

Quantitative chain extensions and integral cohomological fillings

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Abstract

We develop two methods in quantitative homological algebra: A declarative approach to integral cohomological filling problems and a quantitatively controlled replacement of normed chain complexes by their homology complex. Both methods are connected through so-called chain extensions. As applications, we obtain transparent proofs for uniform integral simplicial cohomological fillings for subdivisions of simplicial complexes, for uniform integral cohomological fillings of certain finite coverings, and for lower bounds of the ℓ^1 -semi-norm on product spaces.

1 Introduction

Quantitative homological algebra enriches chain complexes or homology with norms. In geometric topology, quantitative homological algebra is useful in many contexts; examples relevant for this paper are geometric rigidity via simplicial volume or bounded cohomology [Gro82], quantitative cobordism via integral cohomological fillings [CDMW18], and waist inequalities via integral cohomological fillings [BS26].

Fundamental concepts

We develop two methods in quantitative homological algebra:

- A declarative approach to quantitative cohomological lifting over $\mathbb{R}/\mathbb{Z}$ and thus to translating cohomological filling problems over $\mathbb{Z}$ into cohomological filling problems over $\mathbb{R}$.
- Quantitatively controlled replacements of normed $\mathbb{R}$ -chain complexes by their homology complex, i.e., by chain complexes with trivial boundary operator.

Both methods are connected through the simple algebraic concept of *chain extensions* (Section 2). A chain extension in degree k is a map from the k -th

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chain module to the k -th homology extending the canonical projection from the k -cycles to homology. In our quantitative context, we will consider different norms of chain extensions.

In the first method, we introduce size invariants on marked free chain complexes measuring the homological complexity of the boundary operator, while ignoring the size of the homology (Section 4). Size invariants satisfy basic normalisation and monotonicity properties.

A concrete example of such a size invariant is the cohomological lifting volume. By definition, it is the optimal quantity for lifts under cohomological change-of-coefficients maps of the form $H^*(C_*; \mathbb{R}) \rightarrow H^*(C_*; \mathbb{R}/\mathbb{Z})$. The monotonicity of the cohomological lifting volume exhibits (norms of) chain extensions as a pragmatic tool for upper bounds. The cohomological lifting volume controls the passage from cohomological filling problems over $\mathbb{Z}$ to cohomological filling problems over $\mathbb{R}$, which are usually more accessible.

A second example of a size invariant is the so-called gnarledness of chain complexes, which is the optimal quantity for the size of chain extensions with respect to cointegral bases in homology. By design, gnarledness is the universal size invariant. This notion of gnarledness is an algebraic version of the gnarledness of simplicial complexes by Chambers, Dotterrer, Manin, and Weinberger [CDMW18]. In contrast with the geometric setting, algebraic gnarledness enjoys a straightforward monotonicity.

In the second method, we use chain extensions to replace chain complexes by chain complexes with trivial boundary operator. Gnarledness measures the size of such replacements with respect to cointegral bases in homology. Measuring the size of chain extensions of normed chain complexes instead with respect to the induced semi-norm on homology leads to lower bounds for the induced semi-norm on the homology of tensor product complexes.

Applications

We obtain transparent proofs for uniform integral simplicial cohomological fillings for subdivisions of simplicial complexes and for uniform integral cellular cohomological fillings of certain finite coverings. Furthermore, we obtain lower bounds for the ℓ^1 -semi-norm on product spaces. We now outline these applications in more detail.

For simplicial subdivisions, we have the following version of a result of Chambers, Dotterrer, Manin, and Weinberger [CDMW18, Lemma 3.9], which they use to quantitatively control null-cobordisms. The original proof is based on geometric gnarledness and the explicit construction of controlled spanning trees in simplicial subdivisions. Our algebraic approach via the cohomological lifting volume bypasses such geometric constructions and is thus robust enough to work in a uniform way for all simplicial subdivisions.

Corollary 1.1 (integral cohomological fillings for subdivisions; Corollary 6.5). *Let X be a finite simplicial complex, let $k \in \mathbb{N}$, and let $\mathbb{F}$ be a subfield of $\mathbb{R}$. Then there exists a constant $c \in \mathbb{R}_{>0}$ such that: If X' is a simplicial subdivision of X and $f \in C_{\Delta}^k(X'; \mathbb{F})$ is a simplicial cochain such that δf is a coboundary in $C_{\Delta}^*(X'; \mathbb{Z})$, then there exists a cochain $\tilde{f} \in C_{\Delta}^k(X'; \mathbb{Z})$ with*

$$\delta \tilde{f} = \delta f \quad \text{and} \quad |\tilde{f}|_{\infty} \leq |f|_{\infty} + c.$$

For this application, in addition to the basic notions, only Sections 4, 5.1, 5.2, and Section 6 are needed. To put our approach into context, we develop size invariants (Section 4) and algebraic gnarledness of chain complexes (Section 7). This includes the comparison between algebraic gnarledness and the cohomological lifting volume (Section 7.2), the comparison between algebraic gnarledness and geometric gnarledness (Section 7.4), a rephrasing of simplicial volume in terms of algebraic gnarledness of the singular chain complex (Section 7.5), and the universality of gnarledness among size invariants (Section 7.3).

As a second application of the cohomological lifting volume, we obtain a straightforward proof of the relation between integral and real expansion constants of the cellular boundary operator in certain finite coverings:

Corollary 1.2 (Corollary 5.12, Remark 5.13). *Let X be a finite CW-complex. Then there is a constant $c \in \mathbb{R}_{>0}$ such that: For every finite sheeted covering map $p: \bar{X} \rightarrow X$ (where $\bar{X}$ is equipped with the CW-structure induced by p) and every $k \in \mathbb{N}$ with $\dim_{\mathbb{R}} H_k(\bar{X}; \mathbb{R}) = \dim_{\mathbb{R}} H_k(X; \mathbb{R})$, we have*

$$\Xi_k^{\mathbb{Z}}(C_*^{\text{cell}}(\bar{X})) \leq \Xi_k^{\mathbb{R}}(C_*^{\text{cell}}(\bar{X})) + c$$

In particular, this applies in degree 0 and 1 to finite CW-complexes with Kazhdan fundamental group (Example 5.14), as used by Bader and Sauer in their proofs of uniform waist inequalities [BS26]. For this application, in addition to the basic notions, only Sections 4, 5.1, 5.2, and 5.3 are needed.

Finally, using the existence of bounded chain extensions with respect to the induced semi-norm on homology (Section 9.2), we obtain the following result for products. It should be noted that the calculation of the ℓ^1 -semi-norm on products is notoriously difficult and only few exact calculations are known [BK08, HL21].

Corollary 1.3 (Corollary 9.16). *Let X, Y be topological spaces such that X has finite dimensional reduced homology in each degree (e.g., a closed manifold or a CW-complex of finite type). Then, for each $n \in \mathbb{N}$, the Künneth isomorphism*

$$\bigoplus_{p \in \mathbb{N}} H_p(X; \mathbb{R}) \otimes_{\mathbb{R}} H_{n-p}(Y; \mathbb{R}) \rightarrow H_n(X \times Y; \mathbb{R})$$

$$\alpha \otimes \beta \mapsto \alpha \times \beta$$

is bilipschitz.

In particular, this also leads to an inheritance result for the behaviour of product maps with respect to the ℓ^1 -semi-norm (Corollary 9.18). For these applications, in addition to basic notions, only Section 9 is needed.

Contents

1	Introduction	1
2	Chain extensions	4
3	Basic notions in normed homological algebra	6
4	Size invariants on marked free chain complexes	7

5	Cohomological lifting volume	8
5.1	Definition and basic properties	9
5.2	Integral cohomological fillings	11
5.3	Integral cohomological fillings on finite coverings	12
6	Cohomological lifting for subdivisions	14
6.1	Simplicial subdivisions	15
6.2	Cohomological lifting/filling for subdivisions	17
7	Gnarledness of chain complexes	18
7.1	Definition and basic examples	18
7.2	Comparison with the lifting volume	20
7.3	Universality	23
7.4	Comparison with geometric gnarledness	24
7.5	Comparison with simplicial volume	27
8	Irrelevance of coefficients	29
8.1	Cohomological lifting volume	29
8.2	Size invariants	30
8.3	Gnarledness	32
9	Controlling the norm of products	33
9.1	Null spaces and reduced homology	33
9.2	Boundedness of chain extensions with respect to the induced semi-norm on homology	34
9.3	The Künneth isomorphism is bilipschitz	36

2 Chain extensions

Chain extensions are maps from a chain complex to its homology that extend the canonical projection from the space of cycles to the homology.

Definition 2.1 (chain extension). Let C_* be a chain complex over a ring R and $k \in \mathbb{N}$. A *chain extension of C_* in degree k* is an R -linear map $s: C_k \rightarrow H_k(C_*)$ with the following property: For all cycles $c \in C_k$, we have

$$s(c) = [c].$$

In other words, s makes the following diagram commutative, where the left arrow is the inclusion and the right arrow is the canonical projection:

$$\begin{array}{ccc} C_k & \xrightarrow{\quad s \quad} & H_k(C_*) \\ & \nwarrow \quad \nearrow & \\ & \ker \partial_k & \end{array}$$

Clearly not every chain complex admits a chain extension. Chain complexes over fields always admit chain extensions. If C_* is a chain complex and $(s_k)_k$ is a sequence of chain extensions in all degrees k , then $s_*: C_* \rightarrow H_*(C_*)$ is a homology isomorphism, where we view $H_*(C_*)$ as chain complex with trivial boundary operator: under the canonical isomorphism $H_*(H_*(C_*)) \cong H_*(C_*)$, the induced

map $H_*(s_*)$ corresponds to $\text{id}_{H_*(C_*)}$. If the chain complex C_* is free in every degree (and bounded below), then s_* is a chain homotopy equivalence [Bro94, Chapter 1, Theorem 8.4]. Thus, in this situation, chain extensions are a measure gauging “how far” a chain complex is from having a trivial boundary operator.

