pi_i(A)$ that

$$\begin{aligned} \|f(i)_\Gamma(e_{\gamma\Gamma_i})\|_1 &= \left\| \sum_{\lambda \in \Gamma} a_{\gamma\Gamma_i, \lambda} \cdot e_{\lambda^{-1}\gamma\Gamma_i} \right\|_1 = \sum_{\lambda \in \Gamma} |a_{\gamma\Gamma_i, \lambda}| \\ &= [\Gamma : \Gamma_i] \cdot \|(\chi_{\pi_i^{-1}(\gamma\Gamma_i)}, 1) \cdot z \cdot (\chi_B, 1)\|_1 \\ &\leq [\Gamma : \Gamma_i] \cdot \|f\| \cdot \|(\chi_{\pi_i^{-1}(\gamma\Gamma_i)}, 1) \cdot (\chi_A, 1)\|_1 \\ &= [\Gamma : \Gamma_i] \cdot \|f\| \cdot \frac{1}{[\Gamma : \Gamma_i]} = \|f\|. \end{aligned}$$

Hence, $\|f(i)_\Gamma\| \leq \|f\|$.

7.3. From adapted embeddings to homology retracts. The following theorem shows that adapted embeddings allow to construct suitable homology retracts (and so estimates on the dimensions and the torsions).

Theorem 7.6. *In the situation of Setup 7.1, let C_* be a free $Z\Gamma$ -resolution of Z that has finite rank in degrees $\leq n+1$ and let $f_*: C_* \rightarrow D_*$ be a Γ_* -adapted α -embedding. Then, for all large enough $i \in I$:*

- (i) *The complex $D_*(i)$ is defined up to degree $n+1$ and the Z -module $H_n(\Gamma_i; Z)$ is a Z -retract of $H_n(D_*(i)_\Gamma)$.*

(ii) *In particular, we have*

$$\mathrm{rk}_Z H_n(\Gamma_i; Z) \leq [\Gamma : \Gamma_i] \cdot \dim(D_n)$$

and, in the case of $Z = \mathbb{Z}$, we additionally obtain

$$\log \# \text{tors } H_n(\Gamma_i; \mathbb{Z}) \leq \log \# \text{tors}(D_n(i)_\Gamma / \mathrm{im} \partial_{n+1}^D(i)_\Gamma).$$

Proof. Let $\delta \in \mathbb{R}_{>0}$ and let $f_*: C_* \rightarrow D_*$ be such a Γ_* -adapted α -embedding. Let $I' \subset I$ be the set of all $i \in I$ such that all of the (finitely many!) cylinder sets appearing up to degree $n+1$ in D_* , ∂_*^D , and f_* come from Γ/Γ_i . Then I' is non-empty and upwards-closed in I .

For $i \in I'$, the complex $D_*(i)$ is defined up to degree $n+1$ and augments to L_i . Moreover, we write $C_*(i)$ for the R_i -chain complex $(R \otimes_{Z\Gamma} C_*)(i) \cong_{R_i} R_i \otimes_{Z\Gamma} C_*$. As R_i is flat over $Z\Gamma$, this gives an R_i -resolution of $R_i \otimes_{Z\Gamma} Z \cong_{R_i} L_i$. Because $f_*: C_* \rightarrow D_*$ is S_i -adapted, we obtain a corresponding R_i -chain map $f_*(i): C_*(i) \rightarrow D_*(i)$ extending id_{L_i} :

$$\begin{array}{ccc} C_*(i) & \xrightarrow{f_*(i)} & D_*(i) \\ \downarrow & & \downarrow \\ L_i & \xlongequal{\quad} & L_i \end{array}$$

We now make use of the fact that the left hand side is a resolution to obtain the desired retract: By the fundamental lemma of homological algebra, there is an R_i -chain map $g_*(i): D_*(i) \rightarrow C_*(i)$ extending id_{L_i} and $g_*(i) \circ f_*(i) \simeq_{R_i} \mathrm{id}_{C_*(i)}$. Taking Γ -coinvariants shows that hence $H_n(C_*(i)_\Gamma)$ is a Z -retract of $H_n(D_*(i)_\Gamma)$. Moreover,

$$\begin{aligned} H_n(\Gamma_i; Z) &\cong_Z H_n((Z[\Gamma/\Gamma_i] \otimes_Z C_*)_\Gamma) && \text{(Shapiro lemma)} \\ &\cong_Z H_n((L_i \otimes_Z C_*)_\Gamma) \cong_Z H_n((R_i \otimes_{Z\Gamma} C_*)_\Gamma) \\ &\cong_Z H_n(C_*(i)_\Gamma). \end{aligned}$$

This proves the first part.

For the second part, we obtain from the retract in the first part and elementary properties of rk_Z that

$$\mathrm{rk}_Z H_n(\Gamma_i; Z) \leq \mathrm{rk}_Z H_n(D_*(i)_\Gamma) \leq \mathrm{rk}_Z D_n(i)_\Gamma.$$

Moreover, in the case $Z = \mathbb{Z}$, we obtain

$$\log \# \text{tors } H_n(\Gamma_i; \mathbb{Z}) \leq \log \# \text{tors } H_n(D_*(i)_\Gamma) \leq \log \# \text{tors}(D_n(i)_\Gamma / \mathrm{im} \partial_{n+1}^D(i)_\Gamma),$$

as claimed. \square

7.4. Logarithmic torsion estimates. The goal of this section is to establish a logarithmic torsion growth estimate for cokernels in terms of the logarithmic norm:

Theorem 7.7. *In the situation of Setup 7.2 (with $Z = \mathbb{Z}$), let $f: M \rightarrow N$ be a Γ_* -adapted R -homomorphism between Γ_* -adapted marked projective R -modules. Then*

$$\limsup_{i \in I} \frac{\log \# \text{tors}(N(i)_\Gamma / \mathrm{im} f(i)_\Gamma)}{[\Gamma : \Gamma_i]} \leq \log \text{norm}(f).$$

The proof relies on Gabber's estimate for the torsion in cokernels and the fact that the logarithmic norm of adapted homomorphisms can be computed via adapted decompositions. We will use the following version of Gabber's estimate for the torsion part of cokernels.

Proposition 7.8 ([Sou99, Lemma 1][Sau16, Lemma 3.1]). *Let M and N be finitely generated marked free \mathbb{Z} -modules and let $f: M \rightarrow N$ be a \mathbb{Z} -homomorphism. Then*

$$\log \# \text{tors}(N/\text{im } f) \leq \sum_{b \in B} \log_+ \|f(b)\|_1,$$

whenever B is a subset of the marked \mathbb{Z} -basis of M such that $\{f(b) \mid b \in B\}$ generates $\mathbb{C} \otimes_{\mathbb{Z}} \text{im } f$ over \mathbb{C} and where $\|\cdot\|_1$ refers to the ℓ^1 -norm with respect to the chosen basis on N .

Proof. The classical torsion estimate is $\log \# \text{tors}(N/\text{im } f) \leq \sum_{b \in B} \log_+ \|f(b)\|_2$, where $\|\cdot\|_2$ is the ℓ^2 -norm with respect to the chosen basis on N [Sau16, Lemma 3.1]. Because of $\|\cdot\|_2 \leq \|\cdot\|_1$, the claim follows. \square

Corollary 7.9. *Let M, M', N be marked finite rank free \mathbb{Z} -modules and let $f: M \oplus M' \rightarrow N$ be \mathbb{Z} -linear. Moreover, let $M \cong_{\mathbb{Z}} \bigoplus_{i \in I} M_i$ and $M' \cong_{\mathbb{Z}} \bigoplus_{i \in I'} M'_i$ be marked decompositions; for each $i \in I$ let $N_i \subset N$ be a marked direct summand with $f(M_i) \subset N_i$. Then*

$$\log \# \text{tors}(N/\text{im } f) \leq \sum_{i \in I} \text{rk}_{\mathbb{Z}}(N_i) \cdot \log_+ \|f|_{M_i}\| + \sum_{i \in I'} \text{rk}_{\mathbb{Z}}(M'_i) \cdot \log_+ \|f|_{M'_i}\|.$$

Proof. For each $i \in I$, let $B_i \subset M_i$ be a subset of the marked basis such that $\{f(b) \mid b \in B_i\}$ generates $\mathbb{C} \otimes_{\mathbb{Z}} f(M_i)$ over \mathbb{C} . For each $i \in I$, let $B'_i \subset M'_i$ be the marked basis. Then $B := \bigcup_{i \in I} B_i \cup \bigcup_{i \in I'} B'_i$ is a subset of the marked basis of $M \oplus M'$ such that $\{f(b) \mid b \in B\}$ generates $\mathbb{C} \otimes_{\mathbb{Z}} \text{im } f$ over \mathbb{C} . Therefore, Proposition 7.8 shows that

$$\begin{aligned} \log \# \text{tors}(N/\text{im } f) &\leq \sum_{b \in B} \log_+ \|f(b)\|_1 \\ &\leq \sum_{i \in I} \sum_{b \in B_i} \log_+ \|f(b)\|_1 + \sum_{i \in I'} \sum_{b \in B'_i} \log_+ \|f(b)\|_1 \\ &\leq \sum_{i \in I} \#B_i \cdot \log_+ \|f|_{M_i}\| + \sum_{i \in I'} \#B'_i \cdot \log_+ \|f|_{M'_i}\| \\ &\leq \sum_{i \in I} \text{rk}_{\mathbb{Z}}(N_i) \cdot \log_+ \|f|_{M_i}\| + \sum_{i \in I'} \text{rk}_{\mathbb{Z}}(M'_i) \cdot \log_+ \|f|_{M'_i}\|, \end{aligned}$$

as desired. \square

To prove Theorem 7.7, we show the following version in which the limsup is unfolded into an explicit statement:

Theorem 7.10. *In the situation of Setup 7.2 (with $Z = \mathbb{Z}$), let $f: M \rightarrow N$ be a Γ_* -adapted R -homomorphism between Γ_* -adapted marked projective R -modules. Then, for all $\varepsilon \in \mathbb{R}_{>0}$, we have for all large enough $i \in I$:*

$$\frac{\log \# \text{tors}(N(i)_{\Gamma}/\text{im } f(i)_{\Gamma})}{[\Gamma : \Gamma_i]} \leq \text{lognorm}(f) + \varepsilon.$$

Proof. The subalgebra S is dense. Thus, we use the description of lognorm in terms of cylinder sets (Proposition 6.9) and the generic torsion estimate for cokernels (Corollary 7.9).

Let $\varepsilon \in \mathbb{R}_{>0}$. By Proposition 6.9, we find an S -adapted marked decomposition $(M_j)_{j \in J} \in D_S(M)$ with

$$\text{lognorm}'(f, M_*) \leq \text{lognorm}(f) + \varepsilon.$$

Then, for all large enough $i \in I$, the decomposition M_* , as well as M , N , and f all are S_i -adapted. We split $J = J' \sqcup J''$ according to the branches of $\text{lognorm}'$: Let J' be the set of all $j \in J$ with

$$\text{lognorm}'(f|_{M_j}) = \text{rk}(f|_{M_j}) \cdot \log_+ \|f|_{M_j}\|;$$

moreover, we let $N_j \subset N$ be a marked summand satisfying $f(M_j) \subset N_j$ and $\dim(N_j) \leq \text{rk}(f) + \varepsilon/\#J$. Let $J'' := J \setminus J'$.

Using Corollary 7.9 and the dimension/norm compatibility (Remark 7.4, Remark 7.5), we thus obtain

$$\begin{aligned} & \log \# \text{tors}(N(i)_\Gamma / \text{im } f(i)_\Gamma) \\ & \leq \sum_{j \in J'} \text{rk}_{\mathbb{Z}} N_j(i)_\Gamma \cdot \log_+ \|f(i)_\Gamma|_{M_j(i)_\Gamma}\| + \sum_{j \in J''} \text{rk}_{\mathbb{Z}} M_j(i)_\Gamma \cdot \log_+ \|f(i)_\Gamma|_{M_j(i)_\Gamma}\| \\ & \leq \sum_{j \in J'} [\Gamma : \Gamma_i] \cdot \dim N_j \cdot \log_+ \|f|_{M_j}\| + \sum_{j \in J''} [\Gamma : \Gamma_i] \cdot \dim M_j \cdot \log_+ \|f|_{M_j}\| \\ & \leq \sum_{j \in J'} [\Gamma : \Gamma_i] \cdot (\text{rk } f|_{M_j} + \varepsilon/\#J) \cdot \log_+ \|f|_{M_j}\| + \sum_{j \in J''} [\Gamma : \Gamma_i] \cdot \dim M_j \cdot \log_+ \|f|_{M_j}\| \\ & \leq [\Gamma : \Gamma_i] \cdot (\text{lognorm}'(f, M_*) + \varepsilon \cdot \log_+ \|f\|) \\ & \leq [\Gamma : \Gamma_i] \cdot (\text{lognorm}(f) + \varepsilon + \varepsilon \cdot \log_+ \|f\|). \end{aligned}$$

Rescaling ε appropriately gives the claim. \square

This completes the proof of Theorem 7.7.

8. PROOF OF THE DYNAMICAL UPPER BOUNDS

We give proofs for the upper bounds of homological invariants in terms of measured embedding dimension and measured embedding volume (Theorem 1.2 and Theorem 1.3). The key intermediate step is to write the homological terms in question as retracts of the corresponding homology of suitably adapted dynamical embeddings.

For convenience of the reader we recall the definitions of measured embedding dimension and measured embedding volume from the introduction. Let $n \in \mathbb{N}$, let Γ be a residually finite group of type FP_{n+1} , and let α be a standard Γ -action. We denote by $A_n(\alpha)$ the class of all augmented complexes arising in α -embeddings (up to degree n). Let Z be \mathbb{Z} or a finite field. We have the following:

- The *measured embedding dimension* $\text{medim}_n^Z(\alpha)$ over Z in degree n is defined as:

$$\text{medim}_n^Z(\alpha) := \inf_{(D_* \twoheadrightarrow L^\infty(\alpha, Z)) \in A_n(\alpha)} \dim_R(D_n).$$

- The *measured embedding volume* $\text{mevol}_n(\alpha)$ in degree n for $Z = \mathbb{Z}$ is defined as:

$$\text{mevol}_n(\alpha) := \inf_{(D_* \twoheadrightarrow L^\infty(\alpha, \mathbb{Z})) \in A_n(\alpha)} \text{lognorm}(\partial_{n+1}^D).$$

8.1. Homology gradients. In this section we prove Theorem 1.2 that we restate here:

Theorem 8.1 (dynamical upper bounds). *Let $n \in \mathbb{N}$, let Γ be a residually finite group of type FP_{n+1} , let $(\Gamma_i)_{i \in I}$ be a directed system of finite index normal subgroups*

of Γ with $\bigcap_{i \in I} \Gamma_i = 1$ (e.g., a residual chain in Γ or the system of all finite index normal subgroups), and let Z be \mathbb{Z} or a finite field. Then:

$$\begin{aligned}\widehat{b}_n(\Gamma, \Gamma_*; Z) &\leq \text{medim}_n^Z(\Gamma \curvearrowright \widehat{\Gamma}_*) \\ \widehat{t}_n(\Gamma, \Gamma_*) &\leq \text{mevol}_n(\Gamma \curvearrowright \widehat{\Gamma}_*) \quad (\text{if } Z = \mathbb{Z}).\end{aligned}$$

The proof relies on the following input from previous sections:

Setup 8.2. Let $n \in \mathbb{N}$, let Γ be a residually finite group of type FP_{n+1} , let $(\Gamma_i)_{i \in I}$ be a directed system of finite index normal subgroups of Γ with $\bigcap_{i \in I} \Gamma_i = 1$, let $\alpha: \Gamma \curvearrowright \widehat{\Gamma}_*$ be the associated dynamical system, and let Z be \mathbb{Z} or a finite field.

Theorem 8.3. *In the situation of Setup 8.2, let C_* be a free $Z\Gamma$ -resolution of Z that has finite rank in degrees $\leq n+1$ and let $f_*: C_* \rightarrow D_*$ be an α -embedding. Then there exists a $K \in \mathbb{R}_{>0}$ such that: For every $\delta \in \mathbb{R}_{>0}$, there exists a Γ_* -adapted α -embedding $C_* \rightarrow \widehat{D}_*$ with*

$$d_{\text{GH}}^K(\widehat{D}_*, D_*, n) < \delta.$$

Proof. This is the special case of Theorem 5.10 for the action $\Gamma \curvearrowright \widehat{\Gamma}_*$ and the subalgebra of all cylinder sets. \square

Theorem 8.4 (Theorem 7.6). *In the situation of Setup 8.2, let C_* be a free $Z\Gamma$ -resolution of Z that has finite rank in degrees $\leq n+1$ and let $f_*: C_* \rightarrow D_*$ be a Γ_* -adapted α -embedding. Then for all large enough $i \in I$, we have*

$$\text{rk}_Z H_n(\Gamma_i; Z) \leq [\Gamma : \Gamma_i] \cdot \dim(D_n)$$

and, in the case $Z = \mathbb{Z}$,

$$\log \# \text{tors } H_n(\Gamma_i; \mathbb{Z}) \leq \log \# \text{tors}(D_n(i)_\Gamma / \text{im } \partial_{n+1}^D(i)_\Gamma).$$

Theorem 8.5 (Theorem 7.7). *In the situation of Setup 8.2, let $f: M \rightarrow N$ be a Γ_* -adapted R -homomorphism between Γ_* -adapted marked projective R -modules. Then*

$$\limsup_{i \in I} \frac{\log \# \text{tors}(N(i)_\Gamma / \text{im } f(i)_\Gamma)}{[\Gamma : \Gamma_i]} \leq \log \text{norm}(f).$$

Proof of Theorem 8.1. We spell out the proof for \widehat{t}_n . The proof for the Betti gradients works basically in the same way. Because Γ is of type FP_{n+1} , there exists a free $\mathbb{Z}\Gamma$ -resolution C_* of \mathbb{Z} that has finite rank in degrees $\leq n+1$. By definition of the measured embedding volume, it suffices to prove the following: If $f_*: C_* \rightarrow D_*$ is an α -embedding, then

$$\widehat{t}_n(\Gamma, \Gamma_*) \leq \log \text{norm}(\partial_{n+1}^D).$$

Thus, let $f_*: C_* \rightarrow D_*$ be an α -embedding. In view of Theorem 8.3, there exists a constant $K \in \mathbb{R}_{>0}$ such that: For every $\delta \in \mathbb{R}_{>0}$, there exists a Γ_* -adapted α -embedding $C_* \rightarrow \widehat{D}_*$ with

$$d_{\text{GH}}^K(\widehat{D}_*, D_*, n) < \delta.$$

Let $\delta \in \mathbb{R}_{>0}$ and let $\widehat{f}_*: C_* \rightarrow \widehat{D}_*$ be such a Γ_* -adapted α -embedding. Combining the retracts for \widehat{D}_* from Theorem 8.4 and the logarithmic norm estimates from

Theorem 8.5, we obtain

$$\begin{aligned}
\hat{t}_n(\Gamma, \Gamma_*) &= \limsup_{i \in I} \frac{\log \# \text{tors } H_n(\Gamma_i; \mathbb{Z})}{[\Gamma : \Gamma_i]} \\
&\leq \limsup_{i \in I} \frac{\log \# \text{tors}(\hat{D}_n(i)_\Gamma / \text{im } \partial_{n+1}^{\hat{D}}(i)_\Gamma)}{[\Gamma : \Gamma_i]} && \text{(Theorem 8.4)} \\
&\leq \text{lognorm}(\partial_{n+1}^{\hat{D}}) && \text{(Theorem 8.5)} \\
&\leq \text{lognorm}(\partial_{n+1}^D) + \delta \cdot \log_+ K. && \text{(Proposition 6.4 (v))}
\end{aligned}$$

Taking $\delta \rightarrow 0$, we get the desired estimate $\hat{t}_n(\Gamma, \Gamma_*) \leq \text{lognorm}(\partial_{n+1}^D)$. \square

8.2. L^2 -Betti numbers. Analogously to the retraction argument for the gradient estimate, we can apply the retraction argument also on the level of von Neumann algebras. This leads to the L^2 -Betti number estimate (Theorem 1.3).

Theorem 8.6. *Let $n \in \mathbb{N}$, let Γ be a group of type FP_{n+1} , and let α be a standard Γ -action. Then:*

$$b_n^{(2)}(\Gamma) \leq \text{medim}_n^{\mathbb{Z}}(\alpha).$$

Proof. We write $\mathcal{R} := \mathcal{R}_\alpha$ for the orbit relation of α and use the following description of the L^2 -Betti numbers [Sau05]:

$$b_n^{(2)}(\Gamma) = \dim_{N\mathcal{R}} H_n(\Gamma; N\mathcal{R}).$$

Let C_* be a free $\mathbb{Z}\Gamma$ -resolution of \mathbb{Z} that has finite rank in degrees $\leq n+1$ and let $f_*: C_* \rightarrow D_*$ be an α -embedding. It suffices to show that $b_n^{(2)}(\Gamma) \leq \dim_{L^\infty(\alpha)*\Gamma}(D_n)$. Let $\hat{C}_* := (L^\infty(\alpha) * \Gamma) \otimes_{\mathbb{Z}\Gamma} C_*$ and let $\hat{f}_*: \hat{C}_* \rightarrow D_*$ be the chain map induced by f_* .

Because $L^\infty(\alpha) * \Gamma$ is flat over $\mathbb{Z}\Gamma$ (Proposition 2.10), \hat{C}_* is a free $L^\infty(\alpha) * \Gamma$ -resolution of $L^\infty(\alpha)$. Therefore, the fundamental lemma of homological algebra provides us with an $L^\infty(\alpha) * \Gamma$ -chain map $\hat{g}_*: \hat{D}_* \rightarrow \hat{C}_*$ (up to degree $n+1$) that extends $\text{id}_{L^\infty(\alpha)}$ and that satisfies

$$\hat{g}_* \circ \hat{f}_* \simeq_{L^\infty(\alpha)*\Gamma} \text{id}_{\hat{C}_*}.$$

Finally, we pass to the level of the von Neumann algebra $N\mathcal{R}$: Let $\tilde{C}_* := N\mathcal{R} \otimes_{L^\infty(\alpha)*\Gamma} \hat{C}_* \cong_{N\mathcal{R}} N\mathcal{R} \otimes_{\mathbb{Z}\Gamma} C_*$, let $\tilde{D}_* := N\mathcal{R} \otimes_{L^\infty(\alpha)*\Gamma} D_*$, and let

$$\tilde{f}_* := \text{id}_{N\mathcal{R}} \otimes \hat{f}_*: \tilde{C}_* \rightarrow \tilde{D}_*,$$

$$\tilde{g}_* := \text{id}_{N\mathcal{R}} \otimes \hat{g}_*: \tilde{D}_* \rightarrow \tilde{C}_*.$$

In particular, we obtain $\tilde{g}_* \circ \tilde{f}_* \simeq_{N\mathcal{R}} \text{id}_{\tilde{C}_*}$ from the corresponding relation between \hat{g}_* and \hat{f}_* . Therefore, $H_n(\tilde{C}_*)$ is an $N\mathcal{R}$ -retract of $H_n(\tilde{D}_*)$ and the properties of $\dim_{N\mathcal{R}}$ [Sau05] show that

$$\begin{aligned}
b_n^{(2)}(\Gamma) &= \dim_{N\mathcal{R}} H_n(\Gamma; N\mathcal{R}) \\
&= \dim_{N\mathcal{R}} H_n(N\mathcal{R} \otimes_{\mathbb{Z}\Gamma} C_*) = \dim_{N\mathcal{R}} H_n(\tilde{C}_*) && \text{(by definition)} \\
&\leq \dim_{N\mathcal{R}} H_n(\tilde{D}_*) && \text{(by the retract)} \\
&\leq \dim_{N\mathcal{R}} \tilde{D}_n && \text{(properties of } \dim_{N\mathcal{R}}) \\
&= \dim_{L^\infty(\alpha)*\Gamma} D_n.
\end{aligned}$$

Taking the infimum over all α -embeddings proves the claim. \square

Part 2. Examples

We provide examples and computations for the measured embedding dimension and volume: in degree 0 (Section 10), for amenable groups (Section 11), for amalgamated products (Section 12), for products with an amenable factor (Section 13), and for finite index subgroups (Section 14).

Later we will give further examples, especially on hyperbolic 3-manifolds (Section 18.5 and Section 20.2), using dynamical inheritance properties established in Part 3.

9. BASIC PROPERTIES

We collect some basic properties of medim and mevol .

Setup 9.1. Throughout, let Z be the integers (with the usual norm) or a finite field (with the trivial norm).

Lemma 9.2. *Let $n \in \mathbb{N}$ and let Γ be a group of type FP_{n+1} . Let α be a standard Γ -action. Let Z be a finite field (with the trivial norm). Then*

$$\forall_{r \in \{0, \dots, n+1\}} \quad \text{medim}_r^Z(\alpha) \leq \text{medim}_r^{\mathbb{Z}}(\alpha).$$

Proof. Let $C_* \rightarrow D_*$ be an α -embedding over \mathbb{Z} . Then $Z \otimes_{\mathbb{Z}} C_*$ is a $Z\Gamma$ -resolution of Z because C_* is contractible as a \mathbb{Z} -chain complex. Hence $Z \otimes_{\mathbb{Z}} C_* \rightarrow Z \otimes_{\mathbb{Z}} D_*$ is an α -embedding over Z with

$$\dim_{L^\infty(\alpha, Z) * \Gamma}(Z \otimes_{\mathbb{Z}} D_r) = \dim_{L^\infty(\alpha, \mathbb{Z})}(D_r).$$

This proves the claim. \square

Lemma 9.3. *Let $n \in \mathbb{N}$ and let Γ be a group such that there exists a finite free $Z\Gamma$ -resolution of \mathbb{Z} of length n . Let α be a standard Γ -action. Then*

$$\begin{aligned} \forall_{r > n} \quad \text{medim}_r^Z(\alpha) &= 0 \\ \forall_{r \geq n} \quad \text{mevol}_r(\alpha) &= 0. \end{aligned}$$

Proof. By Lemma 9.2, we may assume that $Z = \mathbb{Z}$. Let C_* be a finite free $Z\Gamma$ -resolution of \mathbb{Z} of length n . Then $D_* := \text{Ind}_{Z\Gamma}^{L^\infty(\alpha) * \Gamma} C_*$ is a marked projective chain complex augmented over $L^\infty(\alpha)$. The canonical $Z\Gamma$ -map $C_* \rightarrow D_*$ is an α -embedding. By construction, we have $D_r = 0$ for all $r > n$. Hence for all $r > n$, we have

$$\text{medim}_r^{\mathbb{Z}}(\alpha) \leq \dim(D_r) = 0.$$

For all $r \geq n$, we have

$$\text{mevol}_r(\alpha) \leq \log \text{norm}(\partial_{r+1}^D) \leq \dim(D_{r+1}) \cdot \log_+ \|\partial_{r+1}^D\| = 0.$$

This finishes the proof. \square

10. DEGREE 0

We show that every standard action of an infinite group has medim and mevol equal to zero in degree 0.

Proposition 10.1. *Let Γ be a finitely generated group with finite generating set S . Let (C_*, ζ) be a free $Z\Gamma$ -resolution of \mathbb{Z} with $\partial_1: C_1 \rightarrow C_0$ given by*

$$\partial_1: \bigoplus_{s \in S} Z\Gamma \cdot e_s \rightarrow Z\Gamma, \quad \partial_1(e_s) = 1_\Gamma - s.$$

Let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a standard action. Then the following are equivalent:

- (i) *The group Γ is infinite;*

- (ii) For every $\varepsilon \in \mathbb{R}_{>0}$, there exists an α -embedding $C_* \rightarrow D_*$ with $\dim(D_0) < \varepsilon$ and $\|\partial_1^D\| \leq 2$;
- (iii) For every $\varepsilon \in \mathbb{R}_{>0}$, there exists an α -embedding $C_* \rightarrow D_*$ with $\dim(D_0) < \varepsilon$.

Proof. We show that (i) implies (ii). Let $\varepsilon \in \mathbb{R}_{>0}$. Since Γ is infinite, there exists a measurable subset A of X and a finite subset F of Γ with $\mu(A) < \varepsilon/2$ and $\mu(X \setminus F \cdot A) < \varepsilon/2$ [Lev95, Proposition 1]. Set $B := X \setminus F \cdot A$. The $L^\infty(\alpha) * \Gamma$ -module $D_0 := \langle A \rangle \oplus \langle B \rangle$ satisfies $\dim(D_0) < \varepsilon$. Let $\eta: D_0 \rightarrow L^\infty(\alpha)$ be the $L^\infty(\alpha) * \Gamma$ -linear map that sends χ_A to χ_A and χ_B to χ_B . We construct an element $x \in D_0$ with $\eta(x) = 1$ as follows. Denote the elements of the finite set F by $\gamma_1, \dots, \gamma_k$. Set $A_1 := \gamma_1 \cdot A$ and $A_j := \gamma_j \cdot A \setminus \bigcup_{m=1}^{j-1} A_m$ for $j \in \{2, \dots, k\}$. Then $\bigsqcup_{j=1}^k A_j = F \cdot A$. The element

$$x := \sum_{j=1}^k \gamma_j \cdot \chi_{\gamma_j^{-1} A_j} \cdot \chi_A + \chi_B \in D_0$$

satisfies $\eta(x) = 1$ by construction. Then there is an α -embedding (in low degrees) given by

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \bigoplus_{s \in S} \langle X \rangle \cdot e_s & \xrightarrow{\partial_1^D} & \langle A \rangle \oplus \langle B \rangle & \xrightarrow{\eta} & L^\infty(\alpha) \\ & & \uparrow f_1 & & \uparrow f_0 & & \uparrow \\ \cdots & \longrightarrow & \bigoplus_{s \in S} \mathbb{Z}\Gamma \cdot e_s & \xrightarrow{\partial_1^C} & \mathbb{Z}\Gamma & \xrightarrow{\zeta} & \mathbb{Z} \end{array}$$

where

$$\begin{aligned} \partial_1^D(e_s) &= (1_\Gamma - s)x; \\ f_0(1_\Gamma) &= x; \\ f_1(e_s) &= e_s. \end{aligned}$$

Clearly, (ii) implies (iii). We show that (iii) implies (i). Suppose that Γ is finite. We show that every α -embedding $(C_*, \zeta) \rightarrow (D_*, \eta)$ satisfies $\dim D_0 \geq 1/\#\Gamma$. Indeed, let $D_0 = \bigoplus_{i \in I} \langle A_i \rangle$ and let $x \in D_0$ with $\eta(x) = \chi_X$. We write $x = \sum_{i \in I} g_i \cdot \chi_{A_i} \cdot e_i$ for some $g_i \in L^\infty(\alpha) * \Gamma$. Then

$$\chi_X = \eta(x) = \sum_{i \in I} g_i \cdot \eta(\chi_{A_i} \cdot e_i).$$

Since $\text{supp}(\eta(\chi_{A_i}) \cdot e_i) \subset A_i$, we conclude

$$1 \leq \sum_{i \in I} \#\Gamma \cdot \mu(A_i)$$

and the claim follows. \square

Corollary 10.2. *Let Γ be a finitely generated infinite group and let α be a standard action of Γ . Then*

$$\text{medim}_0^Z(\alpha) = 0 \quad \text{and} \quad \text{mevol}_0(\alpha) = 0.$$

Proof. By Lemma 9.2, we may assume that $Z = \mathbb{Z}$. Then this is a direct consequence of Proposition 10.1 and the definition of the measured embedding dimension/volume. \square

11. AMENABLE GROUPS HAVE CHEAP EMBEDDINGS

We show that every standard action of the integers and, more generally, of an infinite amenable group has medim and mevol equal to zero in all degrees.