We will consider two different ways to quantify the size of chain extensions in the context of marked free chain complexes: Using ℓ^1 -norms on homology with respect to cointegral bases (Sections 4–7) and using the induced semi-norm on homology (Section 9).

In the following, we will mainly be interested in chain complexes obtained by base change from chain complexes over $\mathbb{Z}$.

Notation 2.2 ((co)homology with coefficients). If C_* is a $\mathbb{Z}$ -chain complex, $\mathbb{F}$ is a commutative ring, and $k \in \mathbb{N}$, we abbreviate

$$\begin{aligned} H_k(C_*; \mathbb{F}) &:= H_k(C_* \otimes_{\mathbb{Z}} \mathbb{F}) \\ H^k(C_*; \mathbb{F}) &:= H_k(\text{Hom}_{\mathbb{Z}}(C_*, \mathbb{F})). \end{aligned}$$

If $f_*: C_* \rightarrow D_*$ is a $\mathbb{Z}$ -chain map, then we write

$$\begin{aligned} H_k(f_*; \mathbb{F}) &:= H_k(f_* \otimes_{\mathbb{Z}} \text{id}_{\mathbb{F}}): H_k(C_*; \mathbb{F}) \rightarrow H_k(D_*; \mathbb{F}) \\ H^k(f_*; \mathbb{F}) &:= H^k(\text{Hom}_{\mathbb{Z}}(f_*, \text{id}_{\mathbb{F}})): H^k(D_*; \mathbb{F}) \rightarrow H^k(C_*; \mathbb{F}). \end{aligned}$$

Furthermore, we write $H_k(C_*)_{\mathbb{F}}$ for the integral part of $H_k(C_*; \mathbb{F})$, i.e., for the image of $H_k(C_*; \mathbb{Z})$ under the coefficients change $H_k(C_*; \mathbb{Z}) \rightarrow H_k(C_*; \mathbb{F})$. If $\mathbb{F}$ is a field, an $\mathbb{F}$ -basis B of $H_k(C_*; \mathbb{F})$ is *cointegral* if $H_k(C_*)_{\mathbb{F}} \subset \mathbb{Z}B$.

Definition 2.3 (chain extension, with coefficients). Let C_* be a $\mathbb{Z}$ -chain complex, let $k \in \mathbb{N}$, and let $\mathbb{F}$ be a commutative ring. A *chain extension of C_* in degree k over $\mathbb{F}$* is a chain extension $s: C_k \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow H_k(C_*; \mathbb{F})$ of $C_* \otimes_{\mathbb{Z}} \mathbb{F}$ in degree k of the $\mathbb{F}$ -chain complex $C_* \otimes_{\mathbb{Z}} \mathbb{F}$.

Lemma 2.4 (transport of chain extensions). *Let C_*, D_* be $\mathbb{Z}$ -chain complexes, let $\mathbb{F}$ be a commutative ring, let $k \in \mathbb{N}$, let s be a chain extension of D_* in degree k over $\mathbb{F}$, and let $f_*: C_* \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow D_* \otimes_{\mathbb{Z}} \mathbb{F}$ be an $\mathbb{F}$ -chain map that induces an $\mathbb{F}$ -isomorphism $H_k(f_*): H_k(C_*; \mathbb{F}) \rightarrow H_k(D_*; \mathbb{F})$. Then*

$$H_k(f_*)^{-1} \circ s \circ f_k: C_k \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow H_k(C_*; \mathbb{F})$$

is a chain extension of C_* in degree k over $\mathbb{F}$.

Proof. This is a straightforward calculation: Let $c \in C_k \otimes_{\mathbb{Z}} \mathbb{F}$ be a cycle. Then $f_k(c) \in D_k \otimes_{\mathbb{Z}} \mathbb{F}$ is a cycle and as s is a chain extension of D_* in degree k over $\mathbb{F}$ we obtain (in $H_k(C_*; \mathbb{F})$)

$$[c] = H_k(f_*)^{-1} \circ H_k(f)[c] = H_k(f_*; \mathbb{F})^{-1} \circ s(f_k(c)),$$

as desired. $\square$

The hypotheses on f_* in Lemma 2.4 are, for instance, satisfied if $f_* = \tilde{f}_* \otimes_{\mathbb{Z}} \text{id}_{\mathbb{F}}$, where $\tilde{f}_*: C_* \rightarrow D_*$ is a $\mathbb{Z}$ -chain homotopy equivalence: Indeed, then $f_* = \tilde{f}_* \otimes_{\mathbb{Z}} \text{id}_{\mathbb{F}}: C_* \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow D_* \otimes_{\mathbb{Z}} \mathbb{F}$ is an $\mathbb{F}$ -chain homotopy equivalence and so f_* indeed induces an $\mathbb{F}$ -isomorphism $H_k(f_*; \mathbb{F}): H_k(C_*; \mathbb{F}) \rightarrow H_k(D_*; \mathbb{F})$.

3 Basic notions in normed homological algebra

We recall basic notions and properties of norms, semi-norms, normed chain complexes, and the induced semi-norm on cohomology.

Setup 3.1. In this section, $\mathbb{F}$ will be a subfield of the reals with the standard absolute value.

Definition 3.2 (norm). A *norm on an $\mathbb{F}$ -vector space X* is a map $X \rightarrow \mathbb{R}_{\geq 0}$ that fulfils the triangle inequality, is $\mathbb{F}$ -homogeneous, and positive definite.

Definition 3.3 (semi-norm). Let X be an $\mathbb{F}$ -vector space. A *semi-norm on X* is a map $\|\cdot\|: X \rightarrow \mathbb{R}_{\geq 0}$ that fulfils the triangle inequality and is $\mathbb{F}$ -homogeneous.

More generally, we will also consider norms and semi-norms with values in the extended range $[0, \infty]$ (requiring 0 being mapped to 0).

Definition 3.4 (bounded map). If $f: X \rightarrow Y$ is an $\mathbb{F}$ -linear map, then we write

$$\|f\| := \inf \{c \in \mathbb{R}_{\geq 0} \mid \forall x \in X \quad \|f(x)\| \leq c \cdot \|x\|\} \in [0, \infty].$$

We call f *bounded* if $\|f\| < \infty$.

Normed $\mathbb{F}$ -chain complexes are $\mathbb{F}$ -chain complexes, where all chain modules are equipped with norms (or, more generally, semi-norms) and such that the boundary operators are bounded in each degree with respect to the norms on the chain modules. The norm on a normed chain complex induces a semi-norm on the corresponding homology, the *induced semi-norm*. We will mainly be interested in normed chain complexes arising in the following way:

Definition 3.5 (marked free chain complex). Let R be a ring. A *marked free R -chain complex* is an R -chain complex C_* that is free in each degree and equipped with a fixed (“marked”) basis in every degree.

If C_* is a marked free $\mathbb{Z}$ -chain complex, we view $C_* \otimes_{\mathbb{Z}} \mathbb{F}$ as a marked free $\mathbb{F}$ -chain complex through the induced $\mathbb{F}$ -bases and consider the corresponding ℓ^1 -norm $|\cdot|_1$ on $C_* \otimes_{\mathbb{Z}} \mathbb{F}$.

In particular, singular chain complexes, simplicial chain complexes, cellular chain complexes, and bar complexes carry corresponding ℓ^1 -norms. The boundary operators are bounded in the singular, simplicial, and bar cases or if the underlying chain module is of finite rank.

Notation 3.6. If C_* is a marked free $\mathbb{Z}$ -chain complex with marked basis Σ_k in degree $k \in \mathbb{N}$, then the cochain complex $\text{Hom}_{\mathbb{Z}}(C_*, \mathbb{F})$ carries the dual norm

$$\begin{aligned} |\cdot|_{\infty}: \text{Hom}_{\mathbb{Z}}(C_k, \mathbb{F}) &\rightarrow [0, \infty] \\ f &\mapsto \sup_{b \in \Sigma_k} |f(b)|. \end{aligned}$$

The induced semi-norm on $H^*(C_*; \mathbb{F})$ is denoted by $\|\cdot\|_{\infty}$. The coboundary operator of $\text{Hom}_{\mathbb{Z}}(C_*, \mathbb{F})$ is bounded in degree k if the boundary operator of $C_* \otimes_{\mathbb{Z}} \mathbb{F}$ is bounded in degree $k+1$.

In particular, this construction applies to singular cochains, simplicial cochains, cellular cochains, and bar cochains.

Remark 3.7. Let C_*, D_* be marked free $\mathbb{Z}$ -chain complexes and $f_*: C_* \rightarrow D_*$ a $\mathbb{Z}$ -chain map that is bounded in degree $k \in \mathbb{N}$. Then,

$$\|H_k(f_*; \mathbb{F})\| \leq \|f_k\| \quad \text{and} \quad \|H^k(f_*; \mathbb{F})\| \leq \|f_k\|,$$

where the operator norms are taken with respect to the induced ℓ^1 -semi-norm and ℓ^∞ -semi-norm on (co)homology, respectively.

4 Size invariants on marked free chain complexes

We consider a simple type of size invariants on marked free $\mathbb{Z}$ -chain complexes, satisfying basic normalisation and monotonicity properties. Here, “size” does not refer to the size in terms of the size of homology, but to the homological complexity of the boundary operator. This is reflected in the normalisation condition. The key property for calculations or meaningful estimates is monotonicity, which leads to the chain extension estimate and to finiteness results.

Examples of such size invariants are the cohomological lifting volume (Section 5) and gnarledness of chain complexes (Section 7). The former is optimal for cohomological lifting and related to integral cohomological filling problems; the latter is, by design, universal among size invariants (Section 7.3). The sandwich estimate between the cohomological lifting volume and gnarledness (Proposition 7.5) expresses that both are closely related and the cohomological lifting volume is almost universal.

Notation 4.1. We write $\mathbf{mCh}$ for the class of marked free $\mathbb{Z}$ -chain complexes. We equip chain complexes in $\mathbf{mCh}$ (and their base-changed versions) in each degree with the ℓ^1 -norm $\|\cdot\|_1$ given by the marked basis.

Definition 4.2 (size invariant). Let $\mathbb{F}$ be a subfield of $\mathbb{R}$ and let $k \in \mathbb{N}$. A *size invariant over $\mathbb{F}$ in degree k* is a map $I: \mathbf{mCh} \rightarrow [0, \infty]$ with the following properties:

- *Normalisation.* If C_* is in $\mathbf{mCh}$ with $\partial_k = 0$ and $\partial_{k+1} = 0$, then $I(C_*) \leq 1$.
- *Monotonicity.* If C_*, D_* are in $\mathbf{mCh}$ and $f_*: C_* \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow D_* \otimes_{\mathbb{Z}} \mathbb{F}$ is an $\mathbb{F}$ -chain map inducing an $\mathbb{F}$ -isomorphism $H_k(f_*): H_k(C_*; \mathbb{F}) \rightarrow H_k(D_*; \mathbb{F})$ satisfying the integrality condition $H_k(f_*)(H_k(C_*)_{\mathbb{F}}) \subset H_k(D_*)_{\mathbb{F}}$, then

$$I(C_*) \leq \|f_k\| \cdot I(D_*).$$

Monotonicity includes both an integrality condition and quantitative information (namely, $\|f_k\|$). Taking chain maps as in the monotonicity condition as morphisms on $\mathbf{mCh}$, one could view the monotonicity condition as a functoriality to the poset category $[0, \infty]$, twisted by quantitative data.

Proposition 4.3 (chain extension estimate). *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$, let $k \in \mathbb{N}$, and let I be a size invariant over $\mathbb{F}$ in degree n . Let C_* be a marked free $\mathbb{Z}$ -chain complex, let $s: C_k \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow H_k(C_*; \mathbb{F})$ be a chain extension of C_* over $\mathbb{F}$ in degree k , and let B be a cointegral $\mathbb{F}$ -basis of $H_k(C_*; \mathbb{F})$. Then, we obtain*

$$I(C_*) \leq \|s\|^B,$$

where $\|\cdot\|^B$ refers to the operator norm taken with respect to the ℓ^1 -norms associated with the $\mathbb{F}$ -basis of $C_k \otimes_{\mathbb{Z}} \mathbb{F}$ induced by the marked $\mathbb{Z}$ -basis of C_k and B of $H_k(C_*; \mathbb{F})$, respectively.

Proof. We apply monotonicity to s and normalisation: We consider the $\mathbb{Z}$ -chain complex D_* concentrated in degree k given by

$$D_k := \bigoplus_B \mathbb{Z},$$

with the marked $\mathbb{Z}$ -basis B . The chain extension s can be viewed as a chain map $f_*: C_* \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow D_* \otimes_{\mathbb{Z}} \mathbb{F}$ (through the $\mathbb{F}$ -isomorphism $D_k \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow H_k(C_*; \mathbb{F})$ induced by the basis B and extension by the zero maps in degrees $\neq k$). By construction,

- $H_k(f_*): H_k(C_*; \mathbb{F}) \rightarrow H_k(D_*; \mathbb{F})$ is an $\mathbb{F}$ -isomorphism,
- cointegrality of B ensures that $H_k(f_*)(H_k(C_*)_{\mathbb{F}}) \subset H_k(D)_{\mathbb{F}}$,
- and we have $\|f_k\| = \|s\|^B$.

Therefore, monotonicity applies to f_* and the target complex D_* falls into the normalisation case. We obtain

$$I(C_*) \leq \|f_k\| \cdot I(D_*) \leq \|s\|^B \cdot 1. \quad \square$$

Example 4.4 (vanishing). Let $\mathbb{F}$ be a subfield of $\mathbb{R}$, let $k \in \mathbb{N}$, let I be a size invariant over $\mathbb{F}$ in degree k , and let C_* be a marked free $\mathbb{Z}$ -chain complex with $H_k(C_*; \mathbb{F}) \cong 0$ (e.g., through $C_k \cong 0$). Then $I(C_*) = 0$ by the chain extension estimate (Proposition 4.3), applied to the zero chain extension.

Corollary 4.5 (finiteness). *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$, let $k \in \mathbb{N}$, let I be a size invariant over $\mathbb{F}$ in degree k , and let C_* be a marked free $\mathbb{Z}$ -chain complex. If $\text{rk}_{\mathbb{Z}} C_k < \infty$, then*

$$I(C_*) < \infty.$$

Proof. We use the chain extension estimate (Proposition 4.3): Because $\mathbb{F}$ is a field, there exists a chain extension $s: C_k \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow H_k(C_*; \mathbb{F})$ of C_* over $\mathbb{F}$ in degree k . Let B be a cointegral basis of $H_k(C_*; \mathbb{F})$ (for instance, induced from a $\mathbb{Z}$ -basis of the finitely generated free $\mathbb{Z}$ -module $H_k(C_*)_{\mathbb{F}}$).

We write $\Sigma_k \subset C_k$ for the marked $\mathbb{Z}$ -basis of C_k . Because $C_k \otimes_{\mathbb{Z}} \mathbb{F}$ is equipped with the ℓ^1 -norm given by the finite marked basis Σ_k , we obtain

$$\|s\|^B = \max_{\sigma \in \Sigma_k} |s(\sigma)|_1^B < \infty.$$

The chain extension estimate (Proposition 4.3) shows that $I(C_*) \leq \|s\|^B$ is finite. $\square$

5 Cohomological lifting volume

We introduce the cohomological lifting volume as optimal bound measuring minimal lifts of cohomology classes from $H^*(C_*; \mathbb{R}/\mathbb{Z})$ to $H^*(C_*; \mathbb{R})$. We then establish the fundamental properties of the cohomological lifting volume:

- it is a size invariant in the sense of Section 4 and thus satisfies normalisation, monotonicity, and the chain extension estimate;
- it gives control on integral cohomological fillings.

The applications to finite coverings and simplicial subdivisions are contained in Section 5.3 and Section 6, respectively.

5.1 Definition and basic properties

Definition 5.1 (cohomological lifting volume). Let C_* be a marked free chain complex over $\mathbb{Z}$, let $\mathbb{F}$ be a subfield of $\mathbb{R}$, and let $k \in \mathbb{N}$. We write $\pi: H^k(C_*; \mathbb{F}) \rightarrow H^k(C_*; \mathbb{F}/\mathbb{Z})$ for the map induced by the canonical projection $\mathbb{F} \rightarrow \mathbb{F}/\mathbb{Z}$.

- For $\varphi \in H^k(C_*; \mathbb{F}/\mathbb{Z})$, we define the *lifting volume* of φ by

$$L_k^{\mathbb{F}}(C_*, \varphi) := \inf_{\tilde{\varphi} \in \pi^{-1}(\varphi) \subset H^k(C_*; \mathbb{F})} \|\tilde{\varphi}\|_{\infty} \in [0, \infty].$$

- The *cohomological lifting volume of C_* over $\mathbb{F}$ in degree k* is defined as

$$L_k^{\mathbb{F}}(C_*) := \sup_{\varphi \in \text{im } \pi \subset H^k(C_*; \mathbb{F}/\mathbb{Z})} L_k^{\mathbb{F}}(C_*, \varphi) \in [0, \infty].$$

If $\mathbb{F}$ is a subfield of $\mathbb{R}$, then $\mathbb{F}$ contains $\mathbb{Q}$, the field $\mathbb{F}$ is dense in $\mathbb{R}$, and $[-1/2, 1/2)$ is a fundamental domain in $\mathbb{F}$ for $\mathbb{F}/\mathbb{Z}$.

Remark 5.2. Let C_* be a marked free chain complex over $\mathbb{Z}$, let $\mathbb{F}$ be a subfield of $\mathbb{R}$, and let $k \in \mathbb{N}$. By the universal coefficient theorem, because $\mathbb{F}$ and $\mathbb{F}/\mathbb{Z}$ are divisible $\mathbb{Z}$ -modules, the horizontal evaluation maps in the commutative diagram

$$\begin{array}{ccc} H^k(C_*; \mathbb{F}) & \longrightarrow & \text{Hom}_{\mathbb{Z}}(H_k(C_*), \mathbb{F}) \\ \downarrow \pi & & \downarrow \\ H^k(C_*; \mathbb{F}/\mathbb{Z}) & \longrightarrow & \text{Hom}_{\mathbb{Z}}(H_k(C_*), \mathbb{F}/\mathbb{Z}) \end{array}$$

are isomorphisms over $\mathbb{F}$ and $\mathbb{Z}$, respectively. The upper evaluation map factors over the restriction isomorphisms $\text{Hom}_{\mathbb{F}}(H_k(C_*; \mathbb{F}), \mathbb{F}) \rightarrow \text{Hom}_{\mathbb{Z}}(H_k(C_*)_{\mathbb{F}}, \mathbb{F}) \rightarrow \text{Hom}_{\mathbb{Z}}(H_k(C_*), \mathbb{F})$.

Example 5.3 (normalisation). Let $\mathbb{F}$ be a subfield of $\mathbb{R}$, let $k \in \mathbb{N}$, and let C_* be a marked free $\mathbb{Z}$ -chain complex with $\partial_{k+1} = 0$. Then, we have

$$L_k^{\mathbb{F}}(C_*) = \begin{cases} 0 & \text{if } C_k \cong_{\mathbb{Z}} 0 \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Indeed, this follows from the fact that all cochains in degree k are cocycles and that $[-1/2, 1/2)$ is a fundamental domain in $\mathbb{F}$ for $\mathbb{F}/\mathbb{Z}$.