11.1. The integers.

Setup 11.1. We consider the infinite cyclic group $\Gamma := \mathbb{Z} = \langle t \rangle$ and a standard action $\alpha: \Gamma \curvearrowright (X, \mu)$. Given $\delta \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$, by the Rokhlin lemma [KM04, Theorem 7.5], there exist measurable subsets $A, B \subset X$ with $\mu(B) < \delta$ such that

$$X = A \sqcup tA \sqcup t^2A \sqcup \cdots \sqcup t^{N-1}A \sqcup B.$$

Clearly, $\mu(A) \leq 1/N$. Since $X = tX$, we have

$$(11.1) \quad A \sqcup B = t^N A \sqcup tB,$$

a fact that will be used repeatedly in the sequel.

Consider the marked projective $L^\infty(\alpha) * \Gamma$ -module $\langle A \rangle \oplus \langle B \rangle$ and define the element

$$x := \sum_{j=0}^{N-1} t^j \chi_A + \chi_B \in \langle A \rangle \oplus \langle B \rangle.$$

Proposition 11.2. *In the situation of Setup 11.1, there is an $L^\infty(\alpha) * \Gamma$ -resolution of $L^\infty(\alpha)$ of the form*

$$0 \rightarrow D_1 \xrightarrow{\partial_1} D_0 \xrightarrow{\eta} L^\infty(\alpha) \rightarrow 0,$$

where $D_0 = D_1 = \langle A \rangle \oplus \langle B \rangle$ and the $L^\infty(\alpha) * \Gamma$ -linear maps η and ∂_1 are given on generators by

$$\begin{aligned} \eta(\chi_A) &= \chi_A; \\ \eta(\chi_B) &= \chi_B; \\ \partial_1(\chi_A) &= \chi_A(t^0 - t^1)x; \\ \partial_1(\chi_B) &= \chi_B(t^0 - t^1)x. \end{aligned}$$

Proof. First of all, (D_*, η) is an augmented chain complex, since we have

$$\begin{aligned} \eta(\partial_1(\chi_A)) &= \eta(\chi_A(\chi_A + \chi_B - t^N \chi_A - t\chi_B)) \\ &= \chi_A(\chi_{A \sqcup B} - \chi_{t^N A \sqcup tB}) = 0 \end{aligned}$$

and similarly $\eta(\partial_1(\chi_B)) = 0$. Moreover, η is surjective since $\eta(x) = \chi_X$.

We show that (D_*, η) is a resolution by exhibiting an $L^\infty(\alpha)$ -linear chain contraction $c_*: D_* \rightarrow D_{*+1}$.

$$0 \longrightarrow \langle A \rangle \oplus \langle B \rangle \xrightarrow{\partial_1} \langle A \rangle \oplus \langle B \rangle \xrightarrow{\eta} L^\infty(\alpha) \longrightarrow 0$$

$\xleftarrow{c_0} \quad \quad \quad \xleftarrow{c_{-1}}$

Define the $L^\infty(\alpha)$ -linear maps c_0 and c_{-1} on generators by

$$\begin{aligned} c_{-1}(\chi_X) &= x; \\ c_0(t^m \chi_A) &= \begin{cases} -\chi_{t^m A} \sum_{j=0}^{m-1} t^j \chi_A - \chi_{t^m A} \sum_{j=0}^{m-1} t^j \chi_B & \text{if } m \geq 0; \\ \chi_{t^m A} \sum_{j=m}^{-1} t^j \chi_A + \chi_{t^m A} \sum_{j=m}^{-1} t^j \chi_B & \text{if } m < 0; \end{cases} \\ c_0(t^m \chi_B) &= \begin{cases} -\chi_{t^m B} \sum_{j=0}^{m-1} t^j \chi_A - \chi_{t^m B} \sum_{j=0}^{m-1} t^j \chi_B & \text{if } m \geq 0; \\ \chi_{t^m B} \sum_{j=m}^{-1} t^j \chi_A + \chi_{t^m B} \sum_{j=m}^{-1} t^j \chi_B & \text{if } m < 0. \end{cases} \end{aligned}$$

We verify that c_* is a chain contraction: First, we clearly have $\eta \circ c_{-1} = \text{id}_{L^\infty(\alpha)}$. Second, we have to show that $\partial_1 \circ c_0 = \text{id}_{\langle A \rangle \oplus \langle B \rangle} - c_{-1} \circ \eta$. Indeed, for $m \geq 0$ we

have

$$\begin{aligned}
\partial_1(c_0(t^m \chi_A)) &= -\chi_{t^m A} \left(\sum_{j=0}^{m-1} t^j \right) (\chi_A + \chi_B)(t^0 - t^1)x \\
&= -\chi_{t^m A} \left(\sum_{j=0}^{m-1} t^j \right) (t^0 - t^1)x \\
&= -\chi_{t^m A} (t^0 - t^m)x \\
&= -\chi_{t^m A} x + \chi_{t^m A} t^m x \\
&= -\eta(t^m \chi_A)x + t^m \chi_A x \\
&= -c_{-1}(\eta(t^m \chi_A)) + t^m \chi_A
\end{aligned}$$

and similarly for $m < 0$. The calculation that $\partial_1(c_0(t^m \chi_B)) = t^m \chi_B - c_{-1}(\eta(t^m \chi_B))$ for all $m \in \mathbb{Z}$ is analogous.

Third, we have to show that $c_0 \circ \partial_1 = \text{id}_{\langle A \rangle \oplus \langle B \rangle}$. Indeed, for $m \geq 0$ we have

$$\begin{aligned}
c_0(\partial_1(t^m \chi_A)) &= c_0(t^m \chi_A (\chi_A + \chi_B - t^N \chi_A - t \chi_B)) \\
&= \chi_{t^m A} (c_0(t^m \chi_A) - c_0(t^{m+N} \chi_A) - c_0(t^{m+1} \chi_B)) \\
&= \chi_{t^m A} \left(-\chi_{t^m A} \sum_{j=0}^{m-1} t^j (\chi_A + \chi_B) + \chi_{t^{m+N} A} \sum_{j=0}^{m+N-1} t^j (\chi_A + \chi_B) \right. \\
&\quad \left. + \chi_{t^{m+1} B} \sum_{j=0}^m t^j (\chi_A + \chi_B) \right) \\
&= \chi_{t^m A} \left(-\chi_{t^m A} \sum_{j=0}^{m-1} t^j (\chi_A + \chi_B) + \chi_{t^{m+N} A} \sum_{j=m+1}^{m+N-1} t^j (\chi_A + \chi_B) \right. \\
&\quad \left. + (\chi_{t^{m+N} A} + \chi_{t^{m+1} B}) \sum_{j=0}^m t^j (\chi_A + \chi_B) \right) \\
&= \chi_{t^m A} \left(-\chi_{t^m A} \sum_{j=0}^{m-1} t^j (\chi_A + \chi_B) + \chi_{t^{m+N} A} t^m \sum_{j=1}^{N-1} t^j (\chi_A + \chi_B) \right. \\
&\quad \left. + (\chi_{t^m A} + \chi_{t^m B}) \sum_{j=0}^m t^j (\chi_A + \chi_B) \right) \\
&= \chi_{t^m A} \left(\chi_{t^{m+N} A} t^m \sum_{j=1}^{N-1} t^j \chi_B + t^m \chi_A \right) \\
&= t^m \chi_A,
\end{aligned}$$

where for the last equality we use the following Lemma 11.3, and similarly for $m < 0$. The calculation that $c_0(\partial_1(t^m \chi_B)) = t^m \chi_B$ for all $m \in \mathbb{Z}$ is analogous. This finishes the proof. \square

Lemma 11.3. *For all $j \in \{1, \dots, N-1\}$, we have*

$$\begin{aligned}
A \cap t^N A \cap t^j B &= \emptyset; \\
B \cap t^N A \cap t^j B &= \emptyset.
\end{aligned}$$

Proof. We only prove the first statement, as the second is proved similarly. We proceed by induction on j . For $j = 1$, we have $t^N A \cap tB = \emptyset$. Assume for all $j \in \{1, \dots, N-2\}$ that $A \cap t^N A \cap t^j B = \emptyset$. Let $b \in B$. We have to show that $t^{N-1}b \notin A \cap t^N A$. We have $tb \in tB \subset A \sqcup B$. If $tb \in A$, then $t^{N-1}b \in t^{N-2}A$ and

hence $t^{N-1}b \notin A \cap t^N A$. If $tb \in B$, then $t^{N-1}b \in t^{N-2}B$ and hence by induction $t^{N-1}b \notin A \cap t^N A$. \square

Corollary 11.4. *Let α be a standard action of \mathbb{Z} . For every $n \in \mathbb{N}$, we have*

$$\text{medim}_n^{\mathbb{Z}}(\alpha) = 0 \quad \text{and} \quad \text{mevol}_n(\alpha) = 0.$$

Proof. By Lemma 9.2, we may assume that $Z = \mathbb{Z}$. Since the group $\Gamma := \mathbb{Z}$ is of type F, there exists a finite free $\mathbb{Z}\Gamma$ -resolution C_* of the trivial $\mathbb{Z}\Gamma$ -module \mathbb{Z} . The $L^\infty(\alpha) * \Gamma$ -resolution (D_*, η) from Proposition 11.2 satisfies

$$\dim(D_0) = \dim(D_1) = \mu(A \cup B) \leq 1/N + \delta \quad \text{and} \quad \|\partial_1\| \leq 2.$$

By the fundamental lemma of homological algebra, there exists an $L^\infty(\alpha) * \Gamma$ -chain map $f_*: L^\infty(\alpha) \otimes_{\mathbb{Z}} C_* \rightarrow D_*$ extending $\text{id}_{L^\infty(\alpha)}$. (Since both $L^\infty(\alpha) \otimes_{\mathbb{Z}} C_*$ and D_* are projective resolutions, the map f_* is in fact a chain homotopy equivalence.) Then the composition

$$C_* \rightarrow L^\infty(\alpha) \otimes_{\mathbb{Z}} C_* \xrightarrow{f_*} D_*$$

is an α -embedding. \square

We describe an explicit α -embedding for a standard action of \mathbb{Z} .

Example 11.5. For $\Gamma := \mathbb{Z} = \langle t \rangle$, we consider the usual $\mathbb{Z}\Gamma$ -resolution C_*

$$0 \rightarrow \mathbb{Z}\Gamma \rightarrow \mathbb{Z}\Gamma \rightarrow \mathbb{Z} \rightarrow 0$$

and the induced $L^\infty(\alpha) * \Gamma$ -resolution $L^\infty(\alpha) \otimes_{\mathbb{Z}} C_*$

$$0 \rightarrow L^\infty(\alpha) \otimes_{\mathbb{Z}} \mathbb{Z}\Gamma \xrightarrow{\partial_1^C} L^\infty(\alpha) \otimes_{\mathbb{Z}} \mathbb{Z}\Gamma \xrightarrow{\zeta} L^\infty(\alpha) \otimes_{\mathbb{Z}} \mathbb{Z} \rightarrow 0,$$

where

$$\begin{aligned} \zeta(t^0) &= \chi_X; \\ \partial_1^C(t^0) &= t^0 - t^1. \end{aligned}$$

In the situation of Setup 11.1, let (D_*, η) be the $L^\infty(\alpha) * \Gamma$ -resolution from Proposition 11.2. We exhibit chain maps $f_*: L^\infty(\alpha) \otimes_{\mathbb{Z}} C_* \rightarrow D_*$ and $r_*: D_* \rightarrow L^\infty(\alpha) \otimes_{\mathbb{Z}} C_*$ and a chain homotopy $h_*: L^\infty(\alpha) \otimes_{\mathbb{Z}} C_* \rightarrow L^\infty(\alpha) \otimes_{\mathbb{Z}} C_{*+1}$ between $r_* \circ f_*$ and $\text{id}_{L^\infty(\alpha) \otimes_{\mathbb{Z}} C_*}$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \langle A \rangle \oplus \langle B \rangle & \xrightarrow{\partial_1^D} & \langle A \rangle \oplus \langle B \rangle & \xrightarrow{\eta} & L^\infty(\alpha) \longrightarrow 0 \\ & & f_1 \uparrow \downarrow r_1 & & f_0 \uparrow \downarrow r_0 & & \updownarrow \cong \\ 0 & \longrightarrow & L^\infty(\alpha) \otimes_{\mathbb{Z}} \mathbb{Z}\Gamma & \xrightarrow{\partial_1^C} & L^\infty(\alpha) \otimes_{\mathbb{Z}} \mathbb{Z}\Gamma & \xrightarrow{\zeta} & L^\infty(\alpha) \otimes_{\mathbb{Z}} \mathbb{Z} \longrightarrow 0 \\ & & \swarrow h_0 & & \swarrow h_{-1} & & \end{array}$$

The $L^\infty(\alpha) * \Gamma$ -chain maps f_* and r_* are given on generators by

$$\begin{aligned} f_0(t^0) &= x \quad (\text{where } x \text{ is defined as in Setup 11.1}); \\ f_1(t_0) &= \chi_A + \chi_B; \\ r_0(\chi_A) &= \chi_A \otimes t^0; \\ r_0(\chi_B) &= \chi_B \otimes t^0; \\ r_1(\chi_A) &= \chi_A \tilde{x}; \\ r_1(\chi_B) &= \chi_B \tilde{x}, \end{aligned}$$

where $\tilde{x} \in L^\infty(\alpha) \otimes_{\mathbb{Z}} \mathbb{Z}\Gamma$ is defined as

$$\tilde{x} := \sum_{j=0}^{N-1} \chi_{t^N A} \otimes t^j + \chi_{tB} \otimes t^0.$$

Indeed, f_* and r_* are chain maps, the only non-obvious identity being the following:

$$\begin{aligned} \partial_1^C(r_1(\chi_A)) &= \partial_1^C(\chi_A \tilde{x}) \\ &= \chi_A \left(\sum_{j=0}^{N-1} \chi_{t^N A} \otimes (t^j - t^{j+1}) + \chi_{tB} \otimes (t^0 - t^1) \right) \\ &= \chi_A (\chi_{t^N A} \otimes t^0 - \chi_{t^N A} \otimes t^N + \chi_{tB} \otimes t^0 - \chi_{tB} \otimes t^1) \\ &= \chi_A (\chi_A \otimes t^0 + \chi_B \otimes t^0 - \chi_{t^N A} \otimes t^N - \chi_{tB} \otimes t^1) \\ &= \chi_A r_0(\chi_A + \chi_B - t^N \chi_A - t \chi_B) \\ &= r_0(\chi_A(t^0 - t^1)x) \\ &= r_0(\partial_1^D(\chi_A)) \end{aligned}$$

and similarly $\partial_1^C(r_1(\chi_B)) = r_0(\partial_1^D(\chi_B))$. The $L^\infty(\alpha) * \Gamma$ -chain homotopy h_* is given by $h_{-1} = 0$ and

$$h_0(t^0) = - \sum_{j=0}^{N-1} \sum_{k=0}^{j-1} \chi_{t^j A} \otimes t^k.$$

We have

$$\begin{aligned} \partial_1^C(h_0(t^0)) &= - \sum_{j=0}^{N-1} \sum_{k=0}^{j-1} \chi_{t^j A} \otimes (t^k - t^{k+1}) \\ &= - \sum_{j=0}^{N-1} \chi_{t^j A} \otimes (t^0 - t^j) \\ &= \sum_{j=0}^{N-1} t^j \chi_A \otimes t^0 - \sum_{j=0}^{N-1} \chi_{t^j A} \otimes t^0 \\ &= \sum_{j=0}^{N-1} t^j \chi_A \otimes t^0 + \chi_B \otimes t^0 - t^0 \\ &= r_0(x) - t^0 \\ &= r_0(f_0(t_0)) - t^0 \end{aligned}$$

and

$$\begin{aligned} h_0(\partial_1^C(t^0)) &= h_0(t^0 - t^1) \\ &= - \sum_{j=0}^{N-1} \sum_{k=0}^{j-1} \chi_{t^j A} \otimes t^k + t^1 \sum_{j=0}^{N-1} \sum_{k=0}^{j-1} \chi_{t^j A} \otimes t^k \\ &= - \sum_{j=1}^{N-1} \chi_{t^j A} \otimes t^0 + \sum_{k=1}^{N-1} \chi_{t^N A} \otimes t^k \\ &= \tilde{x} - t^0 \\ &= r_1(f_1(t^0)) - t^0. \end{aligned}$$

Finally, we note that the operator-norms of the above maps satisfy the following estimates

$$\begin{aligned}\|f_0\|, \|f_1\|, \|r_0\| &\leq 1; \\ \|r_1\| &\leq N; \\ \|h_0\| &\leq N^2.\end{aligned}$$

11.2. Amenable groups. We prove that standard actions α of infinite amenable groups have medim and mevol equal to zero in all degrees. We do so by constructing α -embeddings with arbitrarily small dimension and lognorm, to which we refer as “cheap” α -embeddings. We first develop some general preparations. Given a matrix Λ over $\mathbb{Z}\Gamma$ and a marked projective module D_1 , we construct a marked projective module D_2 and a map $D_2 \rightarrow D_1$ given by right multiplication with Λ such that $\dim(D_2)$ is controlled by $\dim(D_1)$.

Remark 11.6. Let $\Lambda = (\lambda_{ij})_{(i,j) \in I \times J}$ be a matrix with entries in $\mathbb{Z}\Gamma$. We set

$$\kappa(\Lambda) := \max_{i,j} |\lambda_{ij}|_1.$$

Then $|\lambda_{ij}|_1 \geq \max\{|\lambda_{ij}|_\infty, \#\text{supp}_\Gamma(\lambda_{ij})\}$. Let $f: \bigoplus_{i \in I} \langle A_i \rangle \rightarrow \bigoplus_{j \in J} \langle B_j \rangle$ be a map between marked projective modules given by right multiplication with the matrix Λ . Then by Lemma 2.31 we have

$$\|f\| \leq N_2(f) \cdot \|f\|_\infty \leq \kappa(\Lambda)^2 \cdot \#J.$$

The upper bound $\kappa(\Lambda)^2 \cdot \#J$ for $\|f\|$ is very coarse but depends only on the matrix Λ and not on the sets $(A_i)_i$ and $(B_j)_j$.

Lemma 11.7. *Let Γ be a group and let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a standard action. Let I and J be finite sets, let $\Lambda = (\lambda_{ij})_{(i,j) \in I \times J}$ be a matrix with entries in $\mathbb{Z}\Gamma$, and let $(B_j)_{j \in J}$ be a family of measurable subsets of X . Then there exists a family $(A_i)_{i \in I}$ of measurable subsets of X with*

$$\mu(A_i) \leq \kappa(\Lambda) \cdot \sum_{j \in J} \mu(B_j)$$

*and an $L^\infty(\alpha) * \Gamma$ -linear map $f: \bigoplus_{i \in I} \langle A_i \rangle \rightarrow \bigoplus_{j \in J} \langle B_j \rangle$ given by right multiplication with Λ satisfying*

$$\|f\| \leq \kappa(\Lambda)^2 \cdot \#J.$$

Proof. For $i \in I$, we consider the element

$$y_i := \sum_{j \in J} \lambda_{ij} \cdot \chi_{B_j} e_j \in \bigoplus_{j \in J} \langle B_j \rangle.$$

By construction, the subset $A_i := \text{supp}_1(y_i) \subset X$ satisfies $\mu(A_i) \leq \kappa(\Lambda) \cdot \sum_{j \in J} \mu(B_j)$. The map $f: \bigoplus_{i \in I} \langle A_i \rangle \rightarrow \bigoplus_{j \in J} \langle B_j \rangle$ defined by $f(\chi_{A_i} e_i) = y_i$ is a well-defined $L^\infty(\alpha) * \Gamma$ -linear map and is given by right multiplication with Λ . The map f satisfies $\|f\| \leq \kappa(\Lambda)^2 \cdot \#J$ by Remark 11.6. \square

We denote by $C_{* \geq 1}$ a chain complex that is concentrated in degrees ≥ 1 .

Lemma 11.8. *Let $C_{* \geq 1}$ be a free $\mathbb{Z}\Gamma$ -chain complex with $C_k \cong \bigoplus_{I_k} \mathbb{Z}\Gamma$ and ∂_k^C given by (right multiplication with) a matrix Λ_k . Let $n \in \mathbb{N}$ and suppose that I_k is finite for all $k \leq n$. Let $(B_{1,j})_{j \in I_1}$ be a family of measurable subsets of X . Then there exists a $L^\infty(\alpha) * \Gamma$ -chain complex $D_{* \geq 1}$ with $D_1 = \bigoplus_{j \in I_1} \langle B_{1,j} \rangle$ satisfying for all $k \leq n$*

$$\dim(D_k) \leq \dim(D_1) \cdot \prod_{m=2}^k \kappa(\Lambda_m) \cdot \#I_m$$

and

$$\|\partial_k^D\| \leq \kappa(\Lambda_k)^2 \cdot \#I_{k-1}$$

and there exists an $L^\infty(\alpha) * \Gamma$ -chain map $f_*: L^\infty(\alpha) \otimes_{\mathbb{Z}} C_* \rightarrow D_*$ given by $f_k(e_i) = \chi_{B_{k,i}} e_i$.

Proof. We construct D_* , ∂_*^D , and f_* inductively. Set $D_1 := \bigoplus_{j \in I_1} \langle B_{1,j} \rangle$ and $f_1(e_j) = \chi_{B_{1,j}} e_j$. Lemma 11.7 yields a module $D_2 = \bigoplus_{i \in I_2} \langle B_{2,i} \rangle$ with $\dim(D_2) \leq \dim(D_1) \cdot \kappa(\Lambda_2) \cdot \#I_2$ and a map $\partial_2^D: D_2 \rightarrow D_1$ with $\|\partial_2^D\| \leq \kappa(\Lambda_2)^2 \cdot \#I_1$. Setting $f_2(e_i) = \chi_{B_{2,i}} e_i$, we have $f_1 \circ \partial_2^D = \partial_2^D \circ f_2$. We apply Lemma 11.7 inductively to obtain D_n , ∂_n^D , and f_n with the desired properties. For $k \geq n+1$ and $i \in I_k$, we simply set $B_{k,i} := X$ and $D_k := \bigoplus_{i \in I_k} \langle B_{k,i} \rangle$. For $\Lambda_k = (\lambda_{k,ij})$, we set $\partial_k^D(\chi_{B_{k,i}} e_i) = \sum_{j \in I_{k-1}} \lambda_{k,ij} \cdot \chi_{B_{k-1,j}} e_j$. We have thus constructed a commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \bigoplus_{I_{n+1}} \langle X \rangle & \xrightarrow{\partial_{n+1}^D} & \bigoplus_{i \in I_n} \langle B_{n,i} \rangle & \xrightarrow{\partial_n^D} & \cdots \xrightarrow{\partial_2^D} \bigoplus_{j \in I_1} \langle B_{1,j} \rangle \\ & & \uparrow f_{n+1} & & \uparrow f_n & & \uparrow f_1 \\ \cdots & \longrightarrow & L^\infty(\alpha) \otimes_{\mathbb{Z}} C_{n+1} & \xrightarrow{\partial_{n+1}^C} & L^\infty(\alpha) \otimes_{\mathbb{Z}} C_n & \xrightarrow{\partial_n^C} & \cdots \xrightarrow{\partial_2^C} L^\infty(\alpha) \otimes_{\mathbb{Z}} C_1 \end{array}$$

Note that (D_*, ∂_*^D) is indeed a chain complex; for every $k \geq 2$ we have $\partial_{k-1}^D \circ \partial_k^D = 0$ because f_k is surjective and $\partial_{k-1}^C \circ \partial_k^C = 0$. \square

The point of Lemma 11.8 is that to construct a cheap α -embedding $f_*: C_* \rightarrow D_*$, it suffices to construct $D_{*\leq 1}$ with $\dim(D_1)$ arbitrarily small and $f_{*\leq 1}$ with f_1 being the obvious projection. For amenable groups this can be achieved using the following strong version of the Rokhlin lemma.

Theorem 11.9 (Rokhlin lemma, [CJK⁺18, Theorem 3.6]). *Let Γ be a countable amenable group, let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a standard action, let $F \subset \Gamma$ be a finite set, and let $\delta \in \mathbb{R}_{>0}$. Then there exists a μ -conull Γ -invariant Borel set $X' \subset X$, a finite set J , a family $(A_j)_{j \in J}$ of Borel subsets of X' , and a family $(T_j)_{j \in J}$ of (F, δ) -invariant non-empty finite subsets of Γ such that $(T_j \cdot x)_{j \in J, x \in A_j}$ partitions X' .*

Here T_j being (F, δ) -invariant means that

$$\frac{\#(F \cdot T_j \triangle T_j)}{\#T_j} \leq \delta.$$

Remark 11.10. If Γ is infinite and F contains a generating set, we have $1/\#T_j \leq \delta$ for all $j \in J$ and hence

$$\mu\left(\bigcup_{j \in J} A_j\right) \leq \sum_{j \in J} \mu(A_j) = \sum_{j \in J} \frac{1}{\#T_j} \cdot \mu(T_j \cdot A_j) \leq \delta \cdot \sum_{j \in J} \mu(T_j \cdot A_j) = \delta.$$

Theorem 11.11. *Let $n \in \mathbb{N}$ and let Γ be an infinite amenable group of type FP_{n+1} . Let α be a standard Γ -action. Then there exists $K \in \mathbb{R}_{>0}$ such that for every $\varepsilon \in \mathbb{R}_{>0}$, there exists an α -embedding $C_* \rightarrow D_*$ such that for all $r \in \{0, \dots, n+1\}$, we have $\dim(D_r) < \varepsilon$ and $\|\partial_r^D\| \leq K$.*

In particular, we have:

$$\begin{aligned} \forall_{r \in \{0, \dots, n+1\}} \quad \text{medim}_r^Z(\alpha) &= 0; \\ \forall_{r \in \{0, \dots, n\}} \quad \text{mevol}_r(\alpha) &= 0. \end{aligned}$$

Proof. By Lemma 9.2, we may assume that $Z = \mathbb{Z}$. Let S be a finite generating set of Γ . There exists a free $\mathbb{Z}\Gamma$ -resolution (C_*, ζ) of \mathbb{Z} of the form

$$\cdots \rightarrow \bigoplus_{I_2} \mathbb{Z}\Gamma \xrightarrow{\partial_2^C} \bigoplus_{I_1} \mathbb{Z}\Gamma \xrightarrow{\partial_1^C} \mathbb{Z}\Gamma \xrightarrow{\zeta} \mathbb{Z} \rightarrow 0$$

with $I_1 = S$, $\partial_1^C(e_s) = 1_\Gamma - s$, and I_k finite for $k \leq n+1$. We construct a chain map $L^\infty(\alpha) \otimes_{\mathbb{Z}} C_* \rightarrow D_*$ from which we then obtain a “cheap” α -embedding $C_* \rightarrow D_*$ by composition with the canonical map $C_* \rightarrow L^\infty(\alpha) \otimes_{\mathbb{Z}} C_*$.

Step 1: $* \leq 1$. This step uses the amenability of Γ and the Rokhlin lemma. Let $F := S \cup S^{-1} \subset \Gamma$ and let $\delta \in \mathbb{R}_{>0}$. Theorem 11.9 yields a finite set J , a family $(A_j)_{j \in J}$ of measurable subsets of X , and a family $(T_j)_{j \in J}$ of finite (F, δ) -invariant finite subsets of Γ such that $(T_j \cdot x)_{j \in J, x \in A_j}$ is a partition of a μ -conull subset of X .