Remark 5.4. The cohomological lifting volume is *not* a chain homotopy invariant: For $d \in \mathbb{N}_{>0}$, we consider the $\mathbb{Z}$ -chain complex C_*

$$\mathbb{Z} \xrightarrow{(d,1)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{0} \mathbb{Z},$$

concentrated in degrees 2, 1, 0 with the marked bases given by $1 \in \mathbb{Z}$ and $(e_1, e_2) := ((1, 0), (0, 1))$ of $\mathbb{Z} \oplus \mathbb{Z}$. This is related to a cellular chain complex of the mapping cylinder of a d -fold covering $S^1 \rightarrow S^1$.

If $f \in \text{Hom}_{\mathbb{Z}}(C_1, \mathbb{F})$ is a cocycle, the cocycle condition $0 = \delta^1 f(1) = f(\partial_2(1)) = d \cdot f(e_1) + f(e_2)$ forces

$$|f(e_2)| = |-d \cdot f(e_1)| = d \cdot |f(e_1)|.$$

From this and an explicit description of $H^1(C_*; \mathbb{F}/\mathbb{Z})$ and $H^1(C_*; \mathbb{F})$, we can calculate $L_1^{\mathbb{F}}(C_*) = d/2$. But the chain homotopy equivalence type of C_* is independent of d .

Proposition 5.5 (monotonicity). *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$, let $k \in \mathbb{N}$, and let C_* and D_* be marked free $\mathbb{Z}$ -chain complexes. Let $f_*: C_* \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow D_* \otimes_{\mathbb{Z}} \mathbb{F}$ be an $\mathbb{F}$ -chain map that induces an $\mathbb{F}$ -isomorphism $H_k(f_*): H_k(C_*; \mathbb{F}) \rightarrow H_k(D_*; \mathbb{F})$ and that satisfies the integrality condition $H_k(f_*)(H_k(C_*)_{\mathbb{F}}) \subset H_k(D_*)_{\mathbb{F}}$. Then, we have*

$$L_k^{\mathbb{F}}(C_*) \leq \|f_k\| \cdot L_k^{\mathbb{F}}(D_*).$$

Proof. In view of the natural isomorphism $\text{Hom}_{\mathbb{Z}}(\cdot, \mathbb{F}) \cong_{\mathbb{F}} \text{Hom}_{\mathbb{F}}(\cdot \otimes_{\mathbb{Z}} \mathbb{F}, \mathbb{F})$, the chain map f_* induces a cochain map $\text{Hom}_{\mathbb{Z}}(D_*, \mathbb{F}) \rightarrow \text{Hom}_{\mathbb{Z}}(C_*, \mathbb{F})$. This cochain map induces, by the homological assumption on f_* and the universal coefficient theorem, an $\mathbb{F}$ -isomorphism $H^k(f_*): H^k(D_*; \mathbb{F}) \rightarrow H^k(C_*; \mathbb{F})$ with $\|H^k(f_*)\| \leq \|f_k\|$.

Let $\varphi \in H^k(C_*; \mathbb{F}/\mathbb{Z})$ be in the image of the reduction homomorphism π , say $\varphi = \pi(\varphi')$ with $\varphi' \in H^k(C_*; \mathbb{F})$. We show that $L_k^{\mathbb{F}}(C_*, \varphi) \leq \|f_k\| \cdot L_k^{\mathbb{F}}(D_*)$. Taking the supremum over all φ then completes the proof.

We consider the transported classes

$$\psi' := H^k(f_*)^{-1}(\varphi') \in H^k(D_*; \mathbb{F}) \quad \text{and} \quad \psi := \pi(\psi') \in H^k(D_*; \mathbb{F}/\mathbb{Z}).$$

and show that $L_k^{\mathbb{F}}(C_*, \varphi) \leq \|f_k\| \cdot L_k^{\mathbb{F}}(D_*, \psi)$: Let $\tilde{\psi} \in H^k(D_*; \mathbb{F})$ with $\pi(\tilde{\psi}) = \psi$. We set

$$\tilde{\varphi} := H^k(f_*)(\tilde{\psi}) \in H^k(C_*; \mathbb{F}).$$

Then, we obtain:

- By construction, $\|\tilde{\varphi}\|_{\infty} \leq \|H^k(f_*)\| \cdot \|\tilde{\psi}\|_{\infty} \leq \|f_k\| \cdot \|\tilde{\psi}\|_{\infty}$.
- Moreover, $\pi(\tilde{\varphi}) = \varphi$, which can be seen as follows: In view of the universal coefficient theorem and $\pi(\varphi') = \varphi$, we only need to show that

$$\forall \alpha \in H_k(C_*) \quad \langle \tilde{\varphi}, \alpha \rangle - \langle \varphi', \alpha \rangle \in \mathbb{Z}$$

(Remark 5.2). Let $\alpha \in H_k(C_*)$. Then, we have

$$\begin{aligned} \langle \tilde{\varphi}, \alpha \rangle - \langle \varphi', \alpha \rangle &= \langle \tilde{\varphi} - \varphi', \alpha \rangle = \langle H^k(f_*)(\tilde{\psi} - \psi'), \alpha \rangle \\ &= \langle \tilde{\psi} - \psi', H_k(f_*)(\alpha) \rangle. \end{aligned}$$

this difference lies in $\mathbb{Z}$, because $\pi(\tilde{\psi}) = \psi = \pi(\psi')$ and $H_k(f_*)(\alpha) \in H_k(D_*)_{\mathbb{F}}$ by the integrality condition $H_k(f_*)(H_k(C_*)_{\mathbb{F}}) \subset H_k(D_*)_{\mathbb{F}}$.

We conclude that $L_k^{\mathbb{F}}(C_*, \varphi) \leq \|f_k\| \cdot \|\tilde{\psi}\|_{\infty}$. Taking the infimum over all $\tilde{\psi}$, we obtain $L_k^{\mathbb{F}}(C_*, \varphi) \leq \|f_k\| \cdot L_k^{\mathbb{F}}(D_*, \psi) \leq \|f_k\| \cdot L_k^{\mathbb{F}}(D_*)$. $\square$

Proposition 5.6. *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$ and let $k \in \mathbb{N}$. Then, $L_k^{\mathbb{F}}$ (more precisely, even $2 \cdot L_k^{\mathbb{F}}$) is a size invariant over $\mathbb{F}$ in degree k in the sense of Section 4.*

Proof. The cohomological lifting volume $L_k^{\mathbb{F}}$ and $2 \cdot L_k^{\mathbb{F}}$ satisfy

- normalisation by Example 5.3 and
- monotonicity by Proposition 5.5. $\square$

In particular, we obtain the corresponding version of the chain extension estimate and finiteness. For convenience, we spell them out explicitly:

Corollary 5.7 (chain extension estimate). *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$, let $k \in \mathbb{N}$, let C_* be a marked free $\mathbb{Z}$ -chain complex, let $s: C_k \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow H_k(C_*; \mathbb{F})$ be a chain extension of C_* over $\mathbb{F}$ in degree k , and let B be a cointegral $\mathbb{F}$ -basis of $H_k(C_*; \mathbb{F})$. Then, we obtain*

$$L_k^{\mathbb{F}}(C_*) \leq \frac{1}{2} \cdot \|s\|^B,$$

where $\|\cdot\|^B$ refers to the operator norm taken with respect to the ℓ^1 -norms associated with the $\mathbb{F}$ -basis of $C_k \otimes_{\mathbb{Z}} \mathbb{F}$ induced by the marked $\mathbb{Z}$ -basis of C_k and B of $H_k(C_*; \mathbb{F})$, respectively.

Proof. Because $2 \cdot L_k^{\mathbb{F}}$ is a size invariant over $\mathbb{F}$ in degree k , this is an instance of the general chain extension estimate (Proposition 4.3). $\square$

Corollary 5.8 (finiteness). *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$, let $k \in \mathbb{N}$, and let C_* be a marked free $\mathbb{Z}$ -chain complex. If $\text{rk}_{\mathbb{Z}} C_k < \infty$, then*

$$L_k^{\mathbb{F}}(C_*) < \infty.$$

Proof. Because $L_k^{\mathbb{F}}$ is a size invariant over $\mathbb{F}$ in degree k , this is an instance of the general finiteness property (Proposition 4.5). $\square$

5.2 Integral cohomological fillings

The cohomological lifting volume gives upper bounds for integral cohomological filling problems.

Proposition 5.9 (integral cohomological fillings). *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$, let $k \in \mathbb{N}$, and let C_* be a marked free $\mathbb{Z}$ -chain complex. Let $f \in \text{Hom}_{\mathbb{Z}}(C_k, \mathbb{F})$ be a cochain such that δf is a coboundary in the cochain complex $\text{Hom}_{\mathbb{Z}}(C_*, \mathbb{Z})$. Then, for every $\varepsilon \in \mathbb{R}_{>0}$, there exists a cochain $\tilde{f} \in \text{Hom}_{\mathbb{Z}}(C_k, \mathbb{Z})$ with*

$$\delta \tilde{f} = \delta f \quad \text{and} \quad |\tilde{f} - f|_{\infty} \leq \frac{1}{2} \cdot \|\partial_k\| + L_k^{\mathbb{F}}(C_*) + \varepsilon.$$

Here, $|\cdot|_{\infty}$ and $\|\cdot\|$ refer to the norms induced by the marked $\mathbb{Z}$ -bases of C_k and C_{k-1} , respectively.

Proof. This is a straightforward translation of the corresponding statement in the simplicial context [CDMW18, Lemma 3.9] to our setting: Let π denote the maps on the level of cochains and cohomology induced by the canonical projection $\mathbb{F} \rightarrow \mathbb{F}/\mathbb{Z}$.