The set $B_0 := \bigcup_{j \in J} A_j \subset X$ satisfies $\mu(B_0) < \delta$ by Remark 11.10. Hence $D_0 := \langle B_0 \rangle$ satisfies $\dim(D_0) \leq \delta$. We consider the element

$$x := \sum_{j \in J} \sum_{t \in T_j} t \chi_{A_j} \in \langle B_0 \rangle.$$

For $s \in S$, let $B_{1,s} := \text{supp}_1((1_\Gamma - s)x) \subset X$. We claim that $\mu(B_{1,s}) \leq 2\delta$. Indeed, let $T_{j,s} := s \cdot T_j \triangle T_j \subset \Gamma$ and observe that

$$\begin{aligned} \#T_{j,s} &= \#(sT_j \setminus T_j) + \#(T_j \setminus sT_j) \\ &= \#(sT_j \setminus T_j) + \#(s^{-1}T_j \setminus T_j) \\ &\leq 2\#(FT_j \setminus T_j) \\ &\leq 2\delta\#T_j. \end{aligned}$$

Since $B_{1,s} \subset \bigcup_{j \in J} \bigcup_{t \in T_{j,s}} tA_j$, we have

$$\mu(B_{1,s}) \leq \sum_{j \in J} \sum_{t \in T_{j,s}} \mu(tA_j) = \sum_{j \in J} \#T_{j,s} \cdot \mu(A_j) \leq 2\delta \cdot \sum_{j \in J} \#T_j \cdot \mu(A_j) = 2\delta.$$

Hence $D_1 := \bigoplus_{s \in S} \langle B_{1,s} \rangle$ satisfies $\dim(D_1) \leq 2\delta \cdot \#S$.

Define the $L^\infty(\alpha) * \Gamma$ -linear map $\partial_1^D : D_1 \rightarrow D_0$ by $\partial_1^D(\chi_{B_{1,s}}) = (1_\Gamma - s)x$, which satisfies $\|\partial_1^D\| \leq 2$.

Then the following diagram commutes

$$\begin{array}{ccccc} \bigoplus_{s \in S} \langle B_{1,s} \rangle & \xrightarrow{\partial_1^D} & \langle B_0 \rangle & \xrightarrow{\eta} & L^\infty(\alpha) \\ f_1 \uparrow & & f_0 \uparrow & & \text{id} \uparrow \\ L^\infty(\alpha) \otimes_{\mathbb{Z}} \bigoplus_S \mathbb{Z}\Gamma & \xrightarrow{\partial_1^C} & L^\infty(\alpha) \otimes_{\mathbb{Z}} \mathbb{Z}\Gamma & \xrightarrow{\zeta} & L^\infty(\alpha) \end{array}$$

where $f_0(1_\Gamma) = x$ and $f_1(e_s) = \chi_{B_{1,s}}e_s$. We have $\eta(x) = \chi_X$ by construction, and $\eta \circ \partial_1^D = 0$ because f_1 is surjective and $\zeta \circ \partial_1^C = 0$.

Step 2: $* \geq 2$. For $k \geq 2$, suppose ∂_k^C is given by (right multiplication with) the matrix Λ_k . Then Lemma 11.8 yields a chain complex $D_{*\geq 1}$ satisfying

$$\dim(D_n) \leq \dim(D_1) \cdot \prod_{m=2}^n \kappa(\Lambda_m) \cdot \#I_m \leq 2\delta \cdot \#S \cdot \prod_{m=2}^n \kappa(\Lambda_m) \cdot \#I_m$$

and

$$\|\partial_{n+1}^D\| \leq \kappa(\Lambda_{n+1}) \cdot \#I_n$$

and a chain map $f_{*\geq 1} : L^\infty(\alpha) \otimes_{\mathbb{Z}} C_* \rightarrow D_{*\geq 0}$. Note that $D_{*\geq 0}$ is indeed a chain complex; we have $\partial_1^D \circ \partial_2^D = 0$ because f_2 is surjective and $\partial_1^C \circ \partial_2^C = 0$. \square

Remark 11.12. Since all standard actions of countable infinite amenable groups are orbit equivalent [OW80, Theorem 6], if Theorem 1.9 could be extended to all orbit equivalences (instead of only weak bounded orbit equivalences), then Theorem 11.11 would be a direct consequence of Corollary 11.4.

Remark 11.13. Let Γ be an infinite amenable group. By Example 15.5, all probability measure preserving actions of Γ are weakly equivalent (see Definition 15.1). Thus, Theorem 15.30 yields that it suffices to show the vanishing of medim and mevol for a single standard action of Γ .

12. AMALGAMATED PRODUCTS

12.1. Amalgamated products. We prove inheritance properties of medim and mevol for actions of amalgamated products. Let Z be the integers (with the usual norm) or a finite field (with the trivial norm).

Recall that the *mapping cone* $\text{Cone}(\varphi)_*$ of a chain map $\varphi_*: D_* \rightarrow E_*$ is the chain complex with chain modules

$$\text{Cone}(\varphi)_n = D_{n-1} \oplus E_n$$

and differentials $\partial_n: \text{Cone}(\varphi)_n \rightarrow \text{Cone}(\varphi)_{n-1}$ given by

$$\partial_n(x, y) = (-\partial_{n-1}^D(x), \partial_n^E(y) + \varphi_{n-1}(x)).$$

Lemma 12.1. *Let R be the ring $L^\infty(\alpha) * \Gamma$. Let $\varphi_*: D_* \rightarrow E_*$ be a map of marked projective R -chain complexes. For all $n \in \mathbb{Z}$, the following hold:*

- (i) $\dim_R(\text{Cone}(\varphi)_n) = \dim_R(D_{n-1}) + \dim_R(E_n)$;
- (ii) $\text{lognorm}(\partial_n^{\text{Cone}(\varphi)}) \leq \dim_R(D_{n-1}) \cdot \log_+(\|\partial_{n-1}^D\| + \|\varphi_{n-1}\|) + \text{lognorm}(\partial_n^E)$.

Proof. (i) This is clear by the definition of $\text{Cone}(f)_n$.

(ii) Using properties of lognorm (Proposition 6.4), we have

$$\begin{aligned} \text{lognorm}(\partial_n^{\text{Cone}(\varphi)}) &\leq \text{lognorm}(\partial_n^{\text{Cone}(\varphi)}|_{D_{n-1}}) + \text{lognorm}(\partial_n^{\text{Cone}(\varphi)}|_{E_n}) \\ &\leq \text{lognorm}(D_{n-1} \rightarrow D_{n-2} \oplus E_{n-1}, x \mapsto (-\partial_{n-1}^D(x), \varphi_{n-1}(x))) \\ &\quad + \text{lognorm}(E_n \rightarrow D_{n-2} \oplus E_{n-1}, y \mapsto (0, \partial_n^E(y))) \\ &\leq \dim_R(D_{n-1}) \cdot \log_+(\|\partial_{n-1}^D\| + \|\varphi_{n-1}\|) + \text{lognorm}(\partial_n^E) \end{aligned}$$

as claimed. \square

An amalgamated product is a pushout of groups along injective structure maps.

Proposition 12.2. *Let $\Gamma \cong \Gamma_1 *_{\Gamma_0} \Gamma_2$ be an amalgamated product, where Γ_i is of type FP_∞ for $i \in \{0, 1, 2\}$. Let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a standard action. For $i \in \{0, 1, 2\}$, we write $\alpha_i := \alpha|_{\Gamma_i}: \Gamma_i \curvearrowright (X, \mu)$ for the restricted action. For all $n \in \mathbb{N}$, the following hold:*

- (i) $\text{medim}_n^Z(\alpha) \leq \text{medim}_n^Z(\alpha_1) + \text{medim}_n^Z(\alpha_2) + \text{medim}_{n-1}^Z(\alpha_0)$;
- (ii) If $\Gamma_0 \cong \{1\}$ and $n \geq 1$, then $\text{mevol}_n(\alpha) \leq \text{mevol}_n(\alpha_1) + \text{mevol}_n(\alpha_2)$;
- (iii) If $\Gamma_0 \cong \mathbb{Z}$, then $\text{mevol}_n(\alpha) \leq \text{mevol}_n(\alpha_1) + \text{mevol}_n(\alpha_2)$.

Proof. For $i \in \{0, 1, 2\}$, let $f(i)_*: C(i)_* \rightarrow D(i)_*$ be an α_i -embedding. It follows from the short exact sequence of $Z\Gamma$ -modules

$$0 \rightarrow Z[\Gamma/\Gamma_0] \rightarrow Z[\Gamma/\Gamma_1] \oplus Z[\Gamma/\Gamma_2] \rightarrow Z \rightarrow 0$$

that there exists a $Z\Gamma$ -chain map

$$g_*: \text{Ind}_{Z\Gamma_0}^{Z\Gamma} C(0)_* \rightarrow \text{Ind}_{Z\Gamma_1}^{Z\Gamma} C(1)_* \oplus \text{Ind}_{Z\Gamma_2}^{Z\Gamma} C(2)_*$$

such that $C_* := \text{Cone}(g)_*$ is a free $Z\Gamma$ -resolution of Z . Consider the following diagram of $L^\infty(\alpha) * \Gamma$ -chain complexes:

$$\begin{array}{ccccc}
\text{Ind}_{L^\infty(\alpha_0)*\Gamma_0}^{L^\infty(\alpha)*\Gamma} D(0)_* & \xrightarrow{\varphi_*} & \text{Ind}_{L^\infty(\alpha_1)*\Gamma_1}^{L^\infty(\alpha)*\Gamma} D(1)_* \oplus \text{Ind}_{L^\infty(\alpha_2)*\Gamma_2}^{L^\infty(\alpha)*\Gamma} D(2)_* & \longrightarrow & D_* \\
\text{Ind } f(0)_* \uparrow \Big\downarrow r(0)_* & & \text{Ind } f(1)_* \oplus \text{Ind } f(2)_* \uparrow & & \uparrow \text{---} \\
\text{Ind}_{Z\Gamma_0}^{L^\infty(\alpha)*\Gamma} C(0)_* & \xrightarrow{\text{Ind } g_*} & \text{Ind}_{Z\Gamma_1}^{L^\infty(\alpha)*\Gamma} C(1)_* \oplus \text{Ind}_{Z\Gamma_2}^{L^\infty(\alpha)*\Gamma} C(2)_* & \longrightarrow & \text{Ind}_{Z\Gamma}^{L^\infty(\alpha)*\Gamma} C_*
\end{array}$$

Here

- $r(0)_*$ is a homotopy left-inverse of $\text{Ind } f(0)_*$;
- $\varphi_* := \text{Ind } f(1)_* \oplus \text{Ind } f(2)_* \circ \text{Ind } g_* \circ r(0)_*$;
- $D_* := \text{Cone}(\varphi)_*$;
- the right vertical map is induced by functoriality of the mapping cone (involving a homotopy making the left square commutative).

Then the composition

$$C_* \rightarrow \text{Ind}_{Z\Gamma}^{L^\infty(\alpha)*\Gamma} C_* \rightarrow D_*$$

is an α -embedding. Since

$$\dim_{L^\infty(\alpha)*\Gamma}(\text{Ind}_{L^\infty(\alpha_i)*\Gamma_i}^{L^\infty(\alpha)*\Gamma} D(i)_n) = \dim_{L^\infty(\alpha_i)*\Gamma_i}(D(i)_n),$$

part (i) follows from Lemma 12.1 (i).

For part (ii), assume that Γ_0 is the trivial group and $n \geq 1$. For $i \in \{1, 2\}$, we fix α_i -embeddings $f(i)_*: C(i)_* \rightarrow D(i)_*$. Let $C(0)_* \rightarrow D(0)_*$ be the obvious α_0 -embedding concentrated in degrees ≤ 0 . For brevity, we write Ind instead of $\text{Ind}_{L^\infty(\alpha_i)*\Gamma_i}^{L^\infty(\alpha)*\Gamma}$. Then, since $\dim(D_n) = 0$, Lemma 12.1 (ii) yields

$$\begin{aligned}
\text{lognorm}(\partial_{n+1}^D) &\leq \text{lognorm}(\text{Ind } \partial_{n+1}^{D(1)} \oplus \text{Ind } \partial_{n+1}^{D(2)}) \\
&\leq \text{lognorm}(\text{Ind } \partial_{n+1}^{D(1)}) + \text{lognorm}(\text{Ind } \partial_{n+1}^{D(2)}) \\
&\leq \text{lognorm}(\partial_{n+1}^{D(1)}) + \text{lognorm}(\partial_{n+1}^{D(2)}).
\end{aligned}$$

Here the last step uses Lemma 6.5 (i).

For part (iii), assume that $\Gamma_0 = \mathbb{Z}$. For $i \in \{1, 2\}$, we fix α_i -embeddings $f(i)_*: C(i)_* \rightarrow D(i)_*$. Let $C(0)_*$ be the usual free $\mathbb{Z}\Gamma_0$ -resolution of \mathbb{Z} and let $K \in \mathbb{N}$ be large. By Example 11.5, there exists an α_0 -embedding $C(0)_* \rightarrow D(0)_*$ satisfying $\dim_{L^\infty(\alpha_0)*\Gamma_0}(D(0)_j) < 1/K$, $\|\partial_j^{D(0)}\| \leq K$, and $\|r(0)_j\| \leq K$. Then Lemma 12.1 (ii) yields

$$\begin{aligned}
\text{lognorm}(\partial_{n+1}^D) &\leq 1/K \cdot \log_+(K + (\|\text{Ind } f(1)_n\| + \|\text{Ind } f(2)_n\|) \cdot \|\text{Ind } g_n\| \cdot K) \\
&\quad + \text{lognorm}(\text{Ind } \partial_{n+1}^{D(1)} \oplus \text{Ind } \partial_{n+1}^{D(2)}) \\
&\leq 1/K \cdot \log_+(LK) + \text{lognorm}(\text{Ind } \partial_{n+1}^{D(1)}) + \text{lognorm}(\text{Ind } \partial_{n+1}^{D(2)}) \\
&\leq 1/K \cdot \log_+(LK) + \text{lognorm}(\partial_{n+1}^{D(1)}) + \text{lognorm}(\partial_{n+1}^{D(2)}).
\end{aligned}$$

Here $L := 1 + (\|f(1)_n\| + \|f(2)_n\|) \cdot \|g_n\|$ is independent of K and $D(0)_*$ and the last step uses Lemma 6.5 (i). Part (iii) follows by sending $K \rightarrow \infty$. \square

Proposition 12.2 generalises to fundamental groups of finite graphs of groups (with infinite cyclic or trivial edge groups).

12.2. Free groups. We compute medim and mevol for standard actions of free groups.

Proposition 12.3. *Let $d \in \mathbb{N}_{>0}$, let F_d be a free group of rank d , and let α be a standard action of F_d . Then:*

- (i) *For all $n \in \mathbb{N}$, we have $\text{mevol}_n(\alpha) = 0$;*
- (ii) *For all $n \in \mathbb{N} \setminus \{1\}$, we have $\text{medim}_n^{\mathbb{Z}}(\alpha) = 0$;*
- (iii) *We have $\text{medim}_1^{\mathbb{Z}}(\alpha) \geq d - 1$;*
- (iv) *If α is the profinite completion with respect to a directed system $(\Gamma_i)_{i \in I}$ of finite index normal subgroups of F_d with $\bigcap_{i \in I} \Gamma_i = 1$, then*

$$\text{medim}_1^{\mathbb{Z}}(\alpha) = d - 1.$$

Proof. Parts (i) and (ii) follow from Corollary 10.2 and Lemma 9.3. Alternatively, one can apply Proposition 12.2 by viewing F_d as a free product of copies of \mathbb{Z} .

(iii) By Theorem 8.6, we have

$$\text{medim}_1^{\mathbb{Z}}(\alpha) \geq b_1^{(2)}(F_d) = d - 1.$$

(iv) Let $(\Gamma_i)_{i \in I}$ be directed system with $\alpha: F_d \curvearrowright \widehat{\Gamma}_*$. A direct computation shows that

$$\widehat{b}_1(F_d, \Gamma_*; Z) = d - 1.$$

Therefore, by the dynamical upper bound (Theorem 8.1) we have

$$d - 1 = \widehat{b}_1(F_d, \Gamma_*; Z) \leq \text{medim}_1^{\mathbb{Z}}(\alpha).$$

Conversely, we obtain $\text{medim}_1^{\mathbb{Z}}(\alpha) \leq d - 1$ from Lemma 14.2. Indeed, since Γ_i is a free group of rank $1 + [F_d : \Gamma_i](d - 1)$, Lemma 14.2 yields

$$\text{medim}_1^{\mathbb{Z}}(\alpha) \leq \frac{1}{[F_d : \Gamma_i]} + d - 1.$$

Since $[F_d : \Gamma_i] \rightarrow \infty$ as $i \rightarrow \infty$, the claim follows. \square

12.3. Surface groups. We compute medim and mevol for standard actions of surface groups.

Proposition 12.4. *Let Σ_g be a closed orientable surface of genus g . Let α be a standard action of $\pi_1(\Sigma_g)$. Then:*

- (i) *For all $n \in \mathbb{N}$, we have $\text{mevol}_n(\alpha) = 0$;*
- (ii) *For all $n \in \mathbb{N} \setminus \{1\}$, we have $\text{medim}_n^{\mathbb{Z}}(\alpha) = 0$;*
- (iii) *We have $\text{medim}_1^{\mathbb{Z}}(\alpha) \geq 2g - 2$;*
- (iv) *If α is the profinite completion with respect to a directed system $(\Gamma_i)_{i \in I}$ of finite index normal subgroup of $\pi_1(\Sigma_g)$ with $\bigcap_{i \in I} \Gamma_i = 1$, then*

$$\text{medim}_1^{\mathbb{Z}}(\alpha) = 2g - 2.$$

Proof. Parts (i) and (ii) in degree 0 follow from Corollary 10.2 since $\pi_1(\Sigma_g)$ is infinite. In positive degrees, the claims follow from Proposition 12.2 and Proposition 12.3 by viewing $\pi_1(\Sigma_g)$ as an iterated amalgamated product of free groups over infinite cyclic subgroups.

(iii) By Theorem 8.6, we have

$$\text{medim}_1^{\mathbb{Z}}(\alpha) \geq b_1^{(2)}(\pi_1(\Sigma_g)) = 2g - 2.$$

(iv) Let $(\Gamma_i)_{i \in I}$ be a directed system with $\alpha: \pi_1(\Sigma_g) \curvearrowright \widehat{\Gamma}_*$. A direct computation shows that

$$\widehat{b}_1(\pi_1(\Sigma_g), \Gamma_*; Z) = 2g - 2.$$

Therefore, by the dynamical upper bound (Theorem 8.1) we have

$$2g - 2 = \widehat{b}_1(\pi_1(\Sigma_g), \Gamma_*; Z) \leq \text{medim}_1^Z(\alpha).$$

Conversely, we obtain $\text{medim}_1^Z(\alpha) \leq 2g - 2$ from Lemma 14.2. Indeed, since Γ_i is the fundamental group of a surface of genus $1 + [\pi_1(\Sigma_g) : \Gamma_i](g - 1)$, Lemma 14.2 yields

$$\text{medim}_1^Z(\alpha) \leq \frac{2}{[\pi_1(\Sigma_g) : \Gamma_i]} + 2g - 2.$$

Since $[\pi_1(\Sigma_g) : \Gamma_i] \rightarrow \infty$ as $i \rightarrow \infty$, the claim follows. \square

13. PRODUCTS WITH AN AMENABLE FACTOR

We prove that product actions groups with an amenable factor have medim and mevol equal to zero. Let Z be the integers (with the usual norm) or a finite field (with the trivial norm).

Proposition 13.1. *Let $n \in \mathbb{N}$. Let Γ_1 be an infinite amenable group of type FP_{n+1} and let Γ_2 be a group of type FP_{n+1} . For $i \in \{1, 2\}$, let $\alpha_i : \Gamma_i \curvearrowright (X_i, \mu_i)$ be a standard action. We denote by $\alpha_1 \times \alpha_2 : \Gamma_1 \times \Gamma_2 \curvearrowright (X_1 \times X_2, \mu_1 \otimes \mu_2)$ the product action. Then*

$$\begin{aligned} \forall_{r \in \{0, \dots, n+1\}} \quad \text{medim}_r^Z(\alpha_1 \times \alpha_2) &= 0; \\ \forall_{r \in \{0, \dots, n\}} \quad \text{mevol}_r(\alpha_1 \times \alpha_2) &= 0. \end{aligned}$$

Proof. By Lemma 9.2, we may assume that $Z = \mathbb{Z}$. For $i \in \{1, 2\}$, we denote $R_i := L^\infty(\alpha_i) * \Gamma_i$. Since Γ_i is of type FP_{n+1} , there exists a free $\mathbb{Z}\Gamma_i$ -resolution $C(i)_*$ of \mathbb{Z} such that the $\mathbb{Z}\Gamma_i$ -module $C(i)_r$ is finitely generated for all $r \leq n + 1$. Then we have an α_2 -embedding $C(2)_* \rightarrow D(2)_* := \text{Ind}_{\mathbb{Z}\Gamma_2}^{R_2} C(2)_*$ with $\dim_{R_2}(D(2)_r) = \text{rk}_{\mathbb{Z}\Gamma_2}(C(2)_r) < \infty$. Since Γ_1 is amenable, by Theorem 11.11 there exists $K \in \mathbb{R}_{>0}$ such that for all $\varepsilon \in \mathbb{R}_{>0}$ there exists an α_1 -embedding $C(1)_* \rightarrow D(1)_*$ such that for all $r \in \{0, \dots, n + 1\}$, we have $\dim_{R_1}(D(1)_r) < \varepsilon$ and $\|\partial_r^{D(1)}\| \leq K$. Consider the ring extension

$$R_1 \otimes_{\mathbb{Z}} R_2 = (L^\infty(\alpha_1) * \Gamma_1) \otimes_{\mathbb{Z}} (L^\infty(\alpha_2) * \Gamma_2) \rightarrow L^\infty(\alpha_1 \times \alpha_2) * (\Gamma_1 \times \Gamma_2) =: R.$$

Then the composition

$$C(1)_* \otimes_{\mathbb{Z}} C(2)_* \rightarrow D(1)_* \otimes_{\mathbb{Z}} D(2)_* \rightarrow \text{Ind}_{R_1 \otimes_{\mathbb{Z}} R_2}^R (D(1)_* \otimes_{\mathbb{Z}} D(2)_*)$$

is an $(\alpha_1 \times \alpha_2)$ -embedding. Indeed, given measurable subsets $A_i \subset X_i$ for $i \in \{1, 2\}$, we have

$$\text{Ind}_{R_1 \otimes_{\mathbb{Z}} R_2}^R (\langle A_1 \rangle \otimes_{\mathbb{Z}} \langle A_2 \rangle) \cong \langle A_1 \times A_2 \rangle.$$

Thus, for every $r \in \{0, \dots, n + 1\}$, we have

$$\begin{aligned} \dim_R((\text{Ind}_{R_1 \otimes_{\mathbb{Z}} R_2}^R D(1)_* \otimes_{\mathbb{Z}} D(2)_*)_r) &= \sum_{p+q=r} \dim_{R_1}(D(1)_p) \cdot \dim_{R_2}(D(2)_q) \\ &\leq \varepsilon \cdot \sum_{q=0}^r \dim_{R_2}(D(2)_q). \end{aligned}$$

We can choose ε arbitrarily small and conclude $\text{medim}_r^{\mathbb{Z}}(\alpha_1 \times \alpha_2) = 0$.

The boundary map of the chain complex $D(1)_* \otimes_{\mathbb{Z}} D(2)_*$ is given by

$$\begin{aligned} \partial_r : \bigoplus_{p+q=r} D(1)_p \otimes_{\mathbb{Z}} D(2)_q &\rightarrow \bigoplus_{p+q=r-1} D(1)_p \otimes_{\mathbb{Z}} D(2)_q, \\ \partial_r &= \bigoplus_{p+q=r} (\partial_p^{D(1)} \otimes \text{id}_{D(2)_q}) \oplus (-1)^p (\text{id}_{D(1)_p} \otimes \partial_q^{D(2)}). \end{aligned}$$

Then we have

$$\begin{aligned}
& \text{lognorm}(\text{Ind}_{R_1 \otimes_{\mathbb{Z}} R_2}^R \partial_{r+1}) \\
& \leq \sum_{p+q=r+1} \dim_R(\text{Ind}_{R_1 \otimes_{\mathbb{Z}} R_2}^R D(1)_p \otimes_{\mathbb{Z}} D(2)_q) \\
& \quad \cdot \log_+(\|\text{Ind}_{R_1 \otimes_{\mathbb{Z}} R_2}^R \partial_p^{D(1)} \otimes \text{id}_{D(2)*} \| + \|\text{Ind}_{R_1 \otimes_{\mathbb{Z}} R_2}^R \text{id}_{D(1)*} \otimes \partial_q^{D(2)}\|) \\
& \leq \sum_{p+q=r+1} \dim_{R_1}(D(1)_p) \cdot \dim_{R_2}(D(2)_q) \cdot \log_+(\|\partial_p^{D(1)}\| + \|\partial_q^{D(2)}\|) \\
& \leq \varepsilon \cdot \sum_{q=1}^{r+1} \dim_{R_2}(D(2)_q) \cdot \log_+(K + \|\partial_q^{D(2)}\|).
\end{aligned}$$

Again, we can choose ε arbitrarily small and conclude $\text{mevol}_r(\alpha_1 \times \alpha_2) = 0$. \square

14. FINITE INDEX SUBGROUPS

14.1. Induction and restriction. We prove proportionality results for medim and mevol of standard actions of finite index subgroups.

Let Λ be a finite index subgroup of Γ . If $\alpha: \Gamma \curvearrowright (X, \mu)$ is a standard Γ -action, then $\alpha|_{\Lambda}: \Lambda \curvearrowright (X, \mu)$ is a standard Λ -action. The ring inclusion $L^\infty(\alpha|_{\Lambda}) * \Lambda \rightarrow L^\infty(\alpha) * \Gamma$ induces a restriction functor $\text{Res}_{L^\infty(\alpha|_{\Lambda}) * \Lambda}^{L^\infty(\alpha) * \Gamma}$ on module categories.

If $\beta: \Lambda \curvearrowright (Y, \nu)$ is a standard Λ -action, then $\text{Ind}_{\Lambda}^{\Gamma} \beta: \Gamma \curvearrowright (\Gamma \times_{\Lambda} Y, \text{Ind}_{\Lambda}^{\Gamma} \nu)$ is a standard Γ -action. Note that we rescale $\text{Ind}_{\Lambda}^{\Gamma} \nu$ so that it is a probability measure.

Proposition 14.1. *Let $n \in \mathbb{N}$, let Γ be a group of type FP_{n+1} , and let $\Lambda \subset \Gamma$ be a subgroup of finite index.*

(i) *If $\alpha: \Gamma \curvearrowright (X, \mu)$ is a standard Γ -action, then*

$$\begin{aligned}
\text{medim}_n^Z(\alpha|_{\Lambda}) & \leq [\Gamma : \Lambda] \cdot \text{medim}_n^Z(\alpha) \\
\text{mevol}_n(\alpha|_{\Lambda}) & \leq [\Gamma : \Lambda] \cdot \text{mevol}_n(\alpha).
\end{aligned}$$

(ii) *If $\beta: \Lambda \curvearrowright (Y, \nu)$ is a standard Λ -action, then*

$$\begin{aligned}
\text{medim}_n^Z(\text{Ind}_{\Lambda}^{\Gamma} \beta) & = \frac{1}{[\Gamma : \Lambda]} \cdot \text{medim}_n^Z(\beta) \\
\text{mevol}_n(\text{Ind}_{\Lambda}^{\Gamma} \beta) & = \frac{1}{[\Gamma : \Lambda]} \cdot \text{mevol}_n(\beta).
\end{aligned}$$

Proof. (i) Let $C_* \rightarrow D_*$ be an α -embedding. Then $\text{Res}_{Z\Lambda}^{\Gamma} C_* \rightarrow \text{Res}_{L^\infty(\alpha|_{\Lambda}) * \Lambda}^{L^\infty(\alpha) * \Gamma} D_*$ is a $\alpha|_{\Lambda}$ -embedding with

$$\begin{aligned}
\dim_{L^\infty(\alpha|_{\Lambda}) * \Lambda}(\text{Res}_{L^\infty(\alpha|_{\Lambda}) * \Lambda}^{L^\infty(\alpha) * \Gamma} D_n) & \leq [\Gamma : \Lambda] \cdot \dim_{L^\infty(\alpha) * \Gamma}(D_n) \\
\text{lognorm}(\text{Res}_{L^\infty(\alpha|_{\Lambda}) * \Lambda}^{L^\infty(\alpha) * \Gamma} \partial_n^D) & \leq [\Gamma : \Lambda] \cdot \text{lognorm}(\partial_n^D)
\end{aligned}$$

by Lemma 6.5 (ii).