As a first step, we consider the cochain $\pi(f) \in \text{Hom}_{\mathbb{Z}}(C_k, \mathbb{F}/\mathbb{Z})$. Because δf is $\mathbb{Z}$ -valued (as coboundary in $\text{Hom}_{\mathbb{Z}}(C_*, \mathbb{Z})$), the cochain $\pi(f)$ in fact is a cocycle. We define

$$\varphi := [\pi(f)] \in H^k(C_*; \mathbb{F}/\mathbb{Z}).$$

We show that φ lies in the image of $\pi: H^k(C_*; \mathbb{F}) \rightarrow H^k(C_*; \mathbb{F}/\mathbb{Z})$: By hypothesis, there is a cochain $f' \in \text{Hom}_{\mathbb{Z}}(C_k, \mathbb{Z})$ with

$$\delta f = \delta f'.$$

Then $f - f' \in \text{Hom}_{\mathbb{Z}}(C_k, \mathbb{F})$ is a cocycle and $f - f'$ lifts $\pi(f)$. Therefore, $\pi([f - f']) = [\pi(f)] = \varphi$, which shows that φ is in the image of π .

Let $\varepsilon \in \mathbb{R}_{>0}$. By definition of the lifting volume, there exists a class $\tilde{\varphi} \in H^k(C_*; \mathbb{F})$ with $\pi(\tilde{\varphi}) = \varphi$ that admits a representing cocycle $g \in \text{Hom}_{\mathbb{Z}}(C_k, \mathbb{F})$ with $|g|_{\infty} \leq L_k^{\mathbb{F}}(C_*) + \varepsilon$. The cochain $f - g$ fills δf , but might not be $\mathbb{Z}$ -valued; this can be adjusted as follows: Because $\pi(g)$ and $\pi(f)$ represent the same class φ , there exists a cochain $h \in \text{Hom}_{\mathbb{Z}}(C_{k-1}, \mathbb{F}/\mathbb{Z})$ with $\delta h = \pi(g) - \pi(f)$. We pick a lift $\tilde{h} \in \text{Hom}_{\mathbb{Z}}(C_{k-1}, \mathbb{F})$ of h with $|\tilde{h}|_{\infty} \leq 1/2$ and finally set

$$\tilde{f} := f - g + \delta \tilde{h} \in \text{Hom}_{\mathbb{Z}}(C_k, \mathbb{F}).$$

Then $\tilde{f}$ has the desired properties:

1. The cochain $\tilde{f}$ is $\mathbb{Z}$ -valued, because

$$\pi(\tilde{f}) = \pi(f - g + \delta \tilde{h}) = \pi(f) - \pi(g) + \pi(\delta \tilde{h}) = \pi(f) - \pi(g) + \delta h = 0.$$

2. We have $\delta \tilde{f} = \delta f$, because

$$\delta \tilde{f} = \delta(f - g + \delta \tilde{h}) = \delta f - \delta g + \delta \circ \delta(\tilde{h}) = \delta f - 0 + 0.$$

3. We have

$$|\tilde{f} - f|_{\infty} = |-g + \delta^{k-1} \tilde{h}|_{\infty} \leq |g|_{\infty} + \|\delta^{k-1}\| \cdot |\tilde{h}|_{\infty} \leq L_k^{\mathbb{F}}(C_*) + \varepsilon + \|\partial_k\| \cdot \frac{1}{2}. \quad \square$$

5.3 Integral cohomological fillings on finite coverings

We reformulate the integral cohomological filling estimate in terms of the expansion terminology of Bader and Sauer [BS26, Section 2] and give examples in the setting of finite coverings.

Definition 5.10 (expansion constant). Let R be a normed ring (e.g., a subring of $\mathbb{R}$), let M, N be normed R -modules, and let $f: M \rightarrow N$ be R -linear.

- For $y \in \text{im } f$, we define the *expansion constant of f at y* by

$$\Xi^R(f, y) := \inf \{ c \in \mathbb{R}_{\geq 0} \mid \exists_{x \in f^{-1}(y) \subset M} \quad |x| \leq c \cdot |y| \}.$$

- The *expansion constant of f* is

$$\Xi^R(f) := \sup_{y \in \text{im } f} \Xi^R(f, y) \in [0, \infty].$$

- If C_* is a marked free $\mathbb{Z}$ -chain complex and $k \in \mathbb{N}$, we equip the dual chain complex with the respective $|\cdot|_{\infty}$ -norms and abbreviate

$$\Xi_k^R(C_*) := \Xi^R(\text{Hom}_{\mathbb{Z}}(\partial_{k+1}, R): \text{Hom}_{\mathbb{Z}}(C_k, R) \rightarrow \text{Hom}_{\mathbb{Z}}(C_{k+1}, R)) \in [0, \infty].$$

Corollary 5.11 (integral cohomological fillings via expansion constants). *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$, let $k \in \mathbb{N}$, and let C_* be a marked free $\mathbb{Z}$ -chain complex. Then:*

$$\Xi_k^{\mathbb{Z}}(C_*) \leq \Xi_k^{\mathbb{F}}(C_*) + \frac{1}{2} \cdot \|\partial_k\| + L_k^{\mathbb{F}}(C_*).$$

Proof. We only need to unfold the definition of the expansion constant and apply the integral cohomological filling estimate (Proposition 5.9):

Without loss of generality, we may assume that $\Xi_k^{\mathbb{F}}(C_*)$ is finite. Let $c \in \mathbb{R}$ with $c > \Xi_k^{\mathbb{F}}(C_*)$. Let $g \in \text{Hom}_{\mathbb{Z}}(C_k, \mathbb{Z})$ be in the image of the coboundary operator $\delta_{\mathbb{Z}}$ on $\text{Hom}_{\mathbb{Z}}(C_*, \mathbb{Z})$, without out loss of generality let $g \neq 0$. Then, there exists a cochain $f \in \text{Hom}_{\mathbb{Z}}(C_*, \mathbb{F})$ with

$$\delta_{\mathbb{F}} f = g \quad \text{and} \quad |f|_{\infty} \leq c \cdot |g|_{\infty}.$$

Given $\varepsilon \in \mathbb{R}_{>0}$, by the integral cohomological filling estimate (Proposition 5.9), there is an integral cochain $\tilde{f} \in \text{Hom}_{\mathbb{Z}}(C_*, \mathbb{Z})$ with $\delta_{\mathbb{Z}} \tilde{f} = g$ and

$$\begin{aligned} |\tilde{f}|_{\infty} &\leq |f|_{\infty} + \frac{1}{2} \cdot \|\partial_k\| + L_k^{\mathbb{F}}(C_*) + \varepsilon \\ &\leq c \cdot |g|_{\infty} + \frac{1}{2} \cdot \|\partial_k\| + L_k^{\mathbb{F}}(C_*) + \varepsilon \\ &\leq \left(c + \frac{1}{2} \cdot \|\partial_k\| + L_k^{\mathbb{F}}(C_*) + \varepsilon\right) \cdot |g|_{\infty}; \end{aligned}$$

for the last step, we used integrality of g : If $g \neq 0$, then $|g|_{\infty} \geq 1$. Taking $\varepsilon \rightarrow 0$ and the infimum over all $c > \Xi_k^{\mathbb{F}}(C_*)$ proves the claim. $\square$

We denote the cellular chain complex functor by C_*^{cell} (marked by the bases of open cells) and implicitly use that cellular homology and singular homology are naturally isomorphic.

Corollary 5.12. *Let X be a finite CW-complex, let $p: \bar{X} \rightarrow X$ be a finite sheeted covering map, and let $\bar{X}$ be equipped with the CW-structure induced by p . Let $k \in \mathbb{N}$, let $\mathbb{F}$ be a subfield of $\mathbb{R}$ and assume that $\dim_{\mathbb{F}} H_k(\bar{X}; \mathbb{F}) = \dim_{\mathbb{F}} H_k(X; \mathbb{F})$.*

1. *Then:*

$$L_k^{\mathbb{F}}(C_*^{\text{cell}}(\bar{X})) \leq L_k^{\mathbb{F}}(C_*^{\text{cell}}(X))$$

2. *In particular, denoting the cellular boundary operator of the universal covering $\tilde{X}$ of X by $\tilde{\partial}_*$, we obtain:*

$$\Xi_k^{\mathbb{Z}}(C_*^{\text{cell}}(\bar{X})) \leq \Xi_k^{\mathbb{F}}(C_*^{\text{cell}}(\bar{X})) + \frac{1}{2} \cdot \|\tilde{\partial}_k\| + L_k^{\mathbb{F}}(C_*^{\text{cell}}(X))$$

Proof. *Ad 1.* This is an instance of monotonicity of the lifting volume: Transfer shows that the induced map $H_k(p; \mathbb{F}): H_k(\bar{X}; \mathbb{F}) \rightarrow H_k(X; \mathbb{F})$ is surjective. As both $\mathbb{F}$ -vector spaces are finite-dimensional (both CW-complexes are finite) and of the same dimension (by assumption), $H_k(p; \mathbb{F})$ is an $\mathbb{F}$ -isomorphism. The integrality condition is satisfied because $H_k(p; \mathbb{F})$ is induced by a continuous map. Therefore, monotonicity (Proposition 5.5) implies that $L_k^{\mathbb{F}}(C_*^{\text{cell}}(\bar{X})) \leq \|C_k^{\text{cell}}(p)\| \cdot L_k^{\mathbb{F}}(C_*^{\text{cell}}(X)) \leq 1 \cdot L_k^{\mathbb{F}}(C_*^{\text{cell}}(X))$.

Ad 2. We combine the estimate of the first part with Corollary 5.11 and the straightforward estimate $\|\partial_k^{\text{cell}, \bar{X}}\| \leq \|\tilde{\partial}_k\|$. $\square$

Remark 5.13 (uniformity). The point of the second part of Corollary 5.12 is that $\frac{1}{2} \cdot \|\tilde{\partial}_k\| + L_k^{\mathbb{F}}(C_*^{\text{cell}}(X))$ does not depend on the finite covering $\bar{X}$, but only on the base complex X (and its universal covering). Also, because X is a finite CW-complex, both $\|\tilde{\partial}_k\|$ and $L_k^{\mathbb{F}}(C_*^{\text{cell}}(X))$ are finite.