(ii) Since $\text{Ind}_{\Lambda}^{\Gamma} \beta$ is weakly bounded orbit equivalent to β (Definition 18.14), this follows from Theorem 18.2. \square

Lemma 14.2. *Let $n \in \mathbb{N}$, let Γ be a residually finite group of type FP_{n+1} , and let $\Gamma_* = (\Gamma_i)_{i \in I}$ be a directed system of finite index normal subgroups of Γ with $\bigcap_{i \in I} \Gamma_i = 1$. Let $\alpha: \Gamma \curvearrowright (\hat{\Gamma}_*, \mu)$ be the profinite completion of Γ with respect to Γ_* . Fix $i \in I$. Let C_* be a free $Z\Gamma_i$ -resolution of Z . Let D_* be a free $Z\Gamma_i$ -chain complex*

augmented over Z and let $f_*: C_* \rightarrow D_*$ be a $Z\Gamma_i$ -chain map. Then we have

$$\begin{aligned} \forall_{r \in \{0, \dots, n+1\}} \quad \text{medim}_r^Z(\alpha) &\leq \frac{1}{[\Gamma : \Gamma_i]} \cdot \text{rk}_{Z\Gamma_i}(D_r) \\ \forall_{r \in \{0, \dots, n\}} \quad \text{mevol}_r(\alpha) &\leq \frac{1}{[\Gamma : \Gamma_i]} \cdot \text{rk}_{Z\Gamma_i}(D_{r+1}) \cdot \log_+ \|\partial_{r+1}^D\|. \end{aligned}$$

Proof. The free $Z\Gamma_i$ -chain complex D_* has chain modules of the form $D_r \cong \bigoplus_{I_r} Z\Gamma_i$ for some index set I_r and differentials ∂_r^D . We define an associated marked projective chain complex \widehat{D}_* over $R := L^\infty(\alpha) * \Gamma$ as follows. Consider the map $p_i: \widehat{\Gamma}_* \rightarrow \Gamma/\Gamma_i$ and set $A := p_i^{-1}(1_\Gamma \Gamma_i)$. The subset A of $\widehat{\Gamma}_*$ is Γ_i -invariant and has measure $\mu(A) = 1/[\Gamma : \Gamma_i]$. We define the R -chain modules $\widehat{D}_r := \bigoplus_{I_r} \langle A \rangle_R$ and differentials $\partial_r^{\widehat{D}}$ given by the same matrix (with entries in $Z\Gamma_i$) as ∂_r^D . Then \widehat{D}_* is a marked projective R -chain complex augmented over $L^\infty(\alpha)$ with

$$\dim_R(\widehat{D}_r) = \mu(A) \cdot \#I_r = \frac{1}{[\Gamma : \Gamma_i]} \cdot \text{rk}_{Z\Gamma_i}(D_r)$$

$$\text{lognorm}(\partial_{r+1}^{\widehat{D}}) \leq \dim_R(\widehat{D}_{r+1}) \cdot \log_+ \|\partial_{r+1}^{\widehat{D}}\| \leq \frac{1}{[\Gamma : \Gamma_i]} \cdot \text{rk}_{Z\Gamma_i}(D_{r+1}) \cdot \log_+ \|\partial_{r+1}^D\|.$$

The composition of $Z\Gamma_i$ -chain maps $C_* \rightarrow D_* \rightarrow \widehat{D}_*$ extends to a $Z\Gamma$ -chain map $Z\Gamma \otimes_{Z\Gamma_i} C_* \rightarrow \widehat{D}_*$. Let C'_* be a free $Z\Gamma$ -resolution of Z . Then we obtain an α -embedding as the composition

$$\begin{array}{ccccc} C'_* & \longrightarrow & Z\Gamma \otimes_{Z\Gamma_i} C_* & \longrightarrow & \widehat{D}_* \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longrightarrow & Z[\Gamma/\Gamma_i] & \longrightarrow & L^\infty(\alpha) \end{array}$$

where the $Z\Gamma$ -chain map $C'_* \rightarrow Z\Gamma \otimes_{Z\Gamma_i} C_*$ is induced by the $Z\Gamma$ -map $Z \rightarrow Z[\Gamma/\Gamma_i]$ that sends $1 \in Z$ to the (finite) sum of all Z -basis elements of $Z[\Gamma/\Gamma_i]$. \square

Remark 14.3. We outline an alternative proof of Lemma 14.2 using results from Part 3. However, it involves actions that are not essentially free to which our setup could be extended.

In the situation of Lemma 14.2, we denote by $\alpha_i: \Gamma \curvearrowright \Gamma/\Gamma_i$ the translation action. Since α_i is weakly contained in α (Example 15.3), by monotonicity of medim and mevol (Theorem 15.30) we have

$$\begin{aligned} \text{medim}_r^Z(\alpha) &\leq \text{medim}_r^Z(\alpha_i) \\ \text{mevol}_r(\alpha) &\leq \text{mevol}_r(\alpha_i). \end{aligned}$$

We denote by $\beta_i: \Gamma_i \curvearrowright \text{pt}$ the trivial action. Then $\alpha_i \cong \text{Ind}_{\Gamma_i}^\Gamma \beta_i$ and by Proposition 14.1 (ii) we have

$$\begin{aligned} \text{medim}_r^Z(\alpha_i) &= \frac{1}{[\Gamma : \Gamma_i]} \cdot \text{medim}_r^Z(\beta_i) \\ \text{mevol}_r(\alpha_i) &= \frac{1}{[\Gamma : \Gamma_i]} \cdot \text{mevol}_r(\beta_i). \end{aligned}$$

Under the ring isomorphism $Z\Gamma_i \cong L^\infty(\beta_i) * \Gamma_i$, the $Z\Gamma_i$ -chain map $f_*: C_* \rightarrow D_*$ is a β_i -embedding and hence

$$\begin{aligned} \text{medim}_r^Z(\beta_i) &\leq \text{rk}_{Z\Gamma_i}(D_r) \\ \text{mevol}_r(\beta_i) &\leq \text{lognorm}(\partial_{r+1}^D) \leq \text{rk}_{Z\Gamma_i}(D_{r+1}) \cdot \log_+ \|\partial_{r+1}^D\|. \end{aligned}$$

Combining the above inequalities proves Lemma 14.2.

14.2. The cheap rebuilding property. We prove that the profinite completion action of groups satisfying (an equivariant version of) the algebraic cheap rebuilding property has medim and mevol equal to zero. The algebraic cheap rebuilding property was introduced by the authors [LLM⁺] modeled after the (geometric) cheap rebuilding property of Abert–Bergeron–Fraczyk–Gaboriau [ABFG25].

Definition 14.4 ([LLM⁺, Definition 4.18]). Let Γ be a group and let $n \in \mathbb{N}$. We say that Γ satisfies the *algebraic cheap n -rebuilding property* if there exist a free $\mathbb{Z}\Gamma$ -resolution C_* of \mathbb{Z} with C_j finitely generated for all $j \leq n$ and $\kappa \in \mathbb{R}_{\geq 1}$ such that for all $T \in \mathbb{R}_{\geq 1}$ and every residual chain Λ_* in Γ there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, the \mathbb{Z} -chain complex of Λ_i -coinvariants $(C_*)_{\Lambda_i}$ is a \mathbb{Z} -chain homotopy retract of a free \mathbb{Z} -chain complex C'_* via \mathbb{Z} -chain maps $f_*: (C_*)_{\Lambda_i} \rightarrow C'_*$ and $g_*: C'_* \rightarrow (C_*)_{\Lambda_i}$ and \mathbb{Z} -chain homotopy $H_*: g_* \circ f_* \simeq \text{id}_{(C_*)_{\Lambda_i}}$ satisfying the following for all $j \leq n$:

$$\begin{aligned} \text{rk}_{\mathbb{Z}}(C'_j) &\leq \frac{\kappa}{T} \cdot \text{rk}_{\mathbb{Z}}((C_j)_{\Lambda_i}) \\ \|\partial_j^{C'}\|, \|f_j\|, \|g_j\|, \|H_j\| &\leq \exp(\kappa) \cdot T^{\kappa}. \end{aligned}$$

The key facts about the algebraic cheap rebuilding property are that it implies the vanishing of torsion homology growth [LLM⁺, Lemma 4.22], admits a bootstrapping theorem [LLM⁺, Proposition 4.23], and is satisfied by the group of integers \mathbb{Z} [LLM⁺, Examples 4.21].

In order to establish a relationship to the vanishing of medim and mevol, we define an equivariant version of the algebraic cheap rebuilding property. The difference is that in Definition 14.5 we require the existence of a $\mathbb{Z}\Lambda_i$ -chain homotopy retraction of $\text{Res}_{\mathbb{Z}\Lambda_i}^{\mathbb{Z}\Gamma} C_*$, while in Definition 14.4 we require only the existence of a (non-equivariant) \mathbb{Z} -chain homotopy retraction of the Λ_i -coinvariants $(C_*)_{\Lambda_i}$.

Definition 14.5. Let Γ be a group and let $n \in \mathbb{N}$. We say that Γ satisfies the *algebraic cheap equivariant n -rebuilding property* (CERP_n for short) if there exists a free $\mathbb{Z}\Gamma$ -resolution C_* of \mathbb{Z} with C_j finitely generated for all $j \leq n$ and $\kappa \in \mathbb{R}_{\geq 1}$ such that for all $T \in \mathbb{R}_{\geq 1}$ and every residual chain Λ_* in Γ there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, the $\mathbb{Z}\Lambda_i$ -chain complex $\text{Res}_{\mathbb{Z}\Lambda_i}^{\mathbb{Z}\Gamma} C_i$ is a $\mathbb{Z}\Lambda_i$ -chain homotopy retract of a free $\mathbb{Z}\Lambda_i$ -chain complex E_* via $\mathbb{Z}\Lambda_i$ -chain maps $f_*: \text{Res}_{\mathbb{Z}\Lambda_i}^{\mathbb{Z}\Gamma} C_* \rightarrow E_*$ and $g_*: E_* \rightarrow \text{Res}_{\mathbb{Z}\Lambda_i}^{\mathbb{Z}\Gamma} C_*$ and a $\mathbb{Z}\Lambda_i$ -chain homotopy $H_*: g_* \circ f_* \simeq \text{id}_{\text{Res}_{\mathbb{Z}\Lambda_i}^{\mathbb{Z}\Gamma} C_*}$ satisfying the following for all $j \leq n$:

$$\begin{aligned} \text{rk}_{\mathbb{Z}\Lambda_i}(E_j) &\leq \frac{\kappa}{T} \cdot \text{rk}_{\mathbb{Z}\Lambda_i}(\text{Res}_{\mathbb{Z}\Lambda_i}^{\mathbb{Z}\Gamma} C_*) \\ \|\partial_j^E\|, \|f_j\|, \|g_j\|, \|H_j\| &\leq \exp(\kappa) \cdot T^{\kappa} \end{aligned}$$

Clearly, the algebraic cheap equivariant rebuilding property implies the non-equivariant one. We do not know if the converse holds.

Remark 14.6. The algebraic cheap equivariant rebuilding property is a bootstrapable property of residually finite groups in the sense of [LLM⁺, Definition 3.4]. The proof is analogous to the non-equivariant case [LLM⁺, Proposition 4.23 (i)]. Hence this property admits a bootstrapping theorem [LLM⁺, Theorem 3.6]. Since the group of integers \mathbb{Z} satisfies CERP_n for all $n \in \mathbb{N}$, see [LLM⁺, Example 4.21], repeated applications of the bootstrapping theorem show that many groups satisfy CERP_n for suitable n . For example, infinite elementary amenable groups of type FP_{∞} satisfy CERP_n for all n and the special linear group $\text{SL}_d(\mathbb{Z})$ satisfies CERP_{d-2} for $d \geq 3$.

Theorem 14.7. *Let $n \in \mathbb{N}$ and let Γ be a group satisfying CERP_{n+1} . Let Λ_* be a residual chain in Γ and let $\alpha: \Gamma \curvearrowright (X, \mu)$ be the profinite completion of Γ with*

respect to Λ_* . Then

$$\begin{aligned} \forall_{r \in \{0, \dots, n+1\}} \quad \text{medim}_r^Z(\alpha) &= 0 \\ \forall_{r \in \{0, \dots, n\}} \quad \text{mevol}_r(\alpha) &= 0. \end{aligned}$$

Proof. By Lemma 9.2, we may assume that $Z = \mathbb{Z}$. We use the notation C_*, κ, T, i_0 from Definition 14.5. For $i \geq i_0$, we apply Lemma 14.2 to the $\mathbb{Z}\Lambda_i$ -chain map $f_* : \text{Res}_{\mathbb{Z}\Lambda_i}^{\mathbb{Z}\Gamma} C_* \rightarrow E_*$, where the free $\mathbb{Z}\Lambda_i$ -chain complex E_* satisfies

$$\begin{aligned} \text{rk}_{\mathbb{Z}\Lambda_i}(E_r) &\leq \frac{\kappa}{T} \cdot [\Gamma : \Lambda_i] \cdot \text{rk}_{\mathbb{Z}\Gamma}(C_r) \\ \|\partial_r^E\| &\leq \exp(\kappa) \cdot T^\kappa. \end{aligned}$$

Hence Lemma 14.2 yields

$$\begin{aligned} \text{medim}_r^{\mathbb{Z}}(\alpha) &\leq \frac{\kappa}{T} \cdot \text{rk}_{\mathbb{Z}\Gamma}(C_r) \\ \text{mevol}_r(\alpha) &\leq \frac{\kappa^2(1 + \log(T))}{T} \cdot \text{rk}_{\mathbb{Z}\Gamma}(C_{r+1}) \end{aligned}$$

Sending $T \rightarrow \infty$ finishes the proof. \square

Part 3. Dynamical inheritance properties

We establish the following dynamical inheritance and computational properties of measured embedding dimension and measured embedding volume: monotonicity under weak containment (Section 15), a disintegration estimate and reduction to ergodic actions (Section 16), estimates via the equivalence relation ring (Section 17), invariance under weak bounded orbit equivalence (Section 18), comparison with cost (Section 19), comparison with integral foliated simplicial volume (Section 20). In particular, this will also allow us to compute further examples.

15. WEAK CONTAINMENT

In this section, we prove monotonicity of measured embedding dimension and measured embedding volume under weak containment of actions (Theorem 15.30). After recalling the definition of weak containment, we introduce an upper bound on the norm and a way of translating chain complexes over a crossed product ring to a different action.

15.1. Preliminaries on weak containment. We briefly recall the definition of weak containment and a few examples.

Definition 15.1 (weak containment, [Kec10, p. 64]). Let Γ be a group and $\alpha: \Gamma \curvearrowright (X, \mu)$ and $\beta: \Gamma \curvearrowright (Y, \nu)$ be probability measure preserving actions of Γ on standard probability spaces. We say that α is *weakly contained* in β (in symbols $\alpha \prec \beta$) if for all $n \in \mathbb{N}$, measurable sets $A_1, \dots, A_n \subset X$, finite sets $F \subset \Gamma$, and $\varepsilon > 0$, there are measurable sets $B_1, \dots, B_n \subset Y$ such that

$$\forall_{\gamma \in F} \forall_{i,j \in \{1, \dots, n\}} |\mu(\gamma^\alpha(A_i) \cap A_j) - \nu(\gamma^\beta(B_i) \cap B_j)| < \varepsilon.$$

Here we write γ^α (resp. γ^β) when $\gamma \in \Gamma$ is acting on (X, μ) via α (resp. on (Y, ν) via β).

Weak containment is transitive on probability measure preserving actions of a given group.

Example 15.2. Let Γ be a group and $\alpha: \Gamma \curvearrowright (X, \mu)$ and $\beta: \Gamma \curvearrowright (Y, \nu)$ be probability measure preserving actions of Γ on standard probability spaces. It is straightforward to check that $\alpha \prec \alpha \times \beta$, where $\alpha \times \beta: \Gamma \curvearrowright (X \times Y, \mu \otimes \nu)$ is the product action.

Example 15.3. Let Γ be a residually finite group and Λ_* be a residual chain of Γ . Let (X, μ) be the inverse limit of the system $(\Gamma/\Lambda_i)_{i \in \mathbb{N}}$, equipped with the Haar measure. Then, we have an action $\Gamma \curvearrowright X$ via left translation. This action is weakly contained in the profinite completion action $\Gamma \curvearrowright \hat{\Gamma}$ [Kec12, Proposition 2.3].

For countably infinite groups, there is a smallest action with respect to weak containment.

Example 15.4 (Bernoulli shift). Let (X, μ) be a non-trivial probability space (i.e., μ is not concentrated in one point) and Γ be a countably infinite group. The *Bernoulli shift* of Γ on X is the action of Γ on $\prod_{\Gamma} X$ (endowed with the product measure) via shifting of the factors. Abért and Weiss proved that the Bernoulli shift is weakly contained in every free probability measure preserving action of Γ [AW13, Theorem 1].

Example 15.5 (amenable groups). Let Γ be an infinite amenable group. Then all free probability measure preserving actions of Γ on standard probability spaces are weakly contained in the Bernoulli shift [Kec10, p. 91]. As a consequence in this situation all free probability measure preserving actions on standard probability spaces are *weakly equivalent* (Example 15.4).

Definition 15.6 (ergodic action). An action $\Gamma \curvearrowright (X, \mu)$ is *ergodic* if for every measurable subset $A \subset X$ with $\Gamma \cdot A = A$, we have

$$\mu(A) = 0 \quad \text{or} \quad \mu(X \setminus A) = 0.$$

Definition 15.7 (EMD*, [Kec12, Definition 4.4, Proposition 4.5]). An infinite countable residually finite group Γ satisfies EMD* if every ergodic standard probability action of Γ is weakly contained in the profinite completion action $\Gamma \curvearrowright \hat{\Gamma}$.

For the examples below, note that Tucker-Drob proved that for all groups property EMD* is equivalent to a similarly defined property MD [TD15, Theorem 1.4].

Example 15.8 (EMD*). The following groups satisfy EMD*:

- countable free groups [Kec12, Theorem 3.1, Proposition 4.5];
- residually finite infinite amenable groups [Kec10, Proposition 13.2];
- free products of non-trivial groups $\Gamma * \Lambda$, where each is either finite or has property EMD* [TD15, Theorem 4.8]
- subgroups of groups with property EMD* [Kec12, p. 486];
- finite index extensions of groups with property EMD* [Kec12, p. 486];
- extensions $1 \rightarrow N \rightarrow \Gamma \rightarrow Q \rightarrow 1$ where N is a finitely generated group with property EMD* and Q is a residually finite amenable group [BTD13, Theorem 1.4].

The previous properties show that also the following geometric families of groups have EMD*:

- fundamental groups of connected closed surfaces [BTD13, Theorem 1.4];
- fundamental groups of connected compact hyperbolic 3-manifolds with empty or toroidal boundary [FLMQ21, Proposition 5.2] (see also [FLPS16, Corollary 3.11]).

More examples can be found in the survey by Burton and Kechris [BK20, pp. 2698f].

Many dynamical invariants are monotone under weak containment (including, e.g., cost [Kec10, Corollary 10.14] and integral foliated simplicial volume [FLPS16, Theorem 1.5], see Sections 19 and 20 for the definitions). In Theorem 15.30, we will prove monotonicity of measured embedding dimension and measured embedding volume under weak containment. In particular, for groups satisfying EMD*, we can bound medim and mevol of the profinite completion by the corresponding invariant of any action (Corollary 16.5).

Instead of working directly with the definition, we will often employ the following characterisation of weak containment using weak neighbourhoods.

Definition 15.9 (weak neighbourhoods, [Kec10, Section 1(B)]). Let Γ be a group and (X, μ) be a standard probability space. The *weak topology* on the space of probability measure preserving actions $\Gamma \curvearrowright (X, \mu)$ is defined by the following basic open neighbourhoods: Let $\alpha: \Gamma \curvearrowright X$, $F \subset \Gamma$ be finite, $n \in \mathbb{N}$, $A_1, \dots, A_n \subset X$ be measurable, and $\varepsilon \in \mathbb{R}_{>0}$. Then,

$$\{\beta: \Gamma \curvearrowright X \mid \forall \gamma \in F \quad \forall_{i \in \{1, \dots, n\}} \quad \mu((\gamma^\alpha A_i) \triangle (\gamma^\beta A_i)) < \varepsilon\}$$

is open in the weak topology.

Proposition 15.10 ([Kec10, Proposition 10.1]). *Let $\alpha: \Gamma \curvearrowright (X, \mu)$ and $\beta: \Gamma \curvearrowright (Y, \nu)$ be probability measure preserving actions. Then, α is weakly contained in β if and only if in every weak neighbourhood U of α , there is $\beta' \in U$ such that β' is isomorphic to β as actions on standard probability spaces, i.e., there is an isomorphism $\varphi: (X, \mu) \rightarrow (Y, \nu)$ of measure spaces such that $\varphi(\gamma^\alpha x) = \gamma^\beta \varphi(x)$ for all $x \in X$ and $\gamma \in \Gamma$.*

15.2. An upper bound on the norm. In the following, we will often use the following upper bound to the norm.

Let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a probability measure preserving action. Since multiple actions will be involved, we stress the action in the notation (we write, e.g., $L^\infty(X) *_\alpha \Gamma := L^\infty(\alpha) * \Gamma$, $\langle A \rangle_\alpha$, \dim_α , rk_α , lognorm_α). We restrict to the special case that $R = R_\alpha := L^\infty(X) *_\alpha \Gamma$.

Definition 15.11. Let $f: M \rightarrow N$ be an R_α -homomorphism between marked projective R_α -modules. Let P be a reduced presentation of f as specified in Setup 2.34. We define

$$Q(f, P) := \sum_{(i,j,k,\gamma) \in I \times J \times K \times F} |a_{i,j,k,\gamma}|$$

and

$$Q(f) := \min_P Q(f, P),$$

where the minimum is taken over all possible reduced presentations of f . Note that this is indeed a minimum, as norms of elements in Z lie in \mathbb{N} . If $z \in R_\alpha$, define $Q(z) := Q(f_z)$, where $f_z: R_\alpha \rightarrow R_\alpha$ is the map given by right multiplication with z .

If $\eta: \langle A \rangle_\alpha \rightarrow L^\infty(\alpha)$ is an R_α -linear map, we define

$$Q(\eta) := Q(\iota(\eta(\chi_A, 1))),$$

where $\iota: L^\infty(\alpha) \hookrightarrow R_\alpha$ is the canonical inclusion into the summand indexed by $1 \in \Gamma$. Finally, if $\eta: M = \bigoplus_{i \in I} \langle A_i \rangle_\alpha \rightarrow L^\infty(\alpha)$ is an R_α -linear map, we define

$$Q(\eta) := \sum_{i \in I} Q(\eta|_{\langle A_i \rangle_\alpha}).$$

Remark 15.12. Proposition 2.35 shows that for every R -homomorphism f between marked projective R -modules, we have $\|f\| \leq Q(f)$. Straightforward calculations also show that Q is an upper bound to the ∞ -norm $\|\cdot\|_\infty$ (Definition 2.25), to N_1 , and to \underline{N}_2 (Definition 2.27).

Lemma 15.13. *We record a few basic properties of this quantity.*

- (i) *Let $f: M \rightarrow N$ be an R_α -homomorphism between marked projective R_α -modules and $M = M_1 \oplus M_2$ be a marked decomposition. Then,*

$$Q(f) \leq Q(f|_{M_1}) + Q(f|_{M_2}).$$

- (ii) *Let $f: M \rightarrow N$ and $g: N \rightarrow P$ be R_α -homomorphisms between marked projective R_α -modules. Then,*

$$Q(g \circ f) \leq Q(g) \cdot Q(f).$$

- (iii) *Let $f_1: M \rightarrow N_1$ and $f_2: M \rightarrow N_2$ be R_α -homomorphisms between marked projective R_α -modules. Then, $(f_1, f_2): M \rightarrow N_1 \oplus N_2$ satisfies*

$$Q((f_1, f_2)) \leq Q(f_1) + Q(f_2).$$

Proof. This follows from straightforward computations. \square

15.3. Translating actions. Let α and β be probability measure preserving actions of a group Γ on a standard probability space (X, μ) . In this section, we consider modules and maps defined over R_α , and we produce “corresponding” R_β -modules and R_β -maps.

We define the translation of modules and homomorphisms as follows.

Definition 15.14 (translation of modules). Let $M = \bigoplus_{i \in I} \langle A_i \rangle_\alpha$ be a marked projective R_α -module. We define the *translation of M to β* as the marked projective R_β -module M_β via

$$M_\beta := \bigoplus_{i \in I} \langle A_i \rangle_\beta.$$

Remark 15.15. Since the definition of dimension (Definition 2.13) does not depend on the action, we have that $\dim_\beta(M_\beta) = \dim_\alpha(M)$.

We can also translate maps to the action β .

Definition 15.16. Let $f: R_\alpha^m \rightarrow R_\alpha^n$ be a linear map between marked free R_α -modules. Recall from Setup 2.34 that f is given by right multiplication with a matrix z over R_α , where

$$z_{i,j} = \sum_{(k,\gamma) \in K \times F} a_{i,j,k,\gamma} \cdot (\chi_{\gamma^\alpha U_k}, \gamma),$$

and $(U_k)_{k \in K}$ is a finite family of disjoint subsets of X , the set $F \subset \Gamma$ is finite, and $a_{i,j,k,\gamma} \in Z$. We define the *translation of f to β* to be the R_β -linear map $f_\beta: R_\beta^m \rightarrow R_\beta^n$ defined by right multiplication with the matrix $z_\beta = ((z_\beta)_{i,j})_{i \in I, j \in J}$ that is defined by

$$(z_\beta)_{i,j} = \sum_{(k,\gamma) \in K \times F} a_{i,j,k,\gamma} \cdot (\chi_{\gamma^\beta U_k}, \gamma).$$

It is straightforward to show that f_β is well-defined and does not depend on the chosen presentation of z in Setup 2.34.

More generally, let $f: M = \bigoplus_{i \in I} \langle A_i \rangle_\alpha \rightarrow N = \bigoplus_{j \in J} \langle B_j \rangle_\alpha$ be a linear map between marked projective R_α -modules. We define its *translation to β* by

$$f_\beta := \pi_{N_\beta} \circ (\iota_N \circ f \circ \pi_M)_\beta \circ \iota_{M_\beta}: M_\beta \rightarrow N_\beta,$$

where $\iota_N: N \rightarrow R_\alpha^{\#J}$ and $\iota_{M_\beta}: M_\beta \rightarrow R_\beta^{\#I}$ denote the canonical marked inclusions and $\pi_{N_\beta}: R_\beta^{\#J} \rightarrow N_\beta$ and $\pi_M: R_\alpha^{\#I} \rightarrow M$ denote the canonical marked projections.

Remark 15.17. The action α is replaced by β in three places:

- (1) In the generation of modules: We generated an R_β -module instead of one over R_α ;
- (2) In the multiplication of the matrix: We multiply over R_β ;
- (3) In the coefficients: We multiply with $\chi_{\gamma^\beta U_k}$, where γ now acts via β on U_k . Previously, we considered the action via α .

Remark 15.18. From Definition 15.11, it follows that Q is translation-invariant, i.e., $Q(f_\beta) \leq Q(f)$.

For complexes obtained from $Z\Gamma$ -chain complexes by tensoring, there is an easy description of the translation.

Lemma 15.19. Let $C_* \rightarrow Z$ be an augmented free $Z\Gamma$ -chain complex. Then, there is a canonical isomorphism of R_β -chain complexes

$$(R_\alpha \otimes_{Z\Gamma} C_*)_\beta \cong R_\beta \otimes_{Z\Gamma} C_*.$$

If $C_0 = Z\Gamma$ and the augmentation map $\eta: Z\Gamma \rightarrow Z$ is given by sending all $\gamma \in \Gamma$ to $1 \in Z$, then we can define an augmentation map $R_\alpha \otimes_{Z\Gamma} C_0 \rightarrow L^\infty(\alpha)$ and the previous isomorphism can be extended by the identity on $L^\infty(\beta)$ in degree -1 .

Proof. The tensor product and the translation are compatible with direct sums, so we can work componentwise. Without loss of generality, we assume $C_j \cong_{Z\Gamma} Z\Gamma$. Then, we have isomorphisms of R_β -modules

$$(R_\alpha \otimes_{Z\Gamma} C_j)_\beta \cong (R_\alpha)_\beta = (\langle X \rangle_\alpha)_\beta = \langle X \rangle_\beta \cong R_\beta \otimes_{Z\Gamma} C_j.$$

For the boundary maps, because all of the above isomorphisms are compatible with direct sums, we can suppose that $\partial_{j+1}: Z\Gamma \rightarrow Z\Gamma$ is given by right multiplication with $\sum_{\gamma \in \Gamma} a_\gamma \cdot \gamma$. Then, $R_\alpha \otimes_{Z\Gamma} \partial_{j+1}$ is given by right multiplication with

$$\sum_{\gamma \in \Gamma} a_\gamma \cdot (\chi_X, \gamma) = \sum_{\gamma \in \Gamma} a_\gamma \cdot (\chi_{\gamma^\alpha X}, \gamma).$$

Thus, its translation $(R_\alpha \otimes_{Z\Gamma} \partial_{j+1})_\beta$ to β is given by

$$\sum_{\gamma \in \Gamma} a_\gamma \cdot (\chi_{\gamma^\beta X}, \gamma).$$

Because $\gamma^\beta X = \gamma^\alpha X$, this agrees with $R_\beta \otimes_{Z\Gamma} \partial_{j+1}$. For the extension to degree -1 , we define $\eta: R_\alpha \otimes_{Z\Gamma} C_0 \cong R_\alpha \rightarrow L^\infty(\alpha)$ as the $L^\infty(\alpha)$ -linear extension of $\eta((\lambda, \gamma)) := \gamma\lambda$ for all $\lambda \in L^\infty(\alpha)$ and $\gamma \in \Gamma$. \square

Compositions behave well under translation in the following sense:

Lemma 15.20 (composition estimate). *Let $f: M \rightarrow N$ and $g: N \rightarrow P$ be linear maps over R_α and $\delta \in \mathbb{R}_{>0}$. Then, there exists a weak neighbourhood U of α such that for all $\beta \in U$, we have*

$$(g \circ f)_\beta =_\delta g_\beta \circ f_\beta.$$

Proof. By Lemma 3.7, Lemma 3.6 (vi), and the definition of the translation via the free case, we can assume that $M = N = P = R_\alpha$. We fix presentations as in Setup 2.34, i.e., f and g are given by multiplication with elements $z_f, z_g \in R_\alpha$, respectively. Since sums behave well with almost equality (Lemma 3.6 (vi)), we can assume without loss of generality that

$$z_f = a \cdot (\chi_{\gamma^\alpha W}, \gamma) \quad \text{and} \quad z_g = b \cdot (\chi_{\lambda^\alpha V}, \lambda),$$

where $a, b \in Z$, $\gamma, \lambda \in \Gamma$ and $W, V \subset X$ are measurable subsets. We define U to be the weak neighbourhood of α defined by setting $F = \{\lambda^{-1}\}$, $n = 2$, $A_1 = W$, $A_2 = V$, and $\varepsilon = \delta$ (in the notation used in Definition 15.9).