Example 5.14. Let X be a finite connected CW-complex whose fundamental group Γ is Kazhdan. Then, there exists a constant $c \in \mathbb{R}_{>0}$ such that: For all finite coverings $\bar{X} \rightarrow X$ (with the induced CW-structure), we have

$$\Xi_0^{\mathbb{Z}}(C_*^{\text{cell}}(\bar{X})) \leq c \quad \text{and} \quad \Xi_1^{\mathbb{Z}}(C_*^{\text{cell}}(\bar{X})) \leq c.$$

Indeed, we only need to consider the case that $\bar{X}$ is connected. As a first step, we apply Corollary 5.12 in this situation. The homology condition is satisfied in degree 0, because X and $\bar{X}$ are connected and thus $H_0(\bar{X}; \mathbb{R}) \cong_{\mathbb{R}} \mathbb{R} \cong_{\mathbb{R}} H_0(X; \mathbb{R})$. The homology condition is satisfied in degree 1, because Γ is Kazhdan, and so $H_1(X; \mathbb{R}) \cong_{\mathbb{R}} H_1(\Gamma; \mathbb{R}) \cong_{\mathbb{R}} 0$; the same reasoning applies to $H_1(\bar{X}; \mathbb{R})$, because the fundamental group of $\bar{X}$ is Kazhdan as finite index subgroup of the Kazhdan group Γ .

We then conclude via the observation on uniformity in Remark 5.13, and the result of Bader and Sauer showing that $\Xi_0^{\mathbb{R}}(C_*^{\text{cell}}(\bar{X}))$ and $\Xi_1^{\mathbb{R}}(C_*^{\text{cell}}(\bar{X}))$ are uniformly bounded in this situation [BS26, Theorem 2.8].

This estimate is slightly weaker than the version of Bader and Sauer [BS26, Theorem 3.13], which gives equality between the integral and the real expansion constants. However, for their proofs of waist inequalities, the above version would be sufficient.

Corollary 5.12 also admits a corresponding version for the singular chain complex (and more general spaces). However, in general, lifting volumes of singular chain complexes will not be finite.

Example 5.15. Let $\bar{M} \rightarrow M$ be a finite sheeted covering of oriented closed connected manifolds. We write $n := \dim M = \dim \bar{M}$ and equip M with a finite CW-structure. Let $\mathbb{F}$ be a subfield of $\mathbb{R}$. Then $\dim_{\mathbb{F}} H_n(\bar{M}; \mathbb{F}) = 1 = \dim_{\mathbb{F}} H_n(M; \mathbb{F})$ and so Corollary 5.12 applies in this situation (both to the cellular chain complex and the singular chain complex). We will return to this in Example 7.19.

For general finite sheeted covering maps, the cohomological lifting volume does not satisfy monotonicity (as can, e.g., be seen in degree 2 for the double covering $S^2 \rightarrow \mathbb{R}P^2$). However, it is unknown, whether the cohomological lifting volume is uniformly bounded over a given base:

Question 5.16. Let X be a finite CW-complex and $k \in \mathbb{N}$. Does there exist a constant $c \in \mathbb{R}_{\geq 0}$ with the following property? For every finite sheeted covering $\bar{X} \rightarrow X$ (where $\bar{X}$ is equipped with the induced CW-structure), we have $L_k^{\mathbb{F}}(C_*^{\text{cell}}(\bar{X})) \leq c$.

6 Cohomological lifting for subdivisions

Monotonicity of the cohomological lifting volume gives uniform control for cohomological liftings and fillings on subdivisions of simplicial complexes (Theorem 6.4, Corollary 6.5). As a preparation, we first recall basics on simplicial subdivisions.

We fix notation on simplicial complexes, in the combinatorial sense: A *simplicial complex* is a set X of finite sets, closed under taking subsets. The elements of X are called *simplices*. If $\sigma \in X$, we write $\dim \sigma := \#\sigma - 1$. For $n \in \mathbb{N}$, we

write $X_n := \{\sigma \in X \mid \dim \sigma = n\}$ for the set of all n -simplices. A *simplicial map* $X \rightarrow Y$ between simplicial complexes X, Y is a map $f: X_0 \rightarrow Y_0$ with

$$\forall \sigma \in X \quad f(\sigma) \in Y.$$

The geometric realisation of simplicial complexes is denoted by $|\cdot|$ and we extend this notation to simplices.

We denote the simplicial chain complex and simplicial homology functors on simplicial complexes by C_*^Δ and H_*^Δ , respectively. We view the simplicial chain complex as marked through the marked bases given by the simplices in the respective dimension; more precisely, to avoid sign ambiguities, we pick an ordering on the set of vertices and consider ordered simplices.

6.1 Simplicial subdivisions

We recall the definition of subdivisions of simplicial complexes and the key fact that each subdivision admits a simplicial map to the original simplicial complex that induces an isomorphism in homology.

Definition 6.1 (subdivision of a simplicial complex). Let X be a simplicial complex. A *subdivision of X* is a pair (X', i) , consisting of a simplicial complex X' and a homeomorphism $i: |X'| \rightarrow |X|$ with the following properties:

- We have $X_0 \subset X'_0$ and $i|_{X_0} = \text{id}_{X_0}$.
- If $\sigma \in X$, then for the subcomplex $Y := \{\tau \mid \tau \in X', i(|\tau|) \subset |\sigma|\}$ of X' , the restriction map $i|_{|Y|}: |Y| \rightarrow |\sigma|$ is a homeomorphism and moreover $Y = \{\tau \mid \tau \in X', i(\tau) \subset |\sigma|\}$ (convexity).

Usually, we suppress i in the notation; we then speak of a subdivision X' of X and write $|\tau| \subset |\sigma|$ if τ and σ are simplices of X' and X , respectively, satisfying $i(|\tau|) \subset |\sigma|$.

As a consequence of this definition, for each simplex in the given subdivision X' , its geometric realisation is contained in the geometric realisation of some simplex of the original simplicial complex X . Moreover, the simplices of X induce a corresponding structure of (finite) subcomplexes of X' .

An example of a systematic simplicial subdivision is the barycentric subdivision.

Lemma 6.2. *Let X be a finite simplicial complex and let X' be a subdivision of X . Then, there exists a simplicial map $f: X' \rightarrow X$ such that: For every field $\mathbb{F}$ and every $k \in \mathbb{N}$, the induced map $H_k^\Delta(f; \mathbb{F}): H_k^\Delta(X'; \mathbb{F}) \rightarrow H_k^\Delta(X; \mathbb{F})$ is an $\mathbb{F}$ -isomorphism.*

Proof. We pick a total ordering on the vertices of X' (and hence also on the vertices and faces of X). For each vertex $x \in X'_0$, we write σ_x for the support simplex of x in X , i.e., for the smallest simplex σ_x in X such that $|x|$ is contained in the relative interior of $|\sigma_x|$. We define

$$\begin{aligned} f: X'_0 &\rightarrow X_0 \\ x &\mapsto \max \sigma_x \end{aligned}$$

and show the following:

1. The map f is a simplicial map $X' \rightarrow X$.
2. If σ is a simplex of X and $n := \dim \sigma$, then there exists precisely one n -simplex σ' of X' with

$$|\sigma'| \subset |\sigma| \quad \text{and} \quad f(\sigma') = \sigma.$$

All other n -simplices τ of X' with $|\tau| \subset |\sigma|$ satisfy $\dim f(\tau) < n$.

3. There exists a chain map $g: C_*^\Delta(X) \rightarrow C_*^\Delta(X')$ with $C_*^\Delta(f) \circ g = \text{id}_{C_*^\Delta(X)}$.

Once these properties are established, we can apply Lemma 6.3 to conclude that $H_k^\Delta(f; \mathbb{F})$ is an isomorphism for every field $\mathbb{F}$ and every $k \in \mathbb{N}$.

Instead of invoking Lemma 6.3, one could also geometrically construct a chain homotopy $g \circ C_*^\Delta(f) \simeq \text{id}_{C_*^\Delta(X')}$.

Ad 1. Because X' is a subdivision of X , for every simplex $\tau = \{x_0, \dots, x_n\} \in X'_n$, there exists a simplex σ of X with $\sigma_{x_0}, \dots, \sigma_{x_n} \subset \sigma$ (every simplex σ with $|\tau| \subset |\sigma|$ will do). Hence, also

$$f(\tau) = \{f(x_0), \dots, f(x_n)\} = \{\max \sigma_{x_0}, \dots, \max \sigma_{x_n}\} \subset \sigma,$$

and so $f(\tau)$ is a simplex of X . Thus, f is a simplicial map $X' \rightarrow X$.

Ad 2. This can be shown by induction on the dimension n : The conditions are clearly satisfied for $n = 0$. For the induction step, let $n \in \mathbb{N}$ and let the claim be shown in dimension n . Let $\sigma \in X_{n+1}$.

We first show existence of $\sigma' \in X'_{n+1}$ with $|\sigma'| \subset |\sigma|$ and $f(\sigma') = \sigma$. By induction, for the 0-face σ_0 of σ , there exists a unique n -simplex $\sigma'_0 \in X'$ with $|\sigma'_0| \subset |\sigma_0|$ and $f(\sigma'_0) = \sigma_0$. Because X' is a simplicial subdivision of X , for topological reasons, there is exactly one $\sigma' \in X'_{n+1}$ such that σ'_0 is a face of σ' and σ' has a vertex v in $|\sigma| \setminus |\sigma'_0|$. By construction, $f(v)$ is the 0-vertex of σ and so $f(\sigma') = \sigma$.

If $\tau \in X'_{n+1}$ with $f(\tau) = \sigma$, then all vertices of τ lie in $|\sigma|$ (by construction of f) and so $|\tau| \subset |\sigma|$ (by convexity). Thus, uniqueness of σ' follows from the next property.