Then, $g \circ f$ is given by right multiplication with

$$\begin{aligned} z_f \cdot z_g &= ab \cdot (\chi_{\gamma^\alpha W \cap \gamma^\alpha \lambda^\alpha V}, \gamma\lambda) \\ &= ab \cdot (\chi_{(\gamma\lambda)^\alpha ((\lambda^{-1})^\alpha W \cap V)}, \gamma\lambda). \end{aligned}$$

Thus, for all $\beta \in U$, the translation $(g \circ f)_\beta$ is given by right multiplication with

$$(15.1) \quad ab \cdot (\chi_{(\gamma\lambda)^\beta ((\lambda^{-1})^\alpha W \cap V)}, \gamma\lambda).$$

Similarly, $g_\beta \circ f_\beta$ is given by right multiplication with

$$(15.2) \quad ab \cdot (\chi_{(\gamma\lambda)^\beta ((\lambda^{-1})^\beta W \cap V)}, \gamma\lambda).$$

Note that the expressions in Equation (15.1) and Equation (15.2) differ only by an α resp. β in the exponent of λ^{-1} . Thus, by Example 3.3, $(g \circ f)_\beta$ and $g_\beta \circ f_\beta$ are almost equal with error at most

$$\begin{aligned} &\mu((\gamma\lambda)^\beta ((\lambda^{-1})^\alpha W \cap V) \triangle (\gamma\lambda)^\beta ((\lambda^{-1})^\beta W \cap V)) \\ &= \mu(((\lambda^{-1})^\alpha W \cap V) \triangle ((\lambda^{-1})^\beta W \cap V)) \leq \delta, \end{aligned}$$

where the last inequality is given by the choice of the weak neighbourhood U . \square

Corollary 15.21 (translation of chain complexes). *Let $n \in \mathbb{N}$, $\delta \in \mathbb{R}_{>0}$, and (D_*, η) be a marked projective R_α -chain complex with an augmentation map $\eta: D_0 \rightarrow L^\infty(\alpha)$. Then, there exists a weak neighbourhood U of α such that for all $\beta \in U$, the translated sequence $((D_*)_\beta, \eta_\beta)$ is a marked projective δ -almost n -chain complex over R_β .*

Proof. We apply Lemma 15.20 multiple times. Define U to be the intersection of the neighbourhoods where the estimate holds for $\partial_r \circ \partial_{r+1}$ for $r \in \{0, \dots, n\}$. Moreover, because η is an augmentation, there exists $z \in D_0$ with $\eta(z) = 1$. Thus, in a suitable neighbourhood, we have $\eta_\beta(z_\beta) =_\delta 1_\beta = 1_\alpha = 1$. \square

Corollary 15.22 (translation of chain maps). *Let $n \in \mathbb{N}$, let $\delta \in \mathbb{R}_{>0}$, and let $f_*: C_* \rightarrow D_*$ be an R_α -chain map between marked projective chain complexes (C_*, ζ) and (D_*, η) extending the identity on $L^\infty(\alpha)$. Then, there exists a weak neighbourhood U of α such that for all $\beta \in U$, the translations of the chain complexes $(C_*)_\beta$ and $(D_*)_\beta$ are marked projective δ -almost n -chain complexes over R_β and moreover, the map $(f_*)_\beta: (C_*)_\beta \rightarrow (D_*)_\beta$ is a δ -almost n -chain map.*

Proof. We apply the above Corollary 15.21 to C_* and D_* and intersect the resulting neighbourhoods. Moreover, we apply Lemma 15.20 with error term $\delta/2$ to the compositions $\eta \circ f_0$, $\partial_r^D \circ f_r$, and $f_{r-1} \circ \partial_r^C$ for $r \in \{1, \dots, n+1\}$. We intersect all resulting neighbourhoods of α to obtain a new neighbourhood U of α . For all $\beta \in U$, we have

$$\eta_\beta \circ (f_0)_\beta =_{\delta/2} (\eta \circ f_0)_\beta = \zeta_\beta.$$

Moreover, for $r \in \{1, \dots, n+1\}$, we have

$$\begin{aligned} (\partial_r^D)_\beta \circ (f_r)_\beta &=_{\delta/2} (\partial_r^D \circ f_r)_\beta \\ &= (f_{r-1} \circ \partial_r^C)_\beta \\ &=_{\delta/2} (f_{r-1})_\beta \circ (\partial_r^C)_\beta \end{aligned}$$

Thus by Lemma 3.6 (i), we obtain that $(\partial_r^D)_\beta \circ (f_r)_\beta =_\delta (f_{r-1})_\beta \circ (\partial_r^C)_\beta$. \square

We show that the marked rank, lognorm (Definition 6.2), and the norm change continuously in the action in the following sense:

Lemma 15.23 (translation and marked rank). *Let $f: M \rightarrow N$ be an R_α -homomorphism between marked projective R_α -modules and $\delta \in \mathbb{R}_{>0}$. Then, there exists a weak neighbourhood U of α such that for all $\beta \in U$, we have $\text{rk}_\beta(f_\beta) \leq \text{rk}_\alpha(f) + \delta$.*

Proof. By Lemma 6.6, we have (in the notation of Setup 2.34)

$$\text{rk}_\alpha(f) = \sum_{j \in J} \mu \left(\bigcup_{\substack{(i,k,\gamma) \in I \times K \times F, \\ a_{i,j,k,\gamma} \neq 0}} ((\gamma^{-1})^\alpha A_i \cap U_k) \right).$$

Thus, for every action β , we have

$$\begin{aligned} \text{rk}_\beta(f_\beta) - \text{rk}_\alpha(f) &\leq \sum_{\substack{(i,j,k,\gamma) \in I \times J \times K \times F, \\ a_{i,j,k,\gamma} \neq 0}} \mu(((\gamma^{-1})^\alpha A_i \cap U_k) \triangle ((\gamma^{-1})^\beta A_i \cap U_k)) \\ &\leq \sum_{(i,j,k,\gamma) \in I \times J \times K \times F} \mu(((\gamma^{-1})^\alpha A_i \cap U_k) \triangle ((\gamma^{-1})^\beta A_i \cap U_k)) \\ &\leq \delta, \end{aligned}$$

where the last inequality holds in a suitable weak neighbourhood U that is defined as in Definition 15.9 with error term $\varepsilon := \delta/(\#I \cdot \#J \cdot \#K \cdot \#F)$ (or $\varepsilon := 1$ if the denominator is zero) and the $(A_i)_{i \in I}$ as the test sets. \square

Lemma 15.24 (translation and norm). *Let $f: M \rightarrow N$ be an R_α -homomorphism between marked projective R_α -modules and $\delta \in \mathbb{R}_{>0}$. Then, there exists a weak neighbourhood U of α such that for all $\beta \in U$, the following holds: There is an R_β -homomorphism $f'_\beta: M_\beta \rightarrow N_\beta$ such that*

$$f_\beta =_{\delta, Q(f)} f'_\beta \quad \text{and} \quad \|f'_\beta\| \leq \|f\| \quad \text{and} \quad f'_\beta(M_\beta) \subset f_\beta(M_\beta).$$

Proof. By an analogue of Lemma 3.7, we can assume that $M = \langle A \rangle_\alpha$. We will thus drop $i \in I$ from the notation. Let P be a presentation representing f as in Setup 2.34 such that $Q(f, P) = Q(f)$. Fix the notation of Setup 2.34. Pick a weak neighbourhood U of α where for all $\gamma \in F, k \in K$, and $\beta \in U$, we have

$$\mu(\gamma^\alpha U_k \triangle \gamma^\beta U_k) \leq \frac{\delta}{\#K \cdot \#F}.$$

Let $\beta \in U$. We define

$$A'' := \bigcup_{(k, \gamma) \in K \times F} \gamma^\alpha U_k \triangle \gamma^\beta U_k$$

and $A' := A \setminus A''$. We set $M' := \langle A' \rangle_\beta$ and $M'' := \langle A'' \cap A \rangle_\beta$. Hence, we have an isomorphism of R_β -modules $M_\beta \cong M' \oplus M''$. By construction, we have that $\dim_\beta(M'') \leq \mu(A'') \leq \delta$. We define $f'_\beta := f_\beta \circ \iota_{A'} \circ \pi_{A'}$, where $\pi_{A'}$ is the projection onto the marked summand M' and $\iota_{A'}: M' \rightarrow M$ is the canonical marked inclusion. In particular,

$$f'_\beta|_{M'} = f_\beta|_{M'} \quad \text{and} \quad f'_\beta|_{M''} = 0$$

and $f'_\beta(M_\beta) \subset f_\beta(M_\beta)$. Moreover, note that for $L \subset K \times F$, we have

$$A' \cap \bigcap_{(k, \gamma) \in L} \gamma^\beta U_k \subset \bigcap_{(k, \gamma) \in L} \gamma^\alpha U_k.$$

Thus, by the explicit description of the operator norm (Proposition 2.35), we have

$$\begin{aligned} \|f'_\beta\| &= \max \left\{ \sum_{j \in J, (k, \gamma) \in L} |a_{j, k, \gamma}| \mid L \subset K \times F \text{ with } \mu \left(A' \cap \bigcap_{(k, \gamma) \in L} \gamma^\beta U_k \right) > 0 \right\} \\ &\leq \max \left\{ \sum_{j \in J, (k, \gamma) \in L} |a_{j, k, \gamma}| \mid L \subset K \times F \text{ with } \mu \left(\bigcap_{(k, \gamma) \in L} \gamma^\alpha U_k \right) > 0 \right\} \\ &= \|f\|. \end{aligned}$$

It remains to estimate $\|f_\beta|_{M''}\|$: we have

$$\begin{aligned} \|f_\beta|_{M''}\| &\leq \|f_\beta\| \\ &\leq Q(f_\beta) && \text{(Remark 15.12)} \\ &\leq Q(f). && \text{(Remark 15.18)} \quad \square \end{aligned}$$

We use these two estimates to show that also lognorm is continuous in the action.

Lemma 15.25 (translation and lognorm). *Let $f: M \rightarrow N$ be an R_α -homomorphism between marked projective R_α -modules and $\varepsilon \in \mathbb{R}_{>0}$. Then, there exists a weak neighbourhood U of α such that for all $\beta \in U$, we have*

$$\text{lognorm}_\beta(f_\beta) \leq \text{lognorm}_\alpha(f) + \varepsilon.$$

Proof. We set $\delta := \varepsilon / (4 \cdot \log_+ Q(f)) > 0$ (or $\delta := 1$ if $\log_+ Q(f) = 0$). By definition of lognorm_α , there exists a marked decomposition $(M_i)_{i \in I}$ of M over R_α with

$$(15.3) \quad \sum_{i \in I} \text{lognorm}'_\alpha(f|_{M_i}: M_i \rightarrow N) = \text{lognorm}'_\alpha(f, (M_i)_{i \in I}) < \text{lognorm}_\alpha(f) + \frac{\varepsilon}{2}.$$

To simplify notation, we will assume that $M = M_i$ consists of a single summand and estimate the change of $\text{lognorm}'_\alpha(f|_{M_i})$ under translation. (The general case will then follow by dividing the allowed error by $\#I$, which stays constant during this proof.)

By Lemma 15.23, there exists a weak neighbourhood U_1 of α , such that for all $\beta \in U_1$, we have $\text{rk}_\beta(f_\beta) \leq \text{rk}_\alpha(f) + \delta$. Moreover, by Lemma 15.24, there exists a weak neighbourhood U_2 of α , such that for all $\beta \in U_2$, there exists $f'_\beta: M_\beta \rightarrow N_\beta$ with

$$f_\beta =_{\delta, Q(f)} f'_\beta \quad \text{and} \quad \|f'_\beta\| \leq \|f\| \quad \text{and} \quad f'_\beta(M_\beta) \subset f_\beta(M_\beta).$$

We define $U := U_1 \cap U_2$. Then, for all $\beta \in U$, we have

$$\begin{aligned} & \text{lognorm}_\beta(f_\beta) \\ & \leq \text{lognorm}_\beta(f'_\beta) + \delta \cdot \log_+ Q(f) && \text{(Lemma 6.4 (iv))} \\ & \leq \text{lognorm}'_\beta(f'_\beta) + \delta \cdot \log_+ Q(f) && \text{(Def. of lognorm)} \\ & = \min\{\dim_\beta M_\beta, \text{rk}_\beta(f'_\beta)\} \cdot \log_+ \|f'_\beta\| + \delta \cdot \log_+ Q(f) \\ & = \min\{\dim_\alpha M, \text{rk}_\beta(f'_\beta)\} \cdot \log_+ \|f'_\beta\| + \delta \cdot \log_+ Q(f) && \text{(Remark 15.15)} \\ & \leq \min\{\dim_\alpha M, \text{rk}_\beta(f_\beta)\} \cdot \log_+ \|f'_\beta\| + \delta \cdot \log_+ Q(f) && (f'_\beta(M_\beta) \subset f_\beta(M_\beta)) \\ & \leq \min\{\dim_\alpha M, \text{rk}_\alpha f + \delta\} \cdot \log_+ \|f'_\beta\| + \delta \cdot \log_+ Q(f) && (\beta \in U_1) \\ & \leq \min\{\dim_\alpha M, \text{rk}_\alpha f + \delta\} \cdot \log_+ \|f\| + \delta \cdot \log_+ Q(f) && (\|f'_\beta\| \leq \|f\|) \\ & \leq \min\{\dim_\alpha M, \text{rk}_\alpha f\} \cdot \log_+ \|f\| + 2\delta \cdot \log_+ Q(f) && \text{(Remark 15.12)} \\ & \leq \text{lognorm}'_\alpha(f) + \frac{\varepsilon}{2} && \text{(Def. of } \delta) \\ & \leq \text{lognorm}_\alpha(f) + \varepsilon. && \text{(Equation (15.3)) } \square \end{aligned}$$

15.4. Strictification and translation. We revisit the results for strictification of chain complexes and chain maps from Section 4. We establish more abstract upper bounds and investigate how these change under translation.

As before, the two main results of this section are the following:

- Every almost chain complex is “close” to a strict chain complex (Theorem 15.28).
- Every almost chain map is “close” to a strict chain map (Theorem 15.29).

Let $\alpha: \Gamma \curvearrowright (X, \mu)$ be an essentially free probability measure preserving action on a standard probability space. We consider modules over the crossed product ring $R_\alpha := L^\infty(X) *_\alpha \Gamma$.

We bound the complexity of the input data by the following notion:

Definition 15.26 (translation-invariant constant). Let (X, μ) be a standard probability space, Γ be a group and $n \in \mathbb{N}$. A *translation-invariant constant* is a family of maps $\kappa = (\kappa_\delta)_{\delta \in \mathbb{R}_{>0}}$ that, given $\delta \in \mathbb{R}_{>0}$, a probability measure preserving action $\alpha: \Gamma \curvearrowright (X, \mu)$, and an (augmented) marked projective δ -almost n -chain complex (D_*, η) over R_α , assigns a positive real number $\kappa_\delta(D_*, \eta)$ such that the following holds:

Let $\delta \in \mathbb{R}_{>0}$, $\alpha: \Gamma \curvearrowright X$ and (D_*, η) be a marked projective $\delta/2$ -almost n -chain complex over R_α . Then, there exists a neighbourhood U of α (see Definition 15.9) such that for every $\beta \in U$, we have that $((D_*)_\beta, \eta_\beta)$ is a δ -almost n -chain complex and

$$\kappa_\delta((D_*)_\beta, \eta_\beta) \leq \kappa_{\delta/2}(D_*, \eta) + 1.$$

We often write $\kappa_\delta(D_*)$ instead of $\kappa_\delta(D_*, \eta)$.

In a similar fashion, we can define translation-invariant constants of almost chain maps between almost chain complexes.

Definition 15.27 (translation-invariant constant of chain complexes). Let (X, μ) be a standard probability space, Γ be a group and $n \in \mathbb{N}$. A *translation-invariant constant* is a family of maps $(\kappa_\delta)_{\delta \in \mathbb{R}_{>0}}$ that, given $\delta \in \mathbb{R}_{>0}$, a probability measure preserving action $\alpha: \Gamma \curvearrowright (X, \mu)$, and a δ -almost n -chain map $f_*: C_* \rightarrow D_*$ between augmented marked projective δ -almost n -chain complexes over R_α , assigns a positive real number $\kappa_\delta(f_*)$ such that the following holds:

Let $\delta \in \mathbb{R}_{>0}$, $\alpha: \Gamma \curvearrowright X$ and $f_*: C_* \rightarrow D_*$ be a $\delta/2$ -almost n -chain map between marked projective $\delta/2$ -almost n -chain complexes over R_α . Then, there exists a neighbourhood U of α such that for every $\beta \in U$, we have that $(f_*)_\beta: (C_*)_\beta \rightarrow (D_*)_\beta$ is a δ -almost n -chain map between δ -almost n -chain complexes over R_β and

$$\kappa_\delta((f_*)_\beta) \leq \kappa_{\delta/2}(f_*) + 1.$$

Theorem 15.28. Let (X, μ) be a standard probability space, Γ be a group, and $n \in \mathbb{N}$. Then, there exist monotone increasing functions $K, p: \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ and a translation-invariant constant $\kappa = (\kappa_\delta)_{\delta \in \mathbb{R}_{>0}}$ such that for every probability measure preserving action $\alpha: \Gamma \curvearrowright (X, \mu)$, $\delta \in \mathbb{R}_{>0}$, and every marked projective δ -almost n -chain complex (D_*, η) over R_α , there exists a marked projective (strict) R_α -chain complex $(\widehat{D}_*, \widehat{\eta})$ (up to degree $n+1$) such that

$$d_{\text{GH}}^{K(\kappa_\delta(D_*))}(\widehat{D}_*, D_*, n) \leq K(\kappa_\delta(D_*)) \cdot \delta.$$

Moreover, \widehat{D}_* can be chosen such that the following hold:

- (i) For each $j \in \{0, \dots, n\}$, the module D_j is a submodule of \widehat{D}_j and the inclusion map $D_* \hookrightarrow \widehat{D}_*$ is a $(K(\kappa_\delta(D_*)) \cdot \delta)$ -almost n -chain map.
- (ii) We have $\kappa_\delta(\widehat{D}_*) \leq p(\kappa_\delta(D_*))$.

Before giving the proof, we recall from Definition 4.7 that

$$\begin{aligned} \kappa_n(D_*) &:= \max\{\|\eta\|, \|\partial_1^D\|, \dots, \|\partial_{n+1}^D\|\} \\ \underline{\nu}_n(D_*) &:= \max\{\|\eta\|_\infty, \underline{N}_1(\partial_1^D), \dots, \underline{N}_1(\partial_{n+1}^D)\} \\ \nu_n(D_*) &:= \max\{\|\eta\|_\infty, N_1(\partial_1^D), \dots, N_1(\partial_{n+1}^D)\}. \end{aligned}$$

Moreover, if $\kappa \in \mathbb{R}_{>0}$, then we say that $\overline{\kappa}_n(D_*) < \kappa$ if

$$\max\{\text{rk}(D_1), \dots, \text{rk}(D_{n+1}), \kappa_n(D_*), \underline{\nu}_n(D_*)\} < \kappa$$

and there exists a $z \in D_0$ with

$$\eta(z) =_\delta 1, \quad N_1(z) < \kappa, \quad N_2(z) < \kappa, \quad |z|_\infty < \kappa.$$

Proof. We employ Theorem 4.8 to define K : For the fixed $n \in \mathbb{N}$ and $x \in \mathbb{R}_{>0}$, we define $K(x)$ to be one of the $K \in \mathbb{R}_{>0}$, for which Theorem 15.28 holds (with $\kappa := x$). Without loss of generality, we can assume that the function K is monotone increasing. We define κ as follows: Let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a probability measure preserving action, $\delta \in \mathbb{R}_{>0}$, and (D_*, η) be a marked projective δ -almost n -chain complex over R_α . We define

$$\kappa'_\delta := \inf_z Q(z),$$

where z ranges over all $z \in D_0$ with $\eta(z) =_\delta 1$. We then define

$$\kappa_\delta(D_*, \eta) := \max\{\kappa'_\delta, Q(\eta), Q(\partial_1^D), \dots, Q(\partial_{n+1}^D), \text{rk}(D_1), \dots, \text{rk}(D_{n+1})\} + 1.$$

Note that Q and rk are invariant under translation (see Remark 15.18). For the (almost-)invariance of κ' , note that if $z \in D_0$ such that $\eta(z) =_{\delta/2} 1$, then the composition estimate (Lemma 15.20) shows that in a suitable neighbourhood U , we have for all $\beta \in U$ that $\eta_\beta(z_\beta) =_\delta 1$. Moreover, $Q(z_\beta) \leq Q(z)$. Thus, we can find a neighbourhood as in the translation-invariance condition. Note that because κ' is defined as an infimum, we add the “+1” in the definition of a translation-invariant

constant to obtain an open neighbourhood. Thus, κ defines a translation-invariant constant.

We show that the functions K and κ satisfy the desired conditions: Let $\alpha: \Gamma \curvearrowright (X, \mu)$, $\delta \in \mathbb{R}_{>0}$, and (D_*, η) be a marked projective δ -almost n -chain complex. We write $\kappa_D := \kappa_\delta(D_*)$ and $K_D := K(\kappa_\delta(D_*))$. We have $\bar{\kappa}_n(D_*) < \kappa_D$, because $N_1(\cdot)$, $N_2(\cdot)$, $|\cdot|_\infty$, and $\|\cdot\|$ are bounded from above by $Q(\cdot)$ (see Remark 15.12). Thus, Theorem 15.28 yields a marked projective R_α -chain complex $(\widehat{D}_*, \widehat{\eta})$ (up to degree $n+1$) with

$$d_{\text{GH}}^{K_D}(\widehat{D}_*, D_*, n) \leq K_D \cdot \delta.$$

Moreover, Theorem 15.28 states that the inclusion map $D_* \hookrightarrow \widehat{D}_*$ is a $(K_D \cdot \delta)$ -almost n -chain map. For the second statement, we have to dive into the details of the proof of Theorem 4.8 and proceed by induction over the degrees. The proof of Lemma 4.10 yields that $\kappa'_\delta(\widehat{D}_*) \leq \kappa'_\delta(D_*)$. Furthermore, that proof shows that $\widehat{D}_0 = D_0 \oplus \langle B \rangle_\alpha$ for some $B \subset X$ and thus,

$$\text{rk}(\widehat{D}_0) = \text{rk}(D_0) + 1.$$

Lemma 15.13 then shows that

$$\begin{aligned} Q(\widehat{\eta}) &\leq Q(\eta) + Q(\widehat{\eta}|_{\langle B \rangle}) \\ &= Q(\eta) + Q(1 - \eta(z)) && \text{(proof of Lemma 4.10)} \\ &\leq Q(\eta) + (1 + Q(\eta) \cdot Q(z)) && \text{(Lemma 15.13)} \\ &\leq p_0(\kappa_\delta(D_*)) \end{aligned}$$

for the function $p_0: x \mapsto 1 + x + x^2$.

For the inductive step, assume that \widehat{D}_{r-1} and $\partial_r^{\widehat{D}}$ have been constructed and satisfy the theorem with the function p_{r-1} . The proof of Lemma 4.10 constructs \widehat{D}_r as $D_r \oplus E_r$, where $\text{rk}(E_r) \leq \text{rk}(D_r)$, and $\dim_\alpha(E_r)$ is bounded by a function in $\kappa_\delta(\widehat{D}_*)$. Moreover, Lemma 15.13 yields that

$$\begin{aligned} Q(\widehat{\partial}_r) &\leq Q(\widetilde{\partial}_r) + \text{rk } E_r \cdot Q(\widetilde{\partial}_r) \cdot Q(\partial_{r+1}) \\ &\leq Q(\widetilde{\partial}_r) \cdot (1 + \text{rk } D_r \cdot Q(\partial_{r+1})) \\ &\leq (Q(\partial_{r+1}) + \text{rk } E_r) \cdot (1 + \text{rk } D_r \cdot Q(\partial_{r+1})) \\ &\leq (Q(\partial_{r+1}) + \text{rk } D_r) \cdot (1 + \text{rk } D_r \cdot Q(\partial_{r+1})), \end{aligned}$$

which is bounded from above by a function in $\kappa_\delta(\widehat{D}_*)$. Moreover, in degree $n+1$, we have

$$\begin{aligned} Q(\widehat{\partial}_{n+1}) &= Q(\widetilde{\partial}_{n+1}) \\ &\leq Q(\partial_{n+1}) + \text{rk}(E_n) \\ &\leq Q(\partial_{n+1}) + \text{rk}(D_n), \end{aligned}$$

which is also bounded from above by $2 \cdot \kappa_\delta(\widehat{D}_*)$. Finally, by construction, we have $\text{rk}(\widehat{D}_r) \leq 2 \cdot \text{rk}(D_r)$. Altogether, we obtain that

$$\kappa_\delta(\widehat{D}_*) \leq p(\kappa_\delta(D_*)),$$

where p is defined to be the maximum of all the upper bounds encountered so far. Note that we can assume p to be monotone increasing, otherwise we set

$$p'(x) := \max_{y \leq x} p(y).$$

□

We can also strictify chain maps between (strict) chain complexes.

Theorem 15.29. *Let (X, μ) be a standard probability space, let Γ be a group, and let $n \in \mathbb{N}$. Then, there exists a monotone increasing function $K: \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ and a translation-invariant constant κ such that for all $\alpha: \Gamma \curvearrowright X$, $\delta \in \mathbb{R}_{>0}$, and for every δ -almost n -chain map $f_*: (C_*, \zeta) \rightarrow (\widehat{D}_*, \widehat{\eta})$ between marked projective (strict) R_α -chain complexes, extending the identity on $L^\infty(\alpha)$, there exists a marked projective strict R_α -chain complex \widetilde{D}_* with*

$$d_{\text{GH}}^{K(\kappa_\delta(f_*))}(\widetilde{D}_*, \widehat{D}_*, n) < K(\kappa_\delta(f_*)) \cdot \delta$$

that admits a chain map $\widetilde{f}_*: C_* \rightarrow \widetilde{D}_*$ extending the identity on $L^\infty(\alpha)$.

Proof. Given $\alpha: \Gamma \curvearrowright (X, \mu)$, $\delta \in \mathbb{R}_{>0}$, and a δ -almost n -chain map $f: C_* \rightarrow \widehat{D}_*$, we define

$$\begin{aligned} \kappa_\delta(f_*) := \max\{Q(\zeta), Q(\partial_1^C), \dots, Q(\partial_{n+1}^C), Q(\widehat{\eta}), Q(\partial_1^{\widehat{D}}), \dots, Q(\partial_{n+1}^{\widehat{D}}), \\ Q(f_0), \dots, Q(f_n)\} + 1. \end{aligned}$$

Because Q is translation-invariant (Remark 15.18), κ is a translation-invariant constant. We employ Theorem 15.28 to define the map $K: \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$, which can be assumed to be monotone.

Because the norm of f is bounded by Q (Remark 15.12), we have the following estimates $\max\{\kappa_n(C_*), \nu_n(C_*)\} < \kappa_\delta(f_*)$ and $\kappa_n(\widehat{D}_*) < \kappa_\delta(f_*)$. Moreover, $\kappa_n(f_*) \leq \kappa_\delta(f_*)$. Thus, Theorem 4.15 yields a marked projective R_α -chain complex $(\widetilde{D}_*, \widetilde{\eta})$ with

$$d_{\text{GH}}^{K(\kappa_\delta(f_*))}(\widetilde{D}_*, \widehat{D}_*, n) < K(\kappa_\delta(f_*)) \cdot \delta$$

that admits a chain map $\widetilde{f}_*: C_* \rightarrow \widetilde{D}_*$ extending the identity on $L^\infty(\alpha)$ such that

$$d_{\text{GH}}^{K(\kappa_\delta(f_*))}(\widetilde{f}_r, f_r) < K(\kappa_\delta(f_*)) \cdot \delta$$

for all $r \in \{0, \dots, n+1\}$. \square

15.5. Proof of monotonicity. The main goal of this section is to prove the following monotonicity result for measured embedding dimension and volume under weak containment of actions.

Theorem 15.30 (weak containment). *Let $n \in \mathbb{N}$, let Γ be a group of type FP_{n+1} , and let $\alpha: \Gamma \curvearrowright (X, \mu)$, $\beta: \Gamma \curvearrowright (Y, \nu)$ be free probability measure preserving actions of Γ on standard probability spaces. Let $\alpha \prec \beta$. Let Z denote the integers (equipped with the standard norm) or a finite field (equipped with the trivial norm). Then, we have*

$$\begin{aligned} \text{medim}_n^Z(\beta) &\leq \text{medim}_n^Z(\alpha), \\ \text{mevol}_n(\beta) &\leq \text{mevol}_n(\alpha). \end{aligned}$$

Remark 15.31. The proof of this theorem consists of several steps. We give a roadmap to the proof outlining the main ideas:

- (1) We fix an α -embedding from C_*^α to an augmented R_α -chain complex D_* with $\dim_\alpha(D_n)$ “close” to $\text{medim}_n^Z(\alpha)$ (resp. $\log\text{norm}_\alpha(\partial_{n+1}^D)$ “close” to $\text{mevol}_n(\alpha)$).
- (2) Because α is weakly contained in β , the action β is (isomorphic to) an action $\beta': \Gamma \curvearrowright X$ “close” to α in the weak topology (see Proposition 15.10).
- (3) We can translate D_* from α to β' (see Section 15.3) and obtain $(D_*)_{\beta'}$. However, in general, $(D_*)_{\beta'}$ will no longer be a chain complex, but only an almost chain complex over $R_{\beta'}$ and the α -embedding $C_*^\alpha \rightarrow D_*$ translates to an almost chain map $C_*^{\beta'} \rightarrow (D_*)_{\beta'}$ over $R_{\beta'}$. The error depends on the previous distances.

- (4) We can strictify $(D_*)_{\beta'}$ to obtain a (strict) $R_{\beta'}$ -chain complex \widehat{D}_* . We obtain an almost β' -embedding to this complex.
- (5) We can also strictify the almost β' -embedding to obtain a strict $R_{\beta'}$ -chain map $C_*^{\beta'} \rightarrow \widehat{D}_*$ to a (different) strict chain complex \widetilde{D}_* .
- (6) If β' is “close” enough to α , the strictified chain complex \widetilde{D}_* is “close” to D_* , thus $\dim_{\beta'}(\widetilde{D}_n)$ is “close” to $\dim_{\alpha}(D_n)$ (resp. $\text{lognorm}_{\beta}(\partial_{n+1}^{\widetilde{D}})$ is “close” to $\text{lognorm}_{\alpha}(\partial_{n+1}^D)$).
- (7) We can make the error arbitrarily small, thus proving the claim.

The main difficulty is making the notions of closeness precise and controlling the distances. These distances depend on one another, thus some work needs to be done to obtain global control on the errors.

The key approximation is contained in the following lemma.

Lemma 15.32. *Let $n \in \mathbb{N}$, let Γ be a group of type FP_{n+1} , and let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a free probability measure preserving standard action. Let $f_*: C_*^{\alpha} \rightarrow D_*$ be an α -embedding and $\varepsilon > 0$. Then, there exists a weak neighbourhood U of α such that for all $\beta \in U$, there exists a β -embedding $C_*^{\beta} \rightarrow \widetilde{D}_*$ satisfying*

$$\dim_{\beta}(\widetilde{D}_n) \leq \dim_{\alpha}(D_n) + \varepsilon \quad \text{and} \quad \text{lognorm}_{\beta}(\partial_{n+1}^{\widetilde{D}}) \leq \text{lognorm}_{\alpha}(\partial_{n+1}^D) + \varepsilon.$$

Proof. We fix a free $Z\Gamma$ -resolution $C_* \twoheadrightarrow Z$ of the trivial $Z\Gamma$ -module Z with finitely generated $Z\Gamma$ -modules in degrees $\leq n+1$. We can additionally assume that $C_0 = Z\Gamma$ and that the augmentation map $\eta: Z\Gamma \rightarrow Z$ is given by mapping all $\gamma \in \Gamma$ to $1 \in Z$. As in the definition of medim_n^Z and mevol_n (Definition 1.1), we set $C_*^{\alpha} := R_{\alpha} \otimes_{Z\Gamma} C_*$. Let $f_*: C_*^{\alpha} \rightarrow D_*$ be the given α -embedding, i.e., an R_{α} -chain map extending the identity on $L^{\infty}(\alpha)$. By Lemma 15.25, we can pick a neighbourhood U such that for all $\beta \in U$, we have

$$\text{lognorm}_{\beta}((\partial_{n+1}^D)_{\beta}) \leq \text{lognorm}_{\alpha}(\partial_{n+1}^D) + \frac{\varepsilon}{2}.$$

We fix a translation-invariant constant κ , and monotone increasing maps K and p as in Theorem 15.28. Moreover, we fix a translation-invariant constant and monotone increasing constant as in Theorem 15.29. By taking the maximum, we can also denote the latter by κ resp. K . We restrict the neighbourhood U to a potentially smaller neighbourhood that additionally satisfies the translation-invariance condition in Definition 15.26 (for $\delta := \varepsilon/2$). We define

$$Q(f, n) := \max\{Q(f_0), \dots, Q(f_n)\}$$

and

$$M = \left(K(\max\{\kappa_{\delta}(f_*), 2 \cdot p(\kappa_{\delta}(D_*))\}) \cdot Q(f, n) + 1 \right) \cdot K(\kappa_{\delta/2}(D_*) + 1).$$

and choose $\delta \in \mathbb{R}_{>0}$ such that

$$M \cdot \delta \leq \varepsilon \quad \text{and} \quad M \cdot \log_+(M \cdot \delta) \leq \frac{\varepsilon}{2} \quad \text{and} \quad \delta \leq \varepsilon.$$

By Corollary 15.22, we can restrict U to a neighbourhood of α such that for all $\beta \in U$ the translated chain complex $(D_*)_{\beta}$ is a δ -almost n -chain complex and $(f_*)_{\beta}: (C_*^{\alpha})_{\beta} \rightarrow (D_*)_{\beta}$ is a δ -almost n -chain map. By Lemma 15.19, we have $(C_*^{\alpha})_{\beta} \cong C_*^{\beta}$.

Let $\beta \in U$. We first strictify $(D_*)_{\beta}$. By Theorem 15.28, we obtain a strict marked projective R_{β} -chain complex \widehat{D}_* such that the inclusion $i_*: (D_*)_{\beta} \hookrightarrow \widehat{D}_*$ is a $(K(\kappa_{\delta}((D_*)_{\beta})) \cdot \delta)$ -almost n -chain map and

$$d_{\text{GH}}^{K(\kappa_{\delta}((D_*)_{\beta}))}(\widehat{D}_*, (D_*)_{\beta}, n) \leq K(\kappa_{\delta}((D_*)_{\beta})) \cdot \delta.$$

By translation-invariance and because of our choice of U , we have

$$\kappa_\delta((D_*)_\beta) \leq \kappa_{\delta/2}(D_*) + 1.$$

Because K is monotone increasing, we can directly use $K(\kappa_{\delta/2}(D_*) + 1)$ as an upper bound in the following. Then, by Lemma 4.4 and Remark 15.12, we have that the composition $i_* \circ (f_*)_\beta: C_*^\beta \rightarrow (\tilde{D}_*)_\beta$ is a $(Q(f_\beta, n) \cdot K(\kappa_{\delta/2}(D_*) + 1) \cdot \delta + \delta)$ -almost n -chain map. Note that $Q(f_\beta, n) \leq Q(f, n)$.

Now, we can strictify the chain map: By Theorem 15.29, there exists a strict marked projective R_β -chain complex \tilde{D}_* admitting a (strict) R_β -chain map $C_*^\beta \rightarrow \tilde{D}_*$ extending the identity on $L^\infty(\alpha)$ with

$$d_{\text{GH}}^{K(\kappa_\delta(i_* \circ (f_*)_\beta))}(\tilde{D}_*, \hat{D}_*, n) \leq K(\kappa_\delta(i_* \circ (f_*)_\beta)) \cdot Q(f, n) \cdot K(\kappa_{\delta/2}(D_*) + 1) \cdot \delta.$$

Thus, $C_*^\beta \rightarrow \tilde{D}_*$ is a β -embedding. From the explicit descriptions of the translation-invariant constants (see the proofs of Theorem 15.28 and Theorem 15.29), we obtain that

$$\kappa_\delta(i_* \circ (f_*)_\beta) \leq \max\{\kappa_\delta(f_*), 2 \cdot \kappa_\delta(\hat{D}_*)\} \leq \max\{\kappa_\delta(f_*), 2 \cdot p(\kappa_\delta(D_*))\}.$$

Because the Gromov-Hausdorff distance satisfies the triangle inequality (Proposition 3.17), we obtain

$$d_{\text{GH}}^M(\tilde{D}_*, (D_*)_\beta, n) \leq M \cdot \delta$$

with

$$M = \left(K(\max\{\kappa_\delta(f_*), 2 \cdot p(\kappa_\delta(D_*))\}) \cdot Q(f, n) + 1 \right) \cdot K(\kappa_{\delta/2}(D_*) + 1),$$

as defined at the beginning of this proof. For the dimension, we thus obtain that

$$\begin{aligned} \dim_\beta(\tilde{D}_n) &\leq \dim_\beta((D_n)_\beta) + M \cdot \delta && \text{(Proposition 6.4 (v))} \\ &= \dim_\alpha(D_n) + M \cdot \delta && \text{(Remark 15.15)} \\ &\leq \dim_\alpha(D_n) + \varepsilon. && \text{(choice of } \delta) \end{aligned}$$

For the lognorm, we have

$$\begin{aligned} \text{lognorm}_\beta(\partial_{n+1}^{\tilde{D}}) &\leq \text{lognorm}_\beta((\partial_{n+1}^D)_\beta) + M \cdot \log_+(M \cdot \delta) && \text{(Proposition 6.4 (v))} \\ &\leq \text{lognorm}_\beta((\partial_{n+1}^D)_\beta) + \frac{\varepsilon}{2} && \text{(choice of } \delta) \\ &\leq \text{lognorm}_\alpha(\partial_{n+1}^D) + \varepsilon. && \text{(choice of } U) \square \end{aligned}$$

We can now prove the theorem that measured embedding dimension and volume are monotone under weak embeddings.

Proof of Theorem 15.30. We show how to deduce the statement for mevol_n . The proof for medim_n^Z works similarly by replacing every occurrence of “ $\text{lognorm}_*(\partial_{n+1})$ ” by “ $\dim_*(D_n)$ ”.

Without loss of generality, we assume that $\text{mevol}_n(\alpha) < \infty$. Let $\varepsilon \in \mathbb{R}_{>0}$. By definition of mevol_n , there is an α -embedding $C_*^\alpha \rightarrow D_*$ with

$$\text{lognorm}_\alpha(\partial_{n+1}^D) \leq \text{mevol}_n(\alpha) + \varepsilon.$$

Because $\alpha \prec \beta$, in every weak neighbourhood U of α , there is $\beta' \in U$ such that $\beta' \cong \beta$ (Proposition 15.10). Thus, Lemma 15.32 yields a weak neighbourhood U of α and $\beta' \in U$ with $\beta' \cong \beta$ such that there is a β' -embedding $C_*^{\beta'} \rightarrow \tilde{D}_*$ with

$$\text{lognorm}_{\beta'}(\partial_{n+1}^{\tilde{D}}) \leq \text{lognorm}_\alpha(\partial_{n+1}^D) + \varepsilon.$$

As $\beta' \cong \beta$, this defines a β -embedding $C_*^\beta \rightarrow \tilde{D}_*$. Thus,

$$\begin{aligned} \text{mevol}_n(\beta) &\leq \log \text{norm}_\beta(\partial_{n+1}^{\tilde{D}}) \\ &\leq \log \text{norm}_\alpha(\partial_{n+1}^D) + \varepsilon \\ &\leq \text{mevol}_n(\alpha) + 2 \cdot \varepsilon. \end{aligned}$$

Taking the limit $\varepsilon \rightarrow 0$ yields the claim. \square

16. THE DISINTEGRATION ESTIMATE

We show a basic disintegration estimate for measured embedding dimension and measured embedding volume (Proposition 16.2). This is useful in the context of orbit equivalence and property EMD*. As usual Z denotes \mathbb{Z} or a finite field.

Definition 16.1 (disintegration). Let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a standard action. We write $\text{Prob}(\alpha)$ for the set of all probability measures on the measurable space X that are invariant under the measurable action underlying α . A *disintegration* of α is a map $m: X \rightarrow \text{Prob}(\alpha)$ with the following properties:

- For every measurable subset $A \subset X$, the evaluation map

$$\begin{aligned} X &\rightarrow [0, 1] \\ x &\rightarrow m_x(A) \end{aligned}$$

is measurable and $\mu(A) = \int_X m_x(A) d\mu(x)$.

- For all $x \in X$ and all $\gamma \in \Gamma$, we have $m_{\gamma \cdot x} = m_x$.
- For all $\nu \in \text{Prob}(\alpha)$, the preimage $X_\nu := m^{-1}(\{\nu\})$ is measurable and $\nu(X_\nu) \in \{0, 1\}$.

Such a disintegration of α is an *ergodic decomposition* of α if m_x is ergodic for all $x \in X$ (with respect to the measurable action underlying α).

Ergodic decompositions always exist [Var63, Section 4]. If m is a disintegration of a standard action $\alpha: \Gamma \curvearrowright (X, \mu)$, then for μ -almost every $x \in X$, the underlying action of Γ on X is essentially free with respect to m_x [LS24, Remark 3.7]. We then also write (α, m_x) for the induced standard action $\Gamma \curvearrowright (X, m_x)$.

For convenience, we introduce the following dual of the abbreviation “almost every”: Given a probability space (X, μ) and a property $P: X \rightarrow \text{Bool}$ (where Bool denotes the Booleans), we say that *there μ -exists an $x \in X$ with property P* if there exists a measurable subset $A \subset X$ with the property that $\mu(A) > 0$ and that $P(x)$ holds for every $x \in A$.

Proposition 16.2. *Let $n \in \mathbb{N}$, let Γ be a group of type FP_{n+1} , let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a standard Γ -action, and let $m: X \rightarrow \text{Prob}(\alpha)$ be a disintegration of α . Then:*

- (i) *For every $\varepsilon \in \mathbb{R}_{>0}$, there μ -exists an $x \in X$ with*

$$\text{medim}_n^Z(\alpha, m_x) \leq \text{medim}_n^Z(\alpha) + \varepsilon.$$

- (ii) *For every $\varepsilon \in \mathbb{R}_{>0}$, there μ -exists an $x \in X$ with*

$$\text{mevol}_n(\alpha, m_x) \leq \text{mevol}_n(\alpha) + \varepsilon.$$

To prepare the proof of Proposition 16.2, we introduce the following notation: We write $R := L^\infty(X, \mu, Z) * \Gamma$ and $R(x) := L^\infty(X, m_x, Z) * \Gamma$ whenever $x \in X$. Implicitly, we only speak of those $x \in X$ for which the Γ -action on X is essentially free with respect to m_x ; this is satisfied for μ -almost every $x \in X$. From marked projective R -modules M and R -linear maps $f: M \rightarrow N$, we obtain associated marked projective $R(x)$ -modules $M(x)$ and $R(x)$ -linear maps $f(x): M(x) \rightarrow N(x)$ for μ -almost every $x \in X$. We record basic observations on dimensions and operator norms:

Lemma 16.3. *Let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a standard Γ -action and let $m: X \rightarrow \text{Prob}(\alpha)$ be a disintegration of α . Let M, N be marked projective R -modules and let $f: M \rightarrow N$ be an R -linear map. Then:*

- (i) *We have $\dim_R(M) = \int_X \dim_{R(x)}(M(x)) \, d\mu(x)$. In particular, there μ -exists an $x \in X$ with*

$$\dim_{R(x)}(M(x)) \leq \dim_R(M).$$

- (ii) *For μ -almost every $x \in X$, we have*

$$\|f(x)\|_{R(x)} \leq \|f\|_R.$$

- (iii) *For every $\varepsilon \in \mathbb{R}_{>0}$, there μ -exists an $x \in X$ with*

$$\text{lognorm}_{R(x)}(f(x)) \leq \text{lognorm}_R(f) + \varepsilon.$$

Proof. (i) As the dimension is additive with respect to marked decompositions, it suffices to consider the case that $M = \langle A \rangle_R$ for some measurable subset $A \subset X$. In this case, by definition, we have

$$\begin{aligned} \dim_R(M) &= \dim_R(\langle A \rangle_R) = \mu(A) = \int_X m_x(A) \, d\mu(x) \\ &= \int_X \dim_{R(x)}(\langle A \rangle_{R(x)}) \, d\mu(x) = \int_X \dim_{R(x)}(M(x)) \, d\mu(x). \end{aligned}$$

(ii) This is a consequence of the explicit description of the operator norm (Proposition 2.35) and the following observation: if $U \subset X$ is a measurable subset such that there μ -exists an $x \in X$ with $m_x(U) > 0$, then this already implies that $\mu(U) > 0$.

Indeed, taking into account that

$$\{x \in X \mid m_x(U) > 0\} = \bigcup_{n \in \mathbb{N}} \{x \in X \mid m_x(U) > 1/n\},$$

we see that there exists a $\delta \in \mathbb{R}_{>0}$ such that there μ -exists an $x \in X$ with measure $m_x(U) > \delta$. Let $A := \{x \in X \mid m_x(U) > \delta\}$. Then A is measurable and $\mu(A) > 0$. By construction, we have

$$\mu(U) = \int_X m_x(U) \, d\mu(x) \geq \int_A m_x(U) \, d\mu(x) > \delta \cdot \mu(A) > 0.$$

(iii) As lognorm is defined as an infimum over all marked decompositions and the different branches of $\text{lognorm}'$, it suffices to consider the following situation: Let $M = \bigoplus_{i \in I} M_i \oplus \bigoplus_{j \in J} M'_j$ be a marked decomposition of M , let N_j be marked direct summands of N with $f(M'_j) \subset N_j$ and let

$$\ell := \sum_{i \in I} \dim_R(M_i) \cdot \log_+ \|f|_{M_i}\|_R + \sum_{j \in J} \dim_R(N_j) \cdot \log_+ \|f|_{M'_j}\|_R.$$

Similarly, for $x \in X$, we define $\ell(x)$ over $R(x)$. Using the second and the first part, we obtain

$$\begin{aligned} \int_X \ell(x) \, d\mu(x) &\leq \int_X \left(\sum_{i \in I} \dim_{R(x)}(M_i(x)) \cdot \log_+ \|f|_{M_i}\|_{R(x)} \right. \\ &\quad \left. + \sum_{j \in J} \dim_{R(x)}(N_j(x)) \cdot \log_+ \|f|_{M'_j}\|_{R(x)} \right) d\mu(x) \\ &= \sum_{i \in I} \dim_R(M_i) \cdot \log_+ \|f|_{M_i}\|_R + \sum_{j \in J} \dim_R(N_j) \cdot \log_+ \|f|_{M'_j}\|_R \\ &= \ell. \end{aligned}$$

In particular, there μ -exists an $x \in X$ with $\ell(x) \leq \ell$; therefore,

$$\text{lognorm}_{R(x)}(f(x)) \leq \ell(x) \leq \ell.$$

Considering all marked situations as in the definition of ℓ (and thus of $\text{lognorm}_R(f)$) proves the claim. \square

Proof of Proposition 16.2. Let

$$\begin{array}{ccc} C_* & \xrightarrow{f_*} & D_* \\ \downarrow & & \downarrow \\ Z & \longrightarrow & L^\infty(X, \mu, Z) \end{array}$$

be an α -embedding (over R). Then, for μ -almost every $x \in X$, we obtain a corresponding (α, m_x) -embedding (over $R(x)$) of the following form:

$$\begin{array}{ccc} C_* & \xrightarrow{f_*(x)} & D_*(x) \\ \downarrow & & \downarrow \\ Z & \longrightarrow & L^\infty(X, m_x, Z) \end{array}$$

By Lemma 16.3, there μ -exists an $x \in X$ with $\dim_{R(x)}(D_n(x)) \leq \dim_R(D_n)$. Analogously, for each $\varepsilon \in \mathbb{R}_{>0}$, by Lemma 16.3, there μ -exists an $x \in X$ that satisfies $\text{lognorm}_{R(x)}(\partial_{n+1}^{D(x)}) \leq \text{lognorm}_R(\partial_{n+1}^D) + \varepsilon$.

Therefore, considering all α -embeddings, we obtain the claimed approximative disintegration estimates. \square

As straightforward consequences of the disintegration estimate (Proposition 16.2), we obtain:

Corollary 16.4 (ergodic actions suffice). *Let $n \in \mathbb{N}$, let Γ be a group of type FP_{n+1} , and let α be a standard Γ -action. Moreover, let $\varepsilon \in \mathbb{R}_{>0}$. Then:*

- (i) *There exists an ergodic standard Γ -action β with*

$$\text{medim}_n^Z(\beta) \leq \text{medim}_n^Z(\alpha) + \varepsilon.$$

- (ii) *There exists an ergodic standard Γ -action β with*

$$\text{mevol}_n(\beta) \leq \text{mevol}_n(\alpha) + \varepsilon.$$

Proof. The standard Γ -action α admits an ergodic decomposition [Var63]. Applying the disintegration estimate (Proposition 16.2) to such an ergodic decomposition proves the claim. \square

Corollary 16.5. *Let $n \in \mathbb{N}$, let Γ be a residually finite group of type FP_{n+1} that satisfies EMD^* , and let α be a standard Γ -action. Then*

$$\text{medim}_n^Z(\Gamma \curvearrowright \hat{\Gamma}) \leq \text{medim}_n^Z(\alpha) \quad \text{and} \quad \text{mevol}_n(\Gamma \curvearrowright \hat{\Gamma}) \leq \text{mevol}_n(\alpha).$$

Proof. We only give the proof for mevol ; the proof for medim works in the same way. We write $\gamma: \Gamma \curvearrowright \hat{\Gamma}$ for the profinite completion action. Let $\varepsilon \in \mathbb{R}_{>0}$. By Corollary 16.4, there exists an ergodic standard Γ -action β with $\text{mevol}_n(\beta) \leq \text{mevol}_n(\alpha) + \varepsilon$. Because Γ satisfies EMD^* and β is ergodic, we have $\beta \prec \gamma$. Therefore, the weak containment estimate (Theorem 15.30) gives

$$\text{mevol}_n(\gamma) \leq \text{mevol}_n(\beta) \leq \text{mevol}_n(\alpha) + \varepsilon.$$

Taking $\varepsilon \rightarrow 0$ shows that $\text{mevol}_n(\gamma) \leq \text{mevol}_n(\alpha)$. \square

Corollary 16.6. *Let $n \in \mathbb{N}$, let Γ be a group of type FP_{n+1} , let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a standard Γ -action, and let $\beta: \Gamma \curvearrowright (Y, \nu)$ be the trivial Γ -action on a standard Borel probability space (Y, ν) . Then, the diagonal action $\alpha \times \beta: \Gamma \curvearrowright (X \times Y, \mu \otimes \nu)$ is a standard Γ -action and*

$$\text{medim}_n^Z(\alpha \times \beta) = \text{medim}_n^Z(\alpha) \quad \text{and} \quad \text{mevol}_n(\alpha \times \beta) = \text{mevol}_n(\alpha).$$

Proof. We obtain “ \leq ” from the weak containment estimate (Theorem 15.30) and the fact that $\alpha \prec \alpha \times \beta$. More elementarily, one can also show this directly, by taking the product with (Y, ν) at every step.

For the estimate “ \geq ”, we use the disintegration

$$\begin{aligned} m: X \times Y &\rightarrow \text{Prob}(\alpha \times \beta) \\ (x, y) &\mapsto (A \mapsto \mu(A_y)) \end{aligned}$$

of $\alpha \times \beta$. Here, $A_y \subset X$ denotes the image of $A \cap (X \times \{y\})$ under the canonical bijection $X \times \{y\} \rightarrow X$. For $\mu \otimes \nu$ -almost every $(x, y) \in X \times Y$, the standard action $(\alpha \times \beta, m_{(x,y)})$ is canonically isomorphic in the measured sense to α (through the canonical projection $X \times Y \rightarrow X$).

By the disintegration estimate, for every $\varepsilon \in \mathbb{R}_{>0}$, there $\mu \otimes \nu$ -exists an $(x, y) \in X \times Y$ with

$$\text{mevol}_n(\alpha) = \text{mevol}_n(\alpha \times \beta, m_{(x,y)}) \leq \text{mevol}_n(\alpha \times \beta) + \varepsilon.$$

Taking $\varepsilon \rightarrow 0$ shows that $\text{mevol}_n(\alpha) \leq \text{mevol}_n(\alpha \times \beta)$. The argument for medim_n^Z can be carried out in the same way. \square

More generally, we expect that measured embedding dimension and measured embedding volume of a disintegrated action is equal to the integral of the corresponding measured embedding dimensions/volumes.

17. WORKING OVER THE EQUIVALENCE RELATION RING

Measured embeddings over the equivalence relation ring can be approximated in a controlled way by measured embeddings over the crossed product ring (Corollary 17.1). The compatibility of marked projective dimensions and logarithmic norms with the Gromov–Hausdorff distance (Proposition 3.14, Proposition 6.4) shows that the measured embedding dimension and the measured embedding volume can alternatively be computed via measured embeddings over the equivalence relation ring.

Corollary 17.1. *Let Z denote \mathbb{Z} (with the usual norm) or a finite field (with the trivial norm). Let $n \in \mathbb{N}$, let Γ be a group of type FP_{n+1} , and let α be a standard Γ -action. Let C_* be a free $Z\Gamma$ -resolution of Z that is of finite rank up to degree $n+1$. Moreover, let \mathcal{R} denote the orbit relation of α , let D_* be a marked projective $Z\mathcal{R}$ -complex, and let $f_*: C_* \rightarrow D_*$ be a $Z\Gamma$ -chain map extending the canonical inclusion $Z \hookrightarrow L^\infty(\alpha, Z)$.*

Then there exists a $K \in \mathbb{R}_{>0}$ such that: For every $\delta \in \mathbb{R}_{>0}$, there exists a marked projective $L^\infty(\alpha, Z)\Gamma$ -chain complex \widehat{D}_* and a $Z\Gamma$ -chain map $\widehat{f}_*: C_* \rightarrow \widehat{D}_*$ extending the canonical inclusion $Z \hookrightarrow L^\infty(\alpha, Z)$ with*

$$\begin{aligned} d_{\text{GH}}^K(\text{Ind}_{L^\infty(\alpha, Z)*\Gamma}^{Z\mathcal{R}} \widehat{D}_*, D_*, n) &< \delta \\ \forall_{r \in \{0, \dots, n+1\}} \quad d_{\text{GH}}^K(\text{Ind}_{Z\Gamma}^{Z\mathcal{R}}(\widehat{f}_r), \text{Ind}_{Z\Gamma}^{Z\mathcal{R}}(f_r)) &< \delta. \end{aligned}$$

Proof. This is the special case of Theorem 5.10, where the subalgebra is the ring $Z\mathcal{R}$ and where the subalgebra S is the algebra of all measurable subsets. \square

Proposition 17.2 (small resolutions over the equivalence relation ring). *Let Z denote \mathbb{Z} (with the standard norm) or a finite field (with the trivial norm). Let \mathcal{R} be a measured standard equivalence relation on a standard Borel probability space (X, μ) , let $n \in \mathbb{N}$, and let D_* be a marked projective $Z\mathcal{R}$ -resolution of $L^\infty(\alpha, Z)$ (up to degree $n + 1$). Then: If Γ is a countable group of type FP_{n+1} and if α is standard probability action of Γ on (X, μ) that induces \mathcal{R} , then*

$$\begin{aligned} \text{medim}_n^Z(\alpha) &\leq \dim_{Z\mathcal{R}}(D_n) \\ \text{mevol}_n(\alpha) &\leq \log \text{norm}(\partial_{n+1}^D) \quad \text{if } Z = \mathbb{Z}. \end{aligned}$$

Proof. Let C_* be a free $Z\Gamma$ -resolution of Z of finite type (up to degree $n + 1$). By Corollary 17.1 and Propositions 3.14/6.4, it suffices to find a $Z\Gamma$ -chain map $C_* \rightarrow D_*$ extending the canonical inclusion $Z \hookrightarrow L^\infty(\alpha, Z)$.

Let $\widehat{C}_* := \text{Ind}_{Z\Gamma}^{Z\mathcal{R}}(C_*)$. Then \widehat{C}_* is a marked projective $Z\mathcal{R}$ -complex (augmented over $L^\infty(\alpha, Z)$, up to degree $n + 1$). Because D_* is a $Z\mathcal{R}$ -resolution of $L^\infty(\alpha, Z)$, by the fundamental lemma of homological algebra, there exists a $Z\mathcal{R}$ -chain map $\widehat{f}_*: \widehat{C}_* \rightarrow D_*$ extending $\text{id}_{L^\infty(\alpha, Z)}$. Hence, the composition of \widehat{f}_* with the canonical chain map $C_* \rightarrow \widehat{C}_*$ induced by the inclusion $Z\Gamma \hookrightarrow Z\mathcal{R}$ has the desired properties. \square

Unfortunately, it is not clear whether/how these considerations lead to a meaningful version of measured embedding dimension/volume that is invariant under orbit equivalence. For instance, it is not clear in how many cases $Z\mathcal{R}$ -resolutions (satisfying the implicit finiteness conditions in marked projectivity) as in Proposition 17.2 actually exist. The case of weakly bounded orbit equivalence is accessible and treated in Section 18.

18. WEAK BOUNDED ORBIT EQUIVALENCE

Because L^2 -Betti numbers are compatible with orbit equivalence [Gab02a], the homology growth over \mathbb{Q} shares the same property for residually finite groups of finite type (via the approximation theorem [Lüc94]). It is an open problem to determine how (vanishing of) homology gradients over finite fields or torsion homology growth behaves under orbit equivalence. As a step towards this problem, we show that measured embedding dimension and measured embedding volume are compatible with weak bounded orbit equivalences. In particular, these invariants provide upper bounds for homology growth over finite fields and for torsion homology growth that are compatible with weak bounded orbit equivalences.

Setup 18.1. In this section, let Z denote \mathbb{Z} with the standard norm or a finite field with the trivial norm.

Theorem 18.2 (weak bounded orbit equivalence and medim , mevol). *Let $n \in \mathbb{N}$, let Γ and Λ be groups of type FP_{n+1} , and let α and β be standard actions of Γ and Λ , respectively, that are weakly bounded orbit equivalent of index c . Then, we have*

$$\begin{aligned} \text{medim}_n^Z(\alpha) &= c \cdot \text{medim}_n^Z(\beta), \\ \text{mevol}_n(\alpha) &= c \cdot \text{mevol}_n(\beta). \end{aligned}$$

The theorem will be derived from a corresponding statement on measured embeddings over truncated crossed product rings. At the moment, the case of general orbit equivalence is out of reach, because the equivalence relation rings do not exhibit the same level of exactness and finiteness properties as the crossed product rings.

As a preparation for the proof, we discuss dimensions and norms over truncated crossed product rings (Section 18.1) as well as truncated version of measured

embeddings, measured embedding dimension, and measured embedding volume (Section 18.2).

18.1. Truncated crossed product rings. Let R be a unital ring. If $p \in R$ is idempotent, then pRp is a unital ring. The idempotent p is called *full* if the multiplication homomorphism $Rp \otimes_{pRp} pR \rightarrow R$ is surjective.

If $p \in R$ is a full idempotent, then R and pRp are Morita equivalent through the functors $pR \otimes_R \cdot$ and $Rp \otimes_{pRp} \cdot$. In particular, we can translate freely between projective resolutions over R and projective resolutions over pRp .

For convenience, we prove the following well-known fact that we need repeatedly.