Let $\tau \in X'$ with $|\tau| \subset |\sigma|$ and $\dim f(\tau) = n + 1$. Then, $\tau = \sigma'$. Indeed, by induction, there is only one n -simplex $\sigma'_0 \in X'$ with $f(\sigma'_0) = \sigma_0$. Hence, $\sigma'_0 \subset \tau$. Moreover, because $|\tau| \subset |\sigma|$, we have $f(\tau) \subset \sigma$. Thus, $\dim f(\tau) = n + 1$ implies that $f(\tau) = \sigma$ and so τ must have a vertex in $|\sigma| \setminus |\sigma'_0|$. But then τ must coincide with σ' , as constructed in the first part of the induction step.

Ad 3. We consider the map

$$g: C_*^\Delta(X) \rightarrow C_*^\Delta(X') \\ X_n \ni \sigma \mapsto \sum_{\tau \in X'_n} \varepsilon_\tau \cdot \tau,$$

where $\varepsilon_\tau \in \{1, -1\}$ corresponds to the orientation of $|\tau|$ relative to $|\sigma|$.

Inductively, we see that for each $\sigma \in X_n$, the simplex $\sigma' \in X'_n$ from Step 2 satisfies $\varepsilon_{\sigma'} = 1$, and so the properties from Step 2 guarantee that

$$C_*^\Delta(f) \circ g(\sigma) = 1 \cdot \sigma + 0 = \text{id}_{C_*^\Delta(X)}(\sigma). \quad \square$$

Lemma 6.3. *Let X, Y be finite simplicial complexes, let $k \in \mathbb{N}$, and let $\mathbb{F}$ be a field with $H_k(X; \mathbb{F}) \cong_{\mathbb{F}} H_k(Y; \mathbb{F})$. Let $f: Y \rightarrow X$ be a simplicial map that admits a chain map $g: C_*^\Delta(X) \rightarrow C_*^\Delta(Y)$ with $C_*^\Delta(f) \circ g \simeq \text{id}_{C_*^\Delta(X)}$. Then $H_*^\Delta(f; \mathbb{F}): H_*^\Delta(Y; \mathbb{F}) \rightarrow H_*^\Delta(X; \mathbb{F})$ is an $\mathbb{F}$ -isomorphism.*

Proof. Every chain homotopy $C_*^\Delta(f) \circ g \simeq \text{id}_{C_*^\Delta(X)}$ induces a chain homotopy $C_*^\Delta(f; \mathbb{F}) \circ (g \otimes_{\mathbb{Z}} \text{id}_{\mathbb{F}}) \simeq \text{id}_{C_*^\Delta(X; \mathbb{F})}$. In particular, $H_k^\Delta(f; \mathbb{F}): H_k^\Delta(Y; \mathbb{F}) \rightarrow H_k^\Delta(X; \mathbb{F})$ is surjective. Because $H_*^\Delta(X; \mathbb{F})$ and $H_*^\Delta(Y; \mathbb{F})$ are finite-dimensional $\mathbb{F}$ -vector spaces of the same dimension, $H_k^\Delta(f; \mathbb{F})$ is an isomorphism. $\square$

6.2 Cohomological lifting/filling for subdivisions

We apply monotonicity of the cohomological lifting volume to simplicial subdivisions to obtain uniform estimates for cohomological liftings and fillings. In contrast with the original proof for estimates of this type [CDMW18, Section 3], the purely algebraic argument in Theorem 6.4 uses the flexibility of the algebraic setting to bypass an intricate explicit construction of suitable spanning trees. Moreover, this argument is robust enough to generalise in a uniform way to all simplicial subdivisions:

Theorem 6.4 (cohomological lifting for subdivisions). *Let X be a finite simplicial complex, let X' be a simplicial subdivision of X , let $k \in \mathbb{N}$, and let $\mathbb{F}$ be a subfield of $\mathbb{R}$. Then*

$$L_k^\mathbb{F}(C_*^\Delta(X')) \leq L_k^\mathbb{F}(C_*^\Delta(X)).$$

The right-hand side is finite by Corollary 5.8.

Proof. Because X' is a subdivision of X , there exists a simplicial map $f: X' \rightarrow X$ such that $H_k^\Delta(f; \mathbb{F}): H_k^\Delta(X'; \mathbb{F}) \rightarrow H_k^\Delta(X; \mathbb{F})$ is an isomorphism (Lemma 6.2). Moreover, f being simplicial implies that $\|C_k^\Delta(f; \mathbb{F})\| \leq 1$ (with respect to the marked bases given by the simplices of X' and X , respectively) and that $H_k^\Delta(f; \mathbb{F})$ maps integral classes to integral classes. Monotonicity of the lifting volume (Proposition 5.5) thus gives

$$L_k^\mathbb{F}(C_*^\Delta(X')) \leq \|C_k^\Delta(f; \mathbb{F})\| \cdot L_k^\mathbb{F}(C_*^\Delta(X)) \leq 1 \cdot L_k^\mathbb{F}(C_*^\Delta(X)). \quad \square$$

Corollary 6.5 (integral cohomological fillings for subdivisions [CDMW18, Lemma 3.9]). *Let X be a finite simplicial complex, let $k \in \mathbb{N}$, and let $\mathbb{F}$ be a subfield of $\mathbb{R}$. Then there exists a constant $c \in \mathbb{R}_{>0}$ such that: If X' is a simplicial subdivision of X and $f \in C_\Delta^k(X'; \mathbb{F})$ is a simplicial cochain such that δf is a coboundary in $C_\Delta^*(X'; \mathbb{Z})$, then there exists a cochain $\tilde{f} \in C_\Delta^k(X'; \mathbb{Z})$ with*

$$\delta \tilde{f} = \delta f \quad \text{and} \quad |\tilde{f}|_\infty \leq |f|_\infty + c.$$

Proof. We combine the lifting volume estimate for subdivisions (Theorem 6.4) with the integral cohomological filling estimate (Proposition 5.9). We obtain a cochain $\tilde{f} \in C_\Delta^*(X'; \mathbb{Z})$ satisfying $\delta \tilde{f} = \delta f$ and

$$\begin{aligned} |\tilde{f}|_\infty &\leq |f|_\infty + \frac{1}{2} \cdot \|\partial_k^{\Delta, X'}\| + L_k^\mathbb{F}(C_*^\Delta(X')) + 1 \\ &\leq |f|_\infty + \frac{1}{2} \cdot (k+1) + L_k^\mathbb{F}(C_*^\Delta(X)) + 1. \end{aligned}$$

The term $1/2 \cdot (k+1) + L_k^{\mathbb{F}}(C_*^{\Delta}(X)) + 1$ is finite (Corollary 5.8) and depends only on k and X . $\square$

In Section 7.4, we relate this approach to the original one via spanning trees.

7 Gnarledness of chain complexes

We introduce a definition of gnarledness of chain complexes, inspired by the notion of gnarledness of simplicial complexes by Chambers, Dotterrer, Manin, and Weinberger [CDMW18]. This algebraic gnarledness is based on chain extensions and measures how close a chain complex is to a chain complex with trivial boundary operator and so, by design, is universal among size invariants. On the one hand, the chain extension estimate relates the gnarledness of chain complexes to the cohomological lifting volume. On the other hand, spanning trees give chain extensions, which provides a connection with the original, geometric, notion of gnarledness of simplicial complexes. Furthermore, we may express the simplicial volume in terms of gnarledness of the singular chain complex.

7.1 Definition and basic examples

We define the gnarledness of chain complexes in terms of the minimal “size” of chain extensions.

Definition 7.1 (gnarledness). Let C_* be a marked free $\mathbb{Z}$ -chain complex, let $k \in \mathbb{N}$, and let $\mathbb{F}$ be a subfield of $\mathbb{R}$. We write $B_k^{\mathbb{F}}(C_*)$ for the set of all cointegral $\mathbb{F}$ -bases of $H_k(C_*; \mathbb{F})$.

- If s is a chain extension of C_* in degree k over $\mathbb{F}$, we set

$$G_k^{\mathbb{F}}(C_*, s) := \inf_{B \in B_k^{\mathbb{F}}(C_*)} \|s\|^B.$$

- The *gnarledness* of C_* in degree k over $\mathbb{F}$ is defined as

$$G_k^{\mathbb{F}}(C_*) := \inf \{ G_k^{\mathbb{F}}(C_*, s) \mid s \text{ is a chain extension of } C_* \text{ in degree } k \text{ over } \mathbb{F} \}.$$

Example 7.2. We consider the $\mathbb{Z}$ -chain complex C_* given by

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z},$$

concentrated in degrees 1, 0 with the marked bases given by $1 \in \mathbb{Z}$. This is the cellular chain complex of S^1 with respect to the CW-structure consisting of a single 0-cell and a single 1-cell.

We have $G_1^{\mathbb{F}}(C_*) = 1$ for every subfield $\mathbb{F}$ of $\mathbb{R}$, which can be calculated as follows: Because $\partial_1 = 0$, there is only one chain extension s of C_* in degree 1 over $\mathbb{F}$, namely the canonical projection map

$$s: C_1 \otimes_{\mathbb{Z}} \mathbb{F} = \ker(\partial_1 \otimes_{\mathbb{Z}} \text{id}_{\mathbb{F}}) \rightarrow H_1(C_*; \mathbb{F}),$$

which is an isomorphism as $\partial_2 = 0$. Let $B = (b) \in B_1^{\mathbb{F}}(C_*)$. Then the coefficient $\lambda \in \mathbb{F}$ with $s(1) = \lambda \cdot b$ is a non-zero integer. Therefore,

$$\|s\|^B = |\lambda| \geq 1.$$

Conversely, $(s(1)) \in B_1^{\mathbb{F}}(C_*)$ and $\|s\|^{(s(1))} = 1$. Hence, $G_1^{\mathbb{F}}(C_*, s) = 1$. Since s is the only chain extension of C_* in degree 1 over $\mathbb{F}$, we conclude $G_1^{\mathbb{F}}(C_*) = 1$.