Lemma 18.3. *If p is a full idempotent of R , then the multiplication homomorphism $Rp \otimes_{pRp} pR \rightarrow R$ is bijective.*

Proof. By definition, the multiplication homomorphism m is surjective. Consider a pre-image $\sum_{i \in I} r_i p \otimes p s_i$ of $1 \in R$. That is, $\sum_{i \in I} r_i p s_i = 1$. There is a homomorphism $f: R \rightarrow Rp \otimes_{pRp} pR$ of left R -modules that maps 1 to $\sum_{i \in I} r_i p \otimes p s_i$. We claim that f is the inverse of m . It is obvious that $m \circ f = \text{id}$. The other composition $f \circ m = \text{id}$ follows from:

$$\begin{aligned} f(m(xp \otimes py)) &= f(xpy) \\ &= xpy \cdot \sum_{i \in I} r_i p \otimes p s_i \\ &= \sum_{i \in I} xp(py r_i p) \otimes p s_i \\ &= \sum_{i \in I} xp \otimes p y r_i p s_i \\ &= xp \otimes py \left(\sum_{i \in I} r_i p s_i \right) \\ &= xp \otimes py. \end{aligned} \quad \square$$

In the context of measured embeddings, we need more refined information: We need to preserve marked projectivity (instead of projectivity) and control over the dimensions and norms of maps.

Definition 18.4. Let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a standard action. A subset $A \subset X$ is α -*cofinite* if it is measurable and if there exists a finite set $F \subset \Gamma$ with $F \cdot A = X$ (up to μ -measure zero).

Clearly, cofinite sets in this sense have non-zero measure.

Remark 18.5 (existence of small cofinite subsets). Let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a standard action of an infinite group Γ and let $\varepsilon \in \mathbb{R}_{>0}$. Then, there exists an α -cofinite subset $A \subset X$ with $\mu(A) < \varepsilon$: Indeed, there exists a measurable subset $A' \subset X$ with $\Gamma \cdot A' = X$ and $\mu(A') < \varepsilon/2$ [Lev95, Proposition 1]. We choose a large enough finite subset $F \subset \Gamma$ with $\mu(X \setminus F \cdot A') < \varepsilon/2$. Then $A := A' \cup (X \setminus F \cdot A')$ is α -cofinite and $\mu(A) < \varepsilon$.

Definition 18.6 (marked projectives, dimensions, lognorm over truncated crossed product rings). Let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a standard action and let $A \subset X$ be α -cofinite. We write $R := L^\infty(\alpha, Z) * \Gamma$ and $p := (\chi_A, 1) \in R$.

- If $B \subset A$ is a measurable subset, we write

$$\langle B \rangle_{pRp} := pRp \cdot \chi_B.$$

- A *marked projective pRp -module* is a triple $(M, (B_i)_{i \in I}, \varphi)$, consisting of a pRp -module M , a finite family $(B_i)_{i \in I}$ of measurable subsets of A , and a pRp -isomorphism $\varphi: M \rightarrow \bigoplus_{i \in I} \langle B_i \rangle_{pRp}$.
- The *dimension* of a marked projective pRp -module $(M, (B_i)_{i \in I}, \varphi)$ is given by

$$\dim_{pRp}(M) := \sum_{i \in I} \frac{\mu(B_i)}{\mu(A)} \in \mathbb{R}_{\geq 0}.$$

- If $f: M \rightarrow N$ is a pRp -linear map between marked projective pRp -modules, then we define $\text{lognorm}_{pRp}(f)$ analogously to the non-truncated case.

As in the case of marked projective modules in the non-truncated case, we will usually leave φ implicit.

Proposition 18.7 (truncation and marked projectivity). *Let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a standard action and let $A \subset X$ be α -cofinite. We write $R := L^\infty(\alpha, Z) * \Gamma$ and $p := (\chi_A, 1) \in R$. Then:*

- (i) *The idempotent p is full in R .*
- (ii) *If M is a marked projective R -module, then pM inherits a decomposition as a marked projective pRp -module with*

$$\dim_{pRp}(pM) = \frac{1}{\mu(A)} \cdot \dim_R(M).$$

- (iii) *If $f: M \rightarrow N$ is an R -linear map between marked projective R -modules, then (with respect to a marked decomposition as in the previous item)*

$$\text{lognorm}_{pRp}(pf) \leq \frac{1}{\mu(A)} \cdot \text{lognorm}_R(f).$$

- (iv) *If M is a marked projective pRp -module, then $Rp \otimes_{pRp} M$ inherits a canonical decomposition as a marked projective R -module with*

$$\dim_R(Rp \otimes_{pRp} M) = \mu(A) \cdot \dim_{pRp}(M).$$

- (v) *If $f: M \rightarrow N$ is a pRp -linear map between marked projective pRp -modules, then (with respect to the marked decomposition as in the previous item)*

$$\text{lognorm}_R(\text{id} \otimes_{pRp} f) \leq \mu(A) \cdot \text{lognorm}_{pRp}(f).$$

The marked projective structure on pM depends on certain choices, but these will not be problematic in our situations.

Proof. Because A is α -cofinite there exists a finite set $F \subset \Gamma$ with $F \cdot A = X$ (up to measure 0). Through inductive removal, we obtain a family $(B_\gamma)_{\gamma \in F}$ of measurable subsets of A with (up to measure 0)

$$X = \bigsqcup_{\gamma \in F} \gamma \cdot B_\gamma.$$

- (i) With this decomposition, we can write the unit in R as

$$1 = \sum_{\gamma \in F} (\chi_{\gamma \cdot B_\gamma}, 1) = \sum_{\gamma \in F} (1, \gamma) \cdot (\chi_{B_\gamma}, 1) \cdot (1, \gamma^{-1}) = \sum_{\gamma \in F} (1, \gamma) \cdot p \cdot (\chi_{B_\gamma}, 1) \cdot (1, \gamma^{-1}),$$

which shows that the idempotent p is full in R .

- (ii) We only need to consider the case that M has rank 1. Thus, let $M = \langle B \rangle_R$. Then (up to measure 0)

$$B = \bigsqcup_{\gamma \in F} \gamma \cdot B_\gamma \cap B = \bigsqcup_{\gamma \in F} \gamma \cdot (B_\gamma \cap \gamma^{-1} \cdot B),$$

and we obtain

$$\begin{aligned}
p\langle B \rangle_R &\cong_{pRp} \bigoplus_{\gamma \in F} p \cdot R \cdot (1, \gamma) \cdot (\chi_{B_\gamma \cap \gamma^{-1} \cdot B}, 1) \cdot (1, \gamma^{-1}) \\
&\cong_{pRp} \bigoplus_{\gamma \in F} p \cdot R \cdot (1, \gamma) \cdot (\chi_{B_\gamma \cap \gamma^{-1} \cdot B}, 1) && ((1, \gamma^{-1}) \text{ is a unit in } R) \\
&\cong_{pRp} \bigoplus_{\gamma \in F} p \cdot R \cdot \chi_{B_\gamma \cap \gamma^{-1} \cdot B} && ((1, \gamma) \text{ is a unit in } R) \\
&= \bigoplus_{\gamma \in F} p \cdot R \cdot p \cdot \chi_{B_\gamma \cap \gamma^{-1} \cdot B} && (B_\gamma \cap \gamma^{-1} \cdot B \subset A) \\
&= \bigoplus_{\gamma \in F} \langle B_\gamma \cap \gamma^{-1} \cdot B \rangle_{pRp}.
\end{aligned}$$

This provides a marked projective decomposition of the pRp -module $p\langle B \rangle_R$. Moreover, with respect to this marked projective decomposition, we have

$$\begin{aligned}
\dim_{pRp}(p\langle B \rangle_R) &= \sum_{\gamma \in F} \frac{1}{\mu(A)} \cdot \mu(B_\gamma \cap \gamma^{-1} \cdot B) = \frac{1}{\mu(A)} \cdot \sum_{\gamma \in F} \mu(\gamma \cdot B_\gamma \cap B) \\
&= \frac{1}{\mu(A)} \cdot \mu\left(\bigcup_{\gamma \in F} \gamma \cdot B_\gamma \cap B\right) = \frac{1}{\mu(A)} \cdot \mu(B).
\end{aligned}$$

(iii) This follows from the dimension estimate in the previous part and the definition of the logarithmic norms.

(iv) We only need to consider the case that M has rank 1. Thus, let $M = \langle B \rangle_{pRp}$ with $B \subset A$ measurable. Because p is an idempotent in R and $(\chi_B, 1) \in pRp$, the canonical R -homomorphism

$$Rp \otimes_{pRp} \langle B \rangle_{pRp} \rightarrow \langle B \rangle_R$$

is an isomorphism. In particular, this gives a canonical marked projective structure on $Rp \otimes_{pRp} M$ and

$$\dim_R(Rp \otimes_{pRp} M) = \mu(A) \cdot \dim_{pRp}(M).$$

(v) This follows from the dimension estimate in the previous part and the definition of the logarithmic norms. \square

18.2. Measured embeddings over truncated crossed product rings. Let R be a unital ring, let $P \subset R$ be a set of idempotents. A *marked projective* (R, P) -module is an R -module M , together with a decomposition

$$M \cong_R \bigoplus_{i \in I} Rp_i,$$

where $(p_i)_{i \in I}$ is a finite family in P .

Definition 18.8 (embedding). Let R be a unital ring and let $P \subset R$ be a set of idempotents. Let L be an R -module. An (R, P) -embedding (up to degree n) of L is a triple $((C_*, \zeta), f_*, (D_*, \eta))$, consisting of

- a free R -resolution (C_*, ζ) of L ,
- an (R, P) -marked projective augmented R -chain complex (D_*, η) augmenting L ,
- and an R -chain map $f: C_* \rightarrow D_*$ up to degree $n+1$ extending $\text{id}_L: L \rightarrow L$.

$$\begin{array}{ccc}
C_* & \xrightarrow{f_*} & D_* \\
\downarrow \zeta & & \downarrow \eta \\
L & \xlongequal{\quad} & L
\end{array}$$

Remark 18.9. We call these constellations embeddings because they are similar to embeddings in a derived sense: In Definition 18.8, if $((C_*, \zeta), f_*, (D_*, \eta))$ is an (R, P) -embedding of the R -module L , then the fundamental lemma of homological algebra provides us with an R -chain map $g_*: D_* \rightarrow C_*$ extending id_L that satisfies $g_* \circ f_* \simeq_R \text{id}$.

Remark 18.10 (change of domain resolution). In the situation of Definition 18.8, if $((C_*, \zeta), f_*, (D_*, \eta))$ is an (R, P) -embedding of L and (C'_*, ζ') is a free R -resolution of L , then there also exists an (R, P) -embedding $((C'_*, \zeta'), f'_*, (D_*, \eta))$ with the same target complex D_* (by the fundamental lemma of homological algebra). However, the quantitative properties of the maps f'_* and f_* might be different; in the following discussion, we will only take quantitative aspects of the target complex into account and therefore the choice of the domain resolution of L will be immaterial. The same argument applies if C'_* consists of (finitely generated) projective R -modules instead of (finitely generated) free R -modules.

Definition 18.11 (measured embeddings, truncated case). Let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a standard action, let $R := L^\infty(\alpha, Z) * \Gamma$, let $A \subset X$ be α -cofinite, and let $p := (\chi_A, 1) \in R$. We write $P_A := \{(\chi_B, 1) \mid B \subset A \text{ measurable}\}$.

- An (α, A, Z) -measured embedding (up to degree n) is a (pRp, P_A) -embedding up to degree n of $pL^\infty(\alpha, Z)$ in the sense of Definition 18.8.
- We write $A_n(\alpha, A, Z)$ for the class of all augmented target complexes arising in (α, A, Z) -measured embeddings up to degree n .

In the situation of Definition 18.11, it might be helpful to understand that the pRp -module $pL^\infty(\alpha, Z)$ is canonically isomorphic to $L^\infty(A, Z)$. The pRp -structure on $L^\infty(A, Z)$ is given by

$$p(f, \gamma)p \cdot (\chi_B, 1) = \chi_{A \cap \gamma \cdot B} \cdot f$$

for all $f \in L^\infty(\alpha, Z)$, $\gamma \in \Gamma$, and all measurable subsets $B \subset A$ (Remark 2.9).

Definition 18.12 (medim and mevol, truncated case). Let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a standard action, let $R := L^\infty(\alpha, Z) * \Gamma$, let $A \subset X$ be α -cofinite, and let $p := (\chi_A, 1) \in R$. Let $n \in \mathbb{N}$ and let Γ be of type FP_{n+1} . We then set

$$\begin{aligned} \text{medim}_n^Z(\alpha, A) &:= \inf_{D_* \in A_n(\alpha, A, Z)} \dim_{pRp}(D_n) \\ \text{mevol}_n(\alpha, A) &:= \inf_{D_* \in A_n(\alpha, A, Z)} \text{lognorm}_{pRp}(\partial_{n+1}^D) \quad \text{if } Z = \mathbb{Z}. \end{aligned}$$

Remark 18.13 (starting from resolutions over the group ring). Let $n \in \mathbb{N}$, let Γ be of type FP_∞ , and let (C_*, ζ) be a free $Z\Gamma$ -resolution of the trivial $Z\Gamma$ -module Z . Let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a standard action and let $R := L^\infty(\alpha, Z) * \Gamma$.

- (1) Then $(R \otimes_{Z\Gamma} C_*, \text{id} \otimes_{Z\Gamma} \zeta)$ is a free R -resolution of the R -module $L^\infty(\alpha, Z) \cong_R R \otimes_{Z\Gamma} Z$ (with the canonical action; Remark 2.9), because R is flat over $Z\Gamma$ (Proposition 2.10).
- (2) If $A \subset X$ is α -cofinite and $p := (\chi_A, 1) \in R$ is the associated full idempotent, then $(pR \otimes_{Z\Gamma} C_*, \text{id} \otimes_{Z\Gamma} \zeta)$ is a projective pRp -resolution of the pRp -module $pR \otimes_{Z\Gamma} Z \cong_{pRp} pL^\infty(\alpha, Z)$ (with the canonical action).

Therefore, we may start from resolutions over the group ring $Z\Gamma$ when computing or estimating measured embedding dimensions/volumes (Remark 18.10).

In particular, for $A = X$, the notions of embeddings and measured embedding dimension/volume coincide with the ones from Section 1.1.

A similar remark applies when Γ is of type FP_{n+1} .

18.3. Weak bounded orbit equivalence. We recall the notion of weak bounded orbit equivalence, which is a more restrictive type of stable orbit equivalence.

Definition 18.14 (weak bounded orbit equivalence). Standard actions $\alpha: \Gamma \curvearrowright (X, \mu)$ and $\beta: \Lambda \curvearrowright (Y, \nu)$ are *weakly bounded orbit equivalent of index $c \in \mathbb{R}_{>0}$* if there exists an α -cofinite set $A \subset X$, a β -cofinite set $B \subset Y$, and a measurable isomorphism $\varphi: A \rightarrow B$ with the following properties:

- We have

$$\frac{1}{\mu(A)} \cdot \varphi_* \mu|_A = \frac{1}{\nu(B)} \cdot \nu|_B \quad \text{and} \quad c = \frac{\mu(A)}{\nu(B)}.$$

- For every $\gamma \in \Gamma$, there is a finite set $F(\gamma) \subset \Lambda$ such that for μ -almost every $x \in A \cap \gamma^{-1} \cdot A$, we have $\varphi(\gamma \cdot x) \in F(\gamma) \cdot \varphi(x)$.
- For every $\lambda \in \Lambda$, there is a finite set $E(\lambda) \subset \Gamma$ such that for ν -almost every $y \in B \cap \lambda^{-1} \cdot B$, we have $\varphi^{-1}(\lambda \cdot y) \in E(\lambda) \cdot \varphi^{-1}(y)$.

Example 18.15 (uniform lattices). Let G be a locally compact second countable Hausdorff topological group and let $\Gamma, \Lambda \subset G$ be uniform lattices in G . Then the standard actions $\Gamma \curvearrowright G/\Lambda$ and $\Lambda \curvearrowright G/\Gamma$ (with respect to the normalised Haar measure) are weakly boundedly orbit equivalent [Sau03, Example 2.31]. For instance, fundamental groups of closed Riemannian manifolds are uniform lattices in the isometry group of the Riemannian universal covering [Sau03, Example 2.31 and Theorem 2.36].

Example 18.16 (amenable groups). Let Γ and Λ be infinite finitely generated amenable groups. Then Γ and Λ admit weakly boundedly orbit equivalent standard actions if and only if Γ and Λ are quasi-isometric [Sau03, Lemma 2.25 and Theorem 2.38].

On the other hand, Gaboriau describes examples of pairs of groups $\Gamma \times F_n$ and $\Gamma \times F_m$ for $n \neq m$ that are non-amenable and quasi-isometric but not weakly orbit equivalent [Gab02b, Section 2.3].

Example 18.17 (weak bounded orbit equivalence vs. weak orbit equivalence). Let Γ be a cocompact lattice in $\mathrm{SL}_n(\mathbb{R})$. Then Γ and $\mathrm{SL}_n(\mathbb{Z})$ are measure equivalent, hence weakly orbit equivalent [Sau03, Theorem 2.33]. However, Γ and $\mathrm{SL}_n(\mathbb{Z})$ are not quasi-isometric and therefore not weakly boundedly orbit equivalent [Sau03, Lemma 2.25]. This can be deduced from the invariance of the cohomological dimension under quasi-isometry [Sau06, Corollary 1.2].

Remark 18.18 (invariants under weak (bounded) orbit equivalence). The following group invariants are invariants – or invariants up to scaling by the index – under weak bounded orbit equivalence:

- the Novikov–Shubin invariants among groups of type FP_∞ [Sau06];
- the L^2 -torsion among groups of type F and actions that satisfy the measure-theoretic determinant conjecture [LSW10];
- the integral foliated simplicial volume among fundamental groups of closed aspherical manifolds [LP16].

The L^2 -torsion and the integral foliated simplicial volume might be invariants of weak orbit equivalence for all we know. The Novikov–Shubin invariants are definitely not as the example of \mathbb{Z} and \mathbb{Z}^2 shows. Finally, L^2 -Betti numbers are invariants up to scaling by the index of weak orbit equivalence by Gaboriau’s theorem [Gab02a].

18.4. Proof of Theorem 18.2. The proof of Theorem 18.2 consists of two components:

- (1) We relate measured embeddings over the full crossed product ring to measured embeddings over truncated crossed product rings (Proposition 18.19).
- (2) We use the given weak bounded orbit equivalence to compare measured embeddings over the corresponding truncated crossed product rings of the two standard actions (Proposition 18.20).

Proposition 18.19. *Let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a standard action and let $A \subset X$ be α -cofinite. Let $n \in \mathbb{N}$ and let Γ be of type FP_{n+1} . Then:*

$$\begin{aligned} \text{medim}_n^Z(\alpha) &= \mu(A) \cdot \text{medim}_n^Z(\alpha, A) \\ \text{mevol}_n(\alpha) &= \mu(A) \cdot \text{mevol}_n(\alpha, A) \quad \text{if } Z = \mathbb{Z}. \end{aligned}$$

More precisely (where $R := L^\infty(\alpha, Z) * \Gamma$ and $p := (\chi_A, 1)$):

- (i) If $(D_*, \eta) \in A_n(\alpha, Z)$, then $(pD_*, p\eta) \in A_n(\alpha, A, Z)$.
- (ii) If $(D_*, \eta) \in A_n(\alpha, A, Z)$, then $(Rp \otimes_{pRp} D_*, \text{id} \otimes_{pRp} \eta) \in A_n(\alpha, Z)$.

Proof. In view of the definition of medim and mevol in terms of measured embeddings and the compatibility of $pR \otimes_R \cdot$ and $Rp \otimes_{pRp} \cdot$ with dimensions and lognorm (Proposition 18.7), it suffices to show the two claims on measured embeddings.

We abbreviate $L := L^\infty(\alpha, Z)$.

(i) Let $(D_*, \eta) \in A_n(\alpha, Z)$. We choose a free R -resolution (C_*, ζ) of L of finite type. Then there exists an R -chain map $f_*: C_* \rightarrow D_*$ extending id_L (Remark 18.10).

Applying the functor $pR \otimes_R \cdot$, we obtain a pRp -chain map $pf_*: pC_* \rightarrow pD_*$ extending id_{pL} . The complexes pC_* and pD_* consist of marked projective pRp -modules (Proposition 18.7). Moreover, $pR \otimes_R \cdot$ is exact, because pR is projective (as p is idempotent); hence, $(pC_*, p\zeta)$ is a projective pRp -resolution of pL and $(pD_*, p\eta)$ augments to pL .

Therefore, pf_* witnesses that there also exists an (α, A, Z) -measured embedding with target $(pD_*, p\eta)$ (Remark 18.10); i.e., $(pD_*, p\eta) \in A_n(\alpha, A, Z)$.

(ii) Conversely, let $(D_*, \eta) \in A_n(\alpha, A, Z)$. Let (C_*, ζ) be a free R -resolution of L . Because pR is projective over R (whence flat), the induced complex $(pC_*, p\zeta)$ is a projective pRp -resolution of pL . Hence, there exists a pRp -chain map $f_*: pC_* \rightarrow D_*$ extending id_{pL} .

Applying the functor $Rp \otimes_{pRp} \cdot$ and the canonical natural isomorphism $Rp \otimes_{pRp} pR \otimes_R \cdot \cong \text{id}$ from Lemma 18.3, we thus obtain an R -chain map $\text{id} \otimes_{pRp} f_*: C_* \cong_R Rp \otimes_{pRp} pC_* \rightarrow Rp \otimes_{pRp} D_*$ extending the identity on $L \cong_R Rp \otimes_{pRp} pL$.

Moreover, $Rp \otimes_{pRp} D_*$ consists of marked projective pRp -modules (Proposition 18.7) and $(Rp \otimes_{pRp} D_*, \text{id} \otimes_{pRp} \eta)$ augments to $L \cong_R Rp \otimes_{pRp} pL$ (by right-exactness of $Rp \otimes_{pRp} \cdot$).

Therefore, $\text{id} \otimes_{pRp} f_*$ witnesses that there also exists an (α, Z) -measured embedding with target $(Rp \otimes_{pRp} D_*, \text{id} \otimes_{pRp} \eta)$ (Remark 18.10); i.e., $(Rp \otimes_{pRp} D_*, \text{id} \otimes_{pRp} \eta)$ lies in $A_n(\alpha, Z)$. \square

Proposition 18.20. *Let $n \in \mathbb{N}$ and let Γ, Λ be groups of type FP_{n+1} . Suppose that there exist standard actions $\alpha: \Gamma \curvearrowright (X, \mu)$ and $\beta: \Lambda \curvearrowright (Y, \nu)$ that admit a weak bounded orbit equivalence $\varphi: A \rightarrow B$. Then:*

$$\begin{aligned} \text{medim}_n^Z(\alpha, A) &= \text{medim}_n^Z(\beta, B) \\ \text{mevol}_n(\alpha, A) &= \text{mevol}_n(\beta, B) \quad \text{if } Z = \mathbb{Z}. \end{aligned}$$

More precisely: The maps φ and $\psi := \varphi^{-1}$ induce mutually inverse isomorphisms

$$\varphi^*: (\chi_B, 1) \cdot (L^\infty(\beta, Z) * \Lambda) \cdot (\chi_B, 1) \leftrightarrow (\chi_A, 1) \cdot (L^\infty(\alpha, Z) * \Gamma) \cdot (\chi_A, 1) : \psi^*$$

of unital rings, which allow to pull back and push forward module structures, such that:

- (i) If $(D_*, \eta) \in A_n(\alpha, A, Z)$, then $(\varphi^* D_*, \varphi^* \eta) \in A_n(\beta, B, Z)$.
- (ii) If $(D_*, \eta) \in A_n(\beta, B, Z)$, then $(\psi^* D_*, \psi^* \eta) \in A_n(\alpha, A, Z)$.

Proof. We abbreviate $R := L^\infty(\alpha, Z) * \Gamma$, $p := (\chi_A, 1) \in R$, $S := L^\infty(\beta, Z) * \Lambda$, and $q := (\chi_B, 1)$. For $\lambda \in \Lambda$, let $E(\lambda) \subset \Gamma$ be a finite set for φ as in Definition 18.14. Then

$$\begin{aligned} \varphi^* : qSq &\rightarrow pRp \\ q \cdot (g, \lambda) \cdot q &\mapsto \sum_{\gamma \in E(\lambda)} p \cdot (g \circ \varphi|_{A_{\gamma, \lambda}}, \gamma) \cdot p \end{aligned}$$

with $A_{\gamma, \lambda} := \{x \in A \cap \gamma^{-1} \cdot A \mid \varphi(\gamma \cdot x) = \lambda \cdot \varphi(x)\}$ gives a well-defined, unital ring homomorphism. The corresponding map ψ^* for ψ witnesses that φ^* is an isomorphism. To see this one passes to the (restricted) equivalence relation rings. Let $R_{\alpha, A} := \{(\gamma \cdot x, x) \mid x \in A, \gamma \cdot x \in A, \gamma \in \Gamma\}$ be the equivalence relation ring of the orbit equivalence relation of α restricted to $A \times A$. Similarly, we define $\mathcal{R}_{\beta, B}$. Clearly, φ induces a measure preserving isomorphism $\mathcal{R}_{\alpha, A} \cong \mathcal{R}_{\beta, B}$ of equivalence relations and thus a ring isomorphism $Z\mathcal{R}_{\beta, B} \xrightarrow{\cong} Z\mathcal{R}_{\alpha, A}$, which restricts to φ^* in the following way. Consider the following commutative square of ring homomorphisms.

$$\begin{array}{ccc} qSq & \xrightarrow{\varphi^*} & pRp \\ \downarrow & & \downarrow \\ Z\mathcal{R}_{\beta, B} & \xrightarrow{\cong} & Z\mathcal{R}_{\alpha, A} \end{array}$$

The left vertical map maps $q(g, \lambda)q = (qg(\lambda \cdot q), \lambda)$ to the function $f : \mathcal{R}_{\beta, B} \rightarrow Z$ such that

$$f(\lambda' \cdot y, y) = \begin{cases} q(\lambda \cdot y)g(\lambda \cdot y)q(y) & \text{if } \lambda' = \lambda; \\ 0 & \text{otherwise.} \end{cases}$$

The left vertical map is a ring homomorphism (cf. Subsection 2.1). The right vertical map is defined similarly. We verify that the diagram commutes. The element f is mapped under the lower horizontal map (induced by φ) to $f' \in Z\mathcal{R}_{\alpha, A}$ with

$$f'(\gamma \cdot x, x) = \begin{cases} q(\lambda \cdot \varphi(x))g(\lambda \cdot \varphi(x))q(\varphi(x)) & \text{if } \varphi(\gamma \cdot x) = \lambda \cdot \varphi(x); \\ 0 & \text{otherwise.} \end{cases}$$

Hence f' can be expressed as a sum of functions f'_γ ranging over $\gamma \in E(\lambda)$ such that f'_γ is supported on $A_{\gamma, \lambda}$. The function f'_γ is the image of the γ -summand in the formula for $\varphi^*(q \cdot (g, \lambda) \cdot q)$. So the square commutes.

Pulling back the module structure along φ^* defines an exact functor (even an equivalence) from the category of pRp -modules to the category of qSq -modules. By construction, $\varphi^*\langle A \rangle_{pRp} \cong_{qSq} \langle B \rangle_{qSq}$; therefore, for all measurable subsets $\tilde{A} \subset A$, we obtain

$$\varphi^*\langle \tilde{A} \rangle_{pRp} \cong_{qSq} \langle \varphi(\tilde{A}) \rangle_{qSq}$$

and

$$\dim_{pRp}(\langle \tilde{A} \rangle_{pRp}) = \frac{\mu(\tilde{A})}{\mu(A)} = \frac{\nu(\varphi(\tilde{A}))}{\nu(B)} = \dim_{qSq}(\langle \varphi(\tilde{A}) \rangle_{qSq}).$$

Hence, this functor canonically turns marked projective pRp -modules into marked projective qSq -modules with the same dimensions; similarly, the logarithmic norm of homomorphisms between marked projective modules is preserved. Furthermore, we have $\varphi^*pL^\infty(\alpha, Z) \cong_{qSq} qL^\infty(\beta, Z)$.

Analogous statements hold for ψ^* .

This shows the claims (i) and (ii). In view of the compatibility with the dimensions and the logarithmic norms, also the statements on measured embedding dimension and measured embedding volume follow. \square

Proof of Theorem 18.2. Let $\alpha: \Gamma \curvearrowright (X, \mu)$ and $\beta: \Lambda \curvearrowright (Y, \nu)$ be the given actions and let $A \subset X$ and $B \subset Y$ be cofinite sets for which there exists a weak bounded orbit equivalence $\varphi: A \rightarrow B$. Furthermore, let $n \in \mathbb{N}$. We then obtain

$$\begin{aligned} \text{mevol}_n(\alpha) &= \mu(A) \cdot \text{mevol}_n(\alpha, A) && \text{(Proposition 18.19)} \\ &= \mu(A) \cdot \text{mevol}_n(\beta, B) && \text{(Proposition 18.20)} \\ &= \frac{\mu(A)}{\nu(B)} \cdot \nu(B) \cdot \text{mevol}_n(\beta, B) \\ &= \frac{\mu(A)}{\nu(B)} \cdot \text{mevol}_n(\beta) && \text{(Proposition 18.19).} \end{aligned}$$

By definition, the first quotient is the index of φ .

The proof for medim_n^Z works in the same way. \square

18.5. Example: Hyperbolic 3-manifolds. Let M be a hyperbolic 3-manifold of finite volume, let $\Gamma := \pi_1(M)$, and let Γ_* be a residual chain in Γ . Conjecturally, it is expected that $\widehat{t}_1(\Gamma, \Gamma_*) = \text{vol}(M)/6\pi$ holds [BV13], where $\text{vol}(M)$ denotes the hyperbolic volume of M . L   [L  18, Theorem 1.1] proved that indeed

$$\widehat{t}_1(\Gamma, \Gamma_*) \leq \frac{\text{vol}(M)}{6 \cdot \pi}$$

holds (even more generally in the context of homology torsion growth of orientable, irreducible, compact 3-manifolds with empty or toroidal boundary).

In the following, we reproduce the existence of a volume-linear upper bound for torsion homology growth of closed hyperbolic 3-manifolds via the dynamical approach. We follow the strategy for the dynamical computation of stable integral simplicial volume of 3-manifolds [LP16, FLMQ21].