Example 7.3. Let $d \in \mathbb{N}_{>0}$. We consider the $\mathbb{Z}$ -chain complex C_* given by

$$\mathbb{Z} \xrightarrow{(d,1)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{0} \mathbb{Z},$$

concentrated in degrees 2, 1, 0 with the marked bases given by $1 \in \mathbb{Z}$ and $(e_1, e_2) := ((1, 0), (0, 1))$ of $\mathbb{Z} \oplus \mathbb{Z}$. This is related to a cellular chain complex of the mapping cylinder of a d -fold covering $S^1 \rightarrow S^1$.

We have $G_1^{\mathbb{F}}(C_*) = d$ for every subfield $\mathbb{F}$ of $\mathbb{R}$, which can be calculated as follows: Because $\partial_1 = 0$, there is again only one chain extension s of C_* in degree 1 over $\mathbb{F}$, namely the canonical projection map

$$s: C_1 \otimes_{\mathbb{Z}} \mathbb{F} = \ker(\partial_1 \otimes_{\mathbb{Z}} \text{id}_{\mathbb{F}}) \rightarrow H_1(C_*; \mathbb{F}).$$

Because s is a chain extension, we have $s((d, 1)) = 0$ and so

$$s(e_2) = s(e_2 - (d, 1)) = -d \cdot s(e_1);$$

in particular, surjectivity of s shows that $s(e_1) \neq 0$. If $B \in B_1^{\mathbb{F}}(C_*)$ is a cointegral basis of $H_1(C_*; \mathbb{F})$, then the above calculation and the integrality condition lead to

$$\|s\|^B = \max(|s(e_1)|_1^B, |s(e_2)|_1^B) = d \cdot |s(e_1)|_1^B \geq d \cdot 1 = d.$$

Hence, $G_1^{\mathbb{F}}(C_*, s) \geq d$. Conversely, $B := (s(e_1))$ lies in $B_1^{\mathbb{F}}(C_*)$ and satisfies $\|s\|^B = d$. Thus, $G_1^{\mathbb{F}}(C_*) = G_1^{\mathbb{F}}(C_*, s) = d$.

Example 7.4. The above Example 7.3 can be modified so that we obtain every rational number as the gnarledness of a finite marked free $\mathbb{Z}$ -chain complex.

We do this by looking at the cellular chain complex of finite coverings of the CW-structure behind Example 7.3: Let $m, d \in \mathbb{N}_{>0}$ and let the $\mathbb{Z}$ -chain complex $C_* \langle m \rangle$ be given by

$$\begin{aligned} \mathbb{Z}^m &\xrightarrow{\partial_2} \mathbb{Z}^m \oplus \mathbb{Z}^m \xrightarrow{\partial_1} \mathbb{Z}^m \\ e_k &\longmapsto \left(\sum_{i=0}^{d-1} e_{k+i}, e_k \right) \\ (e_k, 0) &\longmapsto -e_k + e_{k+1} \\ (0, e_k) &\longmapsto +e_k - e_{k+d} \end{aligned}$$

concentrated in degrees 2, 1, 0 and marked with the standard basis in every degree. Indices shall be considered modulo m . Then $G_1^{\mathbb{F}}(C_* \langle m \rangle) = \frac{d}{m}$.

It is straightforward to check that $H_1(C_* \langle m \rangle; \mathbb{F})$ is one-dimensional and that the integral part $H_1(C_* \langle m \rangle)_{\mathbb{F}}$ is generated by the homology class

$$b := \left[\left(\sum_{i=1}^m e_i, 0 \right) \right].$$

Similarly to Example 7.3, we have

$$\left[\left(0, \sum_{i=1}^m e_i \right) \right] = \left[- \left(\sum_{i=1}^m \sum_{j=1}^{d-1} e_{j+i}, 0 \right) \right] = \left[-d \cdot \left(\sum_{i=1}^m e_i, 0 \right) \right].$$

Let B be a cointegral basis of $H_1(C_*\langle m \rangle; \mathbb{F})$. Then $B = (\frac{1}{\lambda} \cdot b)$ for some non-zero integer λ . If $s: C_1\langle m \rangle \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow H_1(C_*\langle m \rangle; \mathbb{F})$ is a chain extension of $C_*\langle m \rangle$ in degree 1 over $\mathbb{F}$, we obtain

$$\begin{aligned} \sum_{i=1}^m |s(0, e_i)|_1^B &\geq \left| s\left(0, \sum_{i=1}^m e_i\right) \right|_1^B = \left| \left[\left(0, \sum_{i=1}^m e_i\right) \right] \right|_1^B = \left| -d \cdot \left[\left(\sum_{i=1}^m e_i, 0 \right) \right] \right|_1^B \\ &= d \cdot |b|^B = d \cdot |\lambda|. \end{aligned}$$

Hence, we have $\|s\|^B \geq \frac{d \cdot |\lambda|}{m}$. Taking the infimum over all chain extensions and cointegral bases gives $G_1^{\mathbb{F}}(C_*\langle m \rangle) \geq \frac{d}{m}$.

On the other hand, we consider the $\mathbb{F}$ -linear map given by

$$\begin{aligned} s: C_1\langle m \rangle \otimes_{\mathbb{Z}} \mathbb{F} &\rightarrow H_1(C_*\langle m \rangle; \mathbb{F}) \\ (e_k, 0) &\mapsto \frac{1}{m} \cdot b \\ (0, e_k) &\mapsto -\frac{d}{m} \cdot b. \end{aligned}$$

Then s is a chain extension of $C_*\langle m \rangle$ in degree 1 over $\mathbb{F}$ and for the cointegral basis $B = (b)$ we compute $G_1^{\mathbb{F}}(C_*\langle m \rangle) \leq \|s\|^B = \frac{d}{m}$.

This inequality is also a consequence of monotonicity of gnarledness (Proposition 7.9): Since $C_*\langle m \rangle$ is related to the cellular complex of the m -fold covering of the mapping cylinder of Example 7.3, transfer gives us the chain map

$$\begin{aligned} f_*: C_*\langle m \rangle \otimes_{\mathbb{Z}} \mathbb{F} &\rightarrow C_* \otimes_{\mathbb{Z}} \mathbb{F} \\ \sigma &\mapsto \frac{1}{m} \cdot p(\sigma), \end{aligned}$$

where C_* denotes the chain complex of Example 7.3, p is the covering map and σ is a cell of the total space. By monotonicity and Example 7.3, we obtain

$$G_1^{\mathbb{F}}(C_*\langle m \rangle) \leq \frac{1}{m} \cdot G_1^{\mathbb{F}}(C_*) = \frac{d}{m}.$$

7.2 Comparison with the lifting volume

Up to a Betti number factor, gnarledness coincides with the lifting volume. Hence, gnarledness provides good estimates for liftings and integral fillings of cohomology classes.

Proposition 7.5. *Let C_* be a marked free $\mathbb{Z}$ -chain complex, let $k \in \mathbb{N}$, and let $\mathbb{F}$ be a subfield of $\mathbb{R}$. Let $H_k(C_*; \mathbb{Z})$ be finitely generated. Then:*

$$L_k^{\mathbb{F}}(C_*) \leq \frac{1}{2} \cdot G_k^{\mathbb{F}}(C_*) \leq \dim_{\mathbb{F}} H_k(C_*; \mathbb{F}) \cdot L_k^{\mathbb{F}}(C_*).$$

Proof. The first estimate is the chain extension estimate (Corollary 5.7).

For the second estimate, we construct chain extensions from efficient liftings: More precisely, we show the following for each $\varepsilon \in \mathbb{R}_{>0}$:

1. There exists a cointegral basis B of $H_k(C_*; \mathbb{F})$ such that the dual family $(\varphi_{\beta})_{\beta \in B}$ in $H^k(C_*; \mathbb{F})$ admits representing cocycles $(f_{\beta})_{\beta \in B}$ with

$$|f_{\beta}|_{\infty} \leq 2 \cdot (L_k^{\mathbb{F}}(C_*) + \varepsilon).$$

2. The associated map

$$\begin{aligned} s: C_k \otimes_{\mathbb{Z}} \mathbb{F} &\longrightarrow H_k(C_*; \mathbb{F}) \\ x \otimes \lambda &\longmapsto \lambda \cdot \sum_{\beta \in B} f_{\beta}(x) \cdot \beta \end{aligned}$$

is a chain extension of C_* over $\mathbb{F}$ in degree k

3. and $\|s\|^B \leq 2 \cdot \dim_{\mathbb{F}} H_k(C_*; \mathbb{F}) \cdot (L_k^{\mathbb{F}}(C_*) + \varepsilon)$.

Taking $\varepsilon \rightarrow 0$ then proves the claimed estimate.

Ad 1. Let B be a cointegral basis of $H_k(C_*; \mathbb{F})$ obtained from a $\mathbb{Z}$ -basis of $H_k(C_*)_{\mathbb{F}}$. Let $(\psi_{\beta})_{\beta \in B}$ in $H^k(C_*; \mathbb{F})$ be dual to B . We apply the lifting volume to suitable reductions of this family; however, as this family is dual to B its reduction in $H^k(C_*; \mathbb{F}/\mathbb{Z})$ is trivial. Therefore, to get meaningful classes, for $\beta \in B$, let $\bar{\psi}_{\beta} := \pi(1/2 \cdot \psi_{\beta}) \in H^k(C_*; \mathbb{F}/\mathbb{Z})$. By definition of the lifting volume, there exist π -lifts $\varphi_{\beta} \in H^k(C_*; \mathbb{F})$ of $\bar{\psi}_{\beta}$ and cocycles $f_{\beta} \in \text{Hom}_{\mathbb{Z}}(C_k; \mathbb{F})$ representing φ_{β} with

$$|f_{\beta}|_{\infty} \leq L_k^{\mathbb{F}}(C_*) + \varepsilon.$$

In general, the family $(2 \cdot \varphi_{\beta})_{\beta \in B}$ will not be dual to B , but we still have almost-duality of the following form: For all $\beta, \gamma \in B$, we have

$$\langle 2 \cdot \varphi_{\beta}, \gamma \rangle \in 2 \cdot \left(\mathbb{Z} + \left\langle \frac{1}{2} \cdot \psi_{\beta