Theorem 18.21. *Let M and N be oriented closed connected hyperbolic 3-manifolds and let $\Gamma := \pi_1(M)$, $\Lambda := \pi_1(N)$. Then*

$$\begin{aligned} \frac{\text{mevol}_1(\Gamma \curvearrowright \widehat{\Gamma})}{\text{vol}(M)} &= \frac{\text{mevol}_1(\Lambda \curvearrowright \widehat{\Lambda})}{\text{vol}(N)} \\ \frac{\text{medim}_1^Z(\Gamma \curvearrowright \widehat{\Gamma})}{\text{vol}(M)} &= \frac{\text{medim}_1^Z(\Lambda \curvearrowright \widehat{\Lambda})}{\text{vol}(N)}. \end{aligned}$$

Proof. We denote the profinite completion standard actions by $\alpha_\Gamma: \Gamma \curvearrowright \widehat{\Gamma}$ and $\alpha_\Lambda: \Lambda \curvearrowright \widehat{\Lambda}$, respectively. Let $\bar{\alpha}_\Lambda: \Gamma \curvearrowright \widehat{\Lambda}$ denote the trivial action of Γ on $\widehat{\Lambda}$, which is not essentially free.

We compare Γ and Λ through their actions on hyperbolic 3-space: Let $\beta_\Gamma: \Gamma \curvearrowright G/\Lambda$ and $\beta_\Lambda: \Lambda \curvearrowright G/\Gamma$ be the canonical standard actions on $G := \text{Isom}^+(\mathbb{H}^3)$ associated with the hyperbolic 3-manifolds M and N . These actions are mixing [BM00, Theorem III.2.1] and weakly bounded orbit equivalent with index $\text{vol}(M)/\text{vol}(N)$ (Example 18.15).

As the group Γ satisfies EMD* (Example 15.8) we obtain

$$\text{mevol}_1(\alpha_\Gamma) \leq \text{mevol}_1(\bar{\alpha}_\Lambda \times \beta_\Gamma) \quad (\text{Corollary 16.5})$$

for the diagonal action $\bar{\alpha}_\Lambda \times \beta_\Gamma$ of Γ on $\widehat{\Lambda} \times G/\Lambda$. The standard actions $\bar{\alpha}_\Lambda \times \beta_\Gamma$ and $\alpha_\Lambda \times \beta_\Lambda$ are weakly bounded orbit equivalent with index $\text{vol}(M)/\text{vol}(N)$. This follows from that fact that the index of the measure coupling $\widehat{\Lambda} \times G$ that gives rise

to this weak bounded orbit equivalence has index $\text{vol}(M)/\text{vol}(N)$ and from [Fur99, Lemma 3.2]. Therefore, Theorem 18.2 shows that

$$\begin{aligned} \text{mevol}_1(\alpha_\Gamma) &\leq \text{mevol}_1(\bar{\alpha}_\Lambda \times \beta_\Gamma) \\ &\leq \frac{\text{vol}(M)}{\text{vol}(N)} \cdot \text{mevol}_1(\alpha_\Lambda \times \beta_\Lambda) && \text{(Theorem 18.2)} \\ &\leq \frac{\text{vol}(M)}{\text{vol}(N)} \cdot \text{mevol}_1(\alpha_\Lambda). && \text{(Lemma 18.22)} \end{aligned}$$

Symmetrically, we obtain $\text{mevol}_1(\alpha_\Lambda) \leq \text{vol}(N)/\text{vol}(M) \cdot \text{mevol}_1(\alpha_\Gamma)$. The argument for $\text{medim}_1^{\mathbb{Z}}$ works in the same way. \square

Lemma 18.22. *Let Γ be a countable group, let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a standard action and let $\beta: \Gamma \curvearrowright (Y, \nu)$ be a probability measure preserving action (not necessarily essentially free) on a standard Borel probability space. We write $\alpha \times \beta: \Gamma \curvearrowright (X \times Y, \mu \otimes \nu)$ for the associated diagonal standard action. Let $n \in \mathbb{N}$ and let Γ be of type FP_{n+1} . Then*

$$\begin{aligned} \text{medim}_n^{\mathbb{Z}}(\alpha \times \beta) &\leq \text{medim}_n^{\mathbb{Z}}(\alpha) \\ \text{mevol}_n(\alpha \times \beta) &\leq \text{mevol}_n(\alpha). \end{aligned}$$

Proof. We have $\alpha \prec \alpha \times \beta$ and thus the estimates are a consequence of monotonicity under weak containment (Theorem 15.30). Alternatively, one could also prove these estimates by straightforward direct constructions of measured embeddings. \square

Corollary 18.23. *There exists a constant $K \in \mathbb{R}_{>0}$ with the following property: For all oriented closed connected hyperbolic 3-manifolds M , the fundamental group $\Gamma := \pi_1(M)$ satisfies*

$$\hat{t}_1(\Gamma) \leq K \cdot \text{vol}(M).$$

Proof. This is a direct consequence of Theorem 18.21 and the dynamical upper bound for logarithmic torsion growth (Theorem 8.1). \square

In order to obtain the constant $\text{vol}/6\pi$, one would need a single calculation of the measured embedding volume for some closed hyperbolic 3-manifold. Comparison with the simplicial volume estimate gives constant $6 \cdot \log 3/v_3$ (Example 20.4), which is not optimal.

19. THE COST ESTIMATE

The measured embedding dimension in degree 1 is compatible with cost, a dynamical version of the rank of groups [Gab00, KM04]. Because of Lemma 9.2 we state the result only over the integers.

Theorem 19.1. *Let Γ be an infinite group of type FP_2 and let α be a standard action of Γ . Then*

$$\text{medim}_1^{\mathbb{Z}}(\alpha) \leq \text{cost}(\alpha) - 1.$$

In combination with Theorem 8.6, we obtain the sandwich

$$b_1^{(2)}(\Gamma) \leq \text{medim}_1^{\mathbb{Z}}(\alpha) \leq \text{cost}(\alpha) - 1.$$

Gaboriau asked whether the two outer terms are equal for infinite groups [Gab02a, p. 129]; it is thus natural to raise the following question:

Question 19.2. *Let Γ be an infinite group of type FP_2 and let α be a standard action of Γ . Do we always have $\text{medim}_1^{\mathbb{Z}}(\alpha) = \text{cost}(\alpha) - 1$?*

The basic idea to prove the cost estimate (Theorem 19.1) is to construct the low degrees of a resolution over $L^\infty(\alpha, \mathbb{Z}) * \Gamma$ from graphings of the orbit relation of α . The fundamental theorem of homological algebra then provides α -embeddings with such a target. In order to achieve the additive correction term -1 , we do this in the slightly more general case of restricted actions (Proposition 19.3). The scaling properties of cost and measured embedding dimension then give the claimed upper bound from Theorem 19.1. In principle, this method works for all countable groups (not only those of type FP_2), but our setting is not optimised for that level of generality.

Proposition 19.3. *Let Γ be an infinite group of type FP_2 , let $\alpha: \Gamma \curvearrowright (X, \mu)$ be a standard action of Γ , and let $A \subset X$ be α -cofinite. Then*

$$\text{medim}_1^{\mathbb{Z}}(\alpha, A) \leq \text{cost}\left(\mathcal{R}_\alpha|_A, \frac{1}{\mu(A)} \cdot \mu|_A\right).$$

Here, $\mathcal{R}_\alpha|_A := (A \times A) \cap \{(x, \gamma \cdot x) \mid x \in X, \gamma \in \Gamma\}$ denotes the restriction of the orbit relation of α to A .

Proof. We follow the proof of $b_1^{(2)}(\Gamma) \leq \text{cost}(\alpha) - 1$ via resolutions [Löh20b, Chapter 4.3.2]. Let $R := L^\infty(\alpha, \mathbb{Z}) * \Gamma$ and let $p := (\chi_A, 1) \in R$. Building on the analogy of graphings of equivalence relations as dynamical versions of generating sets of groups, we construct the low degrees of resolutions of $pL^\infty(\alpha, \mathbb{Z})$ from graphings of $\mathcal{R}_\alpha|_A$: Let Φ be a graphing of $\mathcal{R}_\alpha|_A$; without loss of generality we may assume that Φ is given by a family $\Phi = (\varphi_i := \gamma_i \cdot - : A_i \rightarrow B_i)_{i \in I}$ of translation maps, where I is countable and where for each $i \in I$, we have $\gamma_i \in \Gamma$ and measurable subsets $A_i \subset A$ with $B_i := \gamma_i \cdot A_i \subset A$.

We define

$$D_0 := pR \quad \text{and} \quad P_1 := \bigoplus_{i \in I} \langle A_i \rangle_{pRp}$$

as well as

$$\begin{aligned} \partial_0^D : D_0 &\rightarrow pL^\infty(\alpha, \mathbb{Z}) \\ p \cdot (f, \gamma) &\mapsto p \cdot f \\ \partial_1^P : P_1 &\rightarrow D_0 \\ \chi_{A_i} \cdot e_i &\mapsto (\chi_{A_i}, 1) \cdot ((1, 1) - (1, \gamma_i)). \end{aligned}$$

By construction, ∂_0^P is surjective and $\partial_0^D \circ \partial_1^P = 0$. However, in general, we may not have that $\text{im } \partial_1^P$ reaches all of $\ker \partial_0^D$. Therefore, we introduce the following correction term:

Let $\varepsilon \in \mathbb{R}_{>0}$ and let $(g_k)_{k \in \mathbb{N}}$ be an enumeration of Γ . For $k, n \in \mathbb{N}$, we set

$$A(k, n) := \left\{ x \in A \mid \exists_{i_1, \dots, i_n \in I} \exists_{\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}} g_k \cdot x = \varphi_{i_n}^{\varepsilon_n} \circ \dots \circ \varphi_{i_1}^{\varepsilon_1}(x) \right\}.$$

Each $A(k, n)$ is a measurable subset of A . Because Φ is a graphing of $\mathcal{R}_\alpha|_A$, we obtain for all $k \in \mathbb{N}$ that $\bigcup_{n \in \mathbb{N}} A(k, n) = A$. Hence, for each $k \in \mathbb{N}$, there is an $n_k \in \mathbb{N}$ such that

$$C_k := A \setminus \bigcup_{n=0}^{n_k} A(k, n)$$

satisfies $\mu(C_k) \leq 1/2^{k+1} \cdot \varepsilon \cdot \mu(A)$. We then set

$$E_1 := \bigoplus_{k \in \mathbb{N}} \langle C_k \rangle_{pRp}$$

and

$$\begin{aligned}\partial_1^E: E_1 &\rightarrow D_0 \\ \chi_{C_k} \cdot e_k &\mapsto (\chi_{C_k}, 1) \cdot ((1, 1) - (1, g_k)).\end{aligned}$$

Finally, we define

$$D_1 := P_1 \oplus E_1 \quad \text{and} \quad \partial_1^D := \partial_1^P \oplus \partial_1^E: D_1 \rightarrow D_0.$$

By construction $\partial_0^D \circ \partial_1^D = 0$. Moreover, the correction term ensures that $\ker \partial_0^D = \text{im } \partial_1^D$; indeed the inclusion $\ker \partial_0^D \subset \text{im } \partial_1^D$ can be shown by a straightforward adaptation of the case $A = X$ [Löh20b, Lemma 4.3.11] via an inductive argument.

The modules D_0 and D_1 are projective. We may extend the low-degree sequence

$$D_1 \xrightarrow{\partial_1^D} D_0 \xrightarrow{\partial_0^D} pL^\infty(\alpha, \mathbb{Z})$$

to a pRp -resolution D_* of $pL^\infty(\alpha, \mathbb{Z})$ that consists of free pRp -modules in degrees greater than or equal to 2. Let C_* be a free R -resolution of $L^\infty(\alpha, \mathbb{Z})$ that is of finite type (in degrees ≤ 2). By the fundamental theorem of homological algebra, there exists a pRp -chain map $f_*: pC_* \rightarrow D_*$ extending the identity on $pL^\infty(\alpha, \mathbb{Z})$. Because pC_* is finitely generated over pRp in degrees ≤ 2 , the images of f_1 and f_2 touch only finitely many of the marked summands in D_1 and D_2 , respectively. Therefore, we can find an (α, A, Z) -embedding to a marked projective target complex \widehat{D}_* with

$$\begin{aligned}\dim_{pRp}(\widehat{D}_1) &\leq \dim_{pRp}(D_1) = \sum_{i \in I} \frac{1}{\mu(A)} \cdot \mu|_A(A_i) + \sum_{k \in \mathbb{N}} \frac{1}{\mu(A)} \cdot \mu|_A(C_k) \\ &\leq \sum_{i \in I} \frac{1}{\mu(A)} \cdot \mu(A_i) + \sum_{k \in \mathbb{N}} \frac{1}{\mu(A)} \cdot \frac{1}{2^{k+1}} \cdot \varepsilon \cdot \mu(A) \\ &= \text{cost}\left(\Phi, \frac{1}{\mu(A)} \cdot \mu|_A\right) + \varepsilon.\end{aligned}$$

Taking $\varepsilon \rightarrow 0$ and then taking the infimum over all graphings Φ of $\mathcal{R}_\alpha|_A$ shows that

$$\text{medim}_1^{\mathbb{Z}}(\alpha, A) \leq \text{cost}\left(\mathcal{R}_\alpha|_A, \frac{1}{\mu(A)} \cdot \mu|_A\right),$$

as claimed. \square

Proof of Theorem 19.1. Let $\varepsilon \in \mathbb{R}_{>0}$. Because Γ is infinite, there exists an α -cofinite subset $A \subset X$ with $\mu(A) < \varepsilon$ (Remark 18.5). Combining the cost estimate for $\mathcal{R}_\alpha|_A$ and the scaling properties of cost [KM04, Theorem 21.1] and $\text{medim}_1^{\mathbb{Z}}$ (Proposition 18.19), we obtain

$$\begin{aligned}\text{medim}_1^{\mathbb{Z}}(\alpha) &= \mu(A) \cdot \text{medim}_1^{\mathbb{Z}}(\alpha, A) && \text{(Proposition 18.19)} \\ &\leq \mu(A) \cdot \text{cost}\left(\mathcal{R}_\alpha|_A, \frac{1}{\mu(A)} \cdot \mu|_A\right) && \text{(Proposition 19.3)} \\ &= \text{cost}(\mathcal{R}_\alpha|_A, \mu|_A) \\ &= \text{cost}(\alpha) - \mu(X \setminus A) && \text{(scaling of cost)} \\ &= \text{cost}(\alpha) - 1 + \mu(A) \\ &\leq \text{cost}(\alpha) - 1 + \varepsilon.\end{aligned}$$

Taking $\varepsilon \rightarrow 0$ gives the claimed estimate. \square

In the proof of Theorem 19.1, we do not obtain any control on ∂_2^D and thus no upper estimate for mevol_1 in terms of cost. This is compatible with the expectation that no such upper bound for mevol_1 (and whence for \widehat{t}_1) should exist, as predicted

by the conjectures on logarithmic torsion homology growth (in degree 1) of closed hyperbolic 3-manifolds [BV13, Lüc13, Lê18] and the computation of cost of their fundamental groups [Ago14, Theorem 8.5].

Example 19.4. If Γ is a lattice in a higher rank semisimple real Lie group, or a lattice in a product of at least two automorphism groups of trees, then Γ has fixed price 1 [FMW, Theorem D]. Hence, Theorem 19.1 shows that $\text{medim}_1^{\mathbb{Z}}(\alpha) = 0$ for every standard action α of Γ . Examples of such groups include, e.g., Burger–Mozes groups [FMW, Corollary 1.1].

20. THE SIMPLICIAL VOLUME ESTIMATE

The stable integral simplicial volume of closed manifolds gives upper bounds on logarithmic torsion homology growth and Betti number growth [Sau16]. Dynamically, the stable integral simplicial volume can be expressed as integral foliated simplicial volume of the profinite completion [LP16, Löh20b] and integral foliated simplicial volume provides upper bounds on the L^2 -Betti numbers [Sch05] and cost [Löh20a]. The following estimates of measured embedding dimension and measured embedding volume complement these connections:

Theorem 20.1. *Let M be an oriented closed connected aspherical n -manifold with fundamental group Γ , let α be a standard Γ -action, and let $k \in \{0, \dots, n\}$. Then*

$$\begin{aligned} \text{medim}_k^{\mathbb{Z}}(\alpha) &\leq \binom{n+1}{k+1} \cdot |M|^{\alpha} \\ \text{mevol}_k(\alpha) &\leq \log(k+2) \cdot \binom{n+1}{k+1} \cdot |M|^{\alpha}. \end{aligned}$$

Here, $|M|^{\alpha}$ is the α -parametrised (integral foliated) simplicial volume, i.e.,

$$\begin{aligned} |M|^{\alpha} &:= \inf \{ |c|_1 \mid c \in L^{\infty}(\alpha, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_n(\widetilde{M}; \mathbb{Z}) \\ &\quad \text{is an } \alpha\text{-parametrised fundamental cycle of } M \}. \end{aligned}$$

We refer to the literature for further details on this definition [Sch05, LP16].

Remark 20.2. More generally, one can also define corresponding simplicial volumes with finite field coefficients (with respect to the trivial norm on the coefficients) [Löh20a, Section 4.6]. The arguments below work verbatim in that setting and thus give estimates of the measured embedding dimension over finite fields in terms of the corresponding parametrised simplicial volume with the same coefficients.

20.1. Proof of Theorem 20.1. For the proof of Theorem 20.1, we construct α -embeddings to marked projective chain complexes defined out of α -parametrised fundamental cycles. For the construction of such α -embeddings, the equivariant chain-level version of Poincaré duality is essential.

Remark 20.3 (Poincaré duality). Let M be an oriented closed connected n -manifold with fundamental group Γ , let Z be \mathbb{Z} (with the usual norm) or a finite field (with the trivial norm), and let α be a standard Γ -action. Let $L = Z$ (as trivial $\mathbb{Z}\Gamma$ -module) or $L = L^{\infty}(\alpha, Z)$ and let $R := L * \Gamma$ (i.e., $R = Z\Gamma$ or $R = L^{\infty}(\alpha, Z) * \Gamma$, respectively). On R , we consider the involution $\bar{\cdot}$ induced by the inversion map on Γ . Let $c = \sum_{j=1}^m a_j \otimes \sigma_j$ be a cycle in $L \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M}; Z)$ with $a_j \in L$ and

$\sigma_j \in \text{map}(\Delta^n, \widetilde{M})$. Then the cap-product

$$\begin{aligned} \cdot \cap c: \text{Hom}_{Z\Gamma}(C_{n-*}(\widetilde{M}; Z), Z\Gamma) &\rightarrow R \otimes_{Z\Gamma} C_*(\widetilde{M}; Z) \\ f &\mapsto \sum_{j=1}^m \overline{a_j \cdot f(\sigma_j|_{n-*})} \cdot * \lfloor \sigma_j \end{aligned}$$

is a well-defined $Z\Gamma$ -chain map with the following additional properties [LM24, Chapter 5.6][BS21, Section 3.3]:

- If $[c] = [c']$ in $H_n(L \otimes_{Z\Gamma} C_*(\widetilde{M}; Z))$, then $\cdot \cap c \simeq_{Z\Gamma} \cdot \cap c'$.
- If $L = Z$ and c represents a Z -fundamental cycle of M (under the canonical chain isomorphism $Z \otimes_{Z\Gamma} C_*(\widetilde{M}; Z) \cong_Z C_*(M; Z)$), then $\cdot \cap c$ is a $Z\Gamma$ -chain homotopy equivalence.

Proof of Theorem 20.1. Let $c \in C_n(M; L^\infty(\alpha))$ be an α -parametrised fundamental cycle of M , say $c = \sum_{j=1}^m a_j \otimes \sigma_j$ with $a_j \in L^\infty(\alpha)$ and $\sigma_j \in \text{map}(\Delta^n, \widetilde{M})$; without loss of generality, we may assume that the σ_j all belong to different Γ -orbits so that $|c|_1 = \sum_{j=1}^m |a_j|_1$. By definition of the integral foliated simplicial volume and medim/mevol, it suffices to construct an α -embedding to a complex D_* , whose “size” is controlled well enough in terms of $|c|_1$. We abbreviate $R := L^\infty(\alpha) * \Gamma$.

Construction of the target complex. For $j \in \{1, \dots, m\}$, we write $A_j := \text{supp}(a_j)$. Let $k \in \mathbb{N}$. Let $S_k(\sigma_j)$ denote the set of all k -faces of σ_j . We define the marked projective R -module

$$D_k := \bigoplus_{\tau \in \bigcup_{j=1}^m S_k(\sigma_j)} \langle A_j \rangle \cdot \tau$$

and, for $k \in \mathbb{N}_{>0}$, we set

$$\begin{aligned} \partial_k^D: D_k &\rightarrow D_{k-1} \\ \chi_{A_j} \cdot \tau &\mapsto \sum_{r=0}^k (-1)^r \cdot \chi_{A_j} \cdot \partial_r \tau \quad \text{for } \tau \in S_k(\sigma_j). \end{aligned}$$

Moreover, we define

$$\begin{aligned} \eta: D_0 &\rightarrow L^\infty(\alpha) \\ \chi_{A_j} \cdot \tau &\mapsto \chi_{A_j}. \end{aligned}$$

Viewing the marked generators of D_* as actual singular simplices on \widetilde{M} produces a canonical $Z\Gamma$ -chain map $s_*: D_* \rightarrow R \otimes_{Z\Gamma} C_*(\widetilde{M}; \mathbb{Z})$, which extends $\text{id}_{L^\infty(\alpha)}$ with respect to η and the canonical augmentation $R \otimes_{Z\Gamma} C_0(\widetilde{M}; \mathbb{Z}) \rightarrow L^\infty(\alpha)$. We will see below that $\eta: D_0 \rightarrow L^\infty(\alpha)$ indeed is surjective.

By construction, for each $k \in \mathbb{N}$, we have

$$\dim(D_k) \leq \sum_{j=1}^m \binom{n+1}{k+1} \cdot \mu(A_j) \leq \binom{n+1}{k+1} \cdot |c|_1,$$

$$\text{lognorm}(\partial_{k+1}^D) \leq \log_+ \|\partial_{k+1}^D\| \cdot \dim(D_k) \leq \log(k+2) \cdot \binom{n+1}{k+1} \cdot |c|_1.$$

Construction of the chain map. Let C_* be a free $Z\Gamma$ -resolution of \mathbb{Z} (that is of finite rank up to degree $n+1$). Because M is aspherical, there exists a $Z\Gamma$ -chain map $f_*: C_* \rightarrow C_*(\widetilde{M}; \mathbb{Z})$ extending $\text{id}_{\mathbb{Z}}$. Let $E_* := \text{Hom}_{Z\Gamma}(C_{n-*}(\widetilde{M}; \mathbb{Z}), Z\Gamma)$. By equivariant Poincaré duality, there is a $Z\Gamma$ -chain homotopy inverse $g_*: C_*(\widetilde{M}; \mathbb{Z}) \rightarrow E_*$ of the map $\cdot \cap c_{\mathbb{Z}}$ induced by an integral fundamental cycle $c_{\mathbb{Z}} \in \mathbb{Z} \otimes_{Z\Gamma} C_n(\widetilde{M}; \mathbb{Z})$

(Remark 20.3). Finally, the cap-product map

$$h_* := \cdot \cap c: E_* \rightarrow D_*$$

$$f \mapsto \sum_{j=1}^m \overline{a_j \cdot f(\sigma_j]_{n-*})} \cdot * \lfloor \sigma_j$$

is well-defined and a $\mathbb{Z}\Gamma$ -chain map. Indeed, by Remark 20.3, this holds for the target $R \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M}; \mathbb{Z})$; by construction of D_* , this map factors over s_* .

We will now explain why $h_* \circ g_* \circ f_*: C_* \rightarrow D_*$ is an α -embedding, i.e., that this composition extends the inclusion $\mathbb{Z} \rightarrow L^\infty(\alpha)$ as constant functions and that $\eta: D_0 \rightarrow L^\infty(\alpha)$ is surjective: We consider the following diagram of $\mathbb{Z}\Gamma$ -chain maps:

$$\begin{array}{ccccccc} C_*(\widetilde{M}; \mathbb{Z}) & \xrightarrow{g_*} & E_* & \xrightarrow{\cdot \cap c} & D_* & \xrightarrow{s_*} & R \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M}; \mathbb{Z}) \\ \parallel & & \parallel & & & & \uparrow \text{canonical map} \\ C_*(\widetilde{M}; \mathbb{Z}) & \xrightarrow{g_*} & E_* & \xrightarrow{\cdot \cap c_{\mathbb{Z}}} & \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M}; \mathbb{Z}) & & \end{array}$$

The right hand square commutes up to $\mathbb{Z}\Gamma$ -chain homotopy because c and $c_{\mathbb{Z}}$ are cycles in $L^\infty(\alpha) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M}; \mathbb{Z})$ that, by definition of α -parametrised fundamental cycles, represent the same class in homology (Remark 20.3). The lower composition is $\mathbb{Z}\Gamma$ -chain homotopic to the identity (by choice of g_*). Taking H_0 of this diagram thus results in the following commutative diagram of \mathbb{Z} -modules:

$$\begin{array}{ccccccc} \mathbb{Z} & \xrightarrow{\cong} & H_0(\widetilde{M}; \mathbb{Z}) & \xrightarrow{H_0(h_* \circ g_*)} & H_0(D_*) & \xrightarrow{H_0(s_*)} & H_0(M; L^\infty(\alpha)) \xrightarrow{\cong} L^\infty(\alpha) \\ \parallel & & \parallel & & & & \uparrow \text{canonical map} \uparrow \\ \mathbb{Z} & \xrightarrow{\cong} & H_0(\widetilde{M}; \mathbb{Z}) & \xlongequal{\quad} & H_0(M; \mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z} \end{array}$$

In particular, we see that $H_0(s_*)$ is surjective (because $H_0(s_*)$ is $L^\infty(\alpha)$ -linear and 1 lies in the image). By construction of η , this shows that also η is surjective and that $h_* \circ g_*$ extends the canonical inclusion $\mathbb{Z} \rightarrow L^\infty(\alpha)$.

Hence, $h_* \circ g_* \circ f_*: C_* \rightarrow D_*$ is an α -embedding and we obtain

$$\text{medim}_k^{\mathbb{Z}}(\alpha) \leq \dim(D_k) \leq \binom{n+1}{k+1} \cdot |c|_1$$

$$\text{mevol}_k(\alpha) \leq \log \text{norm}(\partial_{k+1}^D) \leq \log(k+2) \cdot \binom{n+1}{k+1} \cdot |c|_1.$$

Taking the infimum over all α -parametrised fundamental cycles c of M gives the claimed estimates. \square

20.2. Examples. We combine the simplicial volume estimate (Theorem 20.1) with known computations of integral foliated simplicial volume. While the resulting upper bounds for (torsion) homology growth are not new, they can now be combined with other inheritance results for measured embedding dimension/volume to obtain new results.

Example 20.4. Let Γ be the fundamental group of an oriented closed connected aspherical 3-manifold M . Then, we have

$$\begin{aligned} \widehat{t}_1(\Gamma, \Gamma_*) &\leq \text{mevol}_1(\Gamma \curvearrowright \widehat{\Gamma}_*) && \text{(Theorem 1.2)} \\ &\leq 6 \cdot \log(3) \cdot |M|^{\Gamma \curvearrowright \widehat{\Gamma}_*} && \text{(Theorem 20.1)} \\ &= 6 \cdot \log(3) \cdot \frac{\text{hypvol}(M)}{v_3}, && [\text{FLMQ21, Theorem 1.7}] \end{aligned}$$

where hypvol denotes the total volume of the hyperbolic pieces in the JSJ decomposition of M .

In the case of hyperbolic 3-manifolds, this is a coarser version of the estimate obtained in Theorem 18.21. It should be noted that the estimates for mevol_1 in the proof of Theorem 18.21 and the computation of $|M|^{\Gamma \curvearrowright \widehat{\Gamma}^*}$ are based on the same, dynamical, principles (weak bounded orbit equivalence and approximation).

Example 20.5. More generally, one obtains also upper bounds for the measured embedding dimension and measured embedding volume in terms of the Riemannian volume and a bound on the volume of small balls [BS21, Theorem 1.5]: Let $V_1 \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}_{>0}$. Then, there exists a constant $\text{const}(n, V_1) \in \mathbb{R}_{>0}$ with the following property: For every aspherical oriented closed connected Riemannian n -manifold that satisfies $\text{vol}_{\widehat{M}}(B) \leq V_1$ for all balls $B \subset \widehat{M}$ of radius at most 1 and every standard $\pi_1(M)$ -space α , we have for all $k \in \{0, \dots, n\}$:

$$\begin{aligned} \text{mevol}_k(\alpha) &\leq \log(k+2) \cdot \binom{n+1}{k+1} \cdot |M|^\alpha && \text{(Theorem 20.1)} \\ &\leq \log(k+2) \cdot \binom{n+1}{k+1} \cdot \text{const}(n, V_1) \cdot \text{vol}(M). && [\text{BS21, Theorem 1.5}] \end{aligned}$$

For $\text{medim}_k^{\mathbb{Z}}(\alpha)$, an analogous version holds.

Example 20.6. Let M be an oriented closed connected aspherical n -manifold (with $n > 0$) with fundamental group Γ . Suppose that there exists an open cover of M by amenable subsets of multiplicity at most n . Then, for all standard Γ -spaces α and all $k \in \mathbb{N}$, we have

$$\text{medim}_k^{\mathbb{Z}}(\alpha) = 0 \quad \text{and} \quad \text{mevol}_k(\alpha) = 0,$$

because $|M|^\alpha = 0$ [LMS22]. This type of vanishing arises in many geometric situations, e.g., manifolds with amenable fundamental group, graph 3-manifolds, smooth manifolds that admit smooth circle actions without fixed points and manifolds admitting an F -structure [LMS22, Section 1.1][Sau09].

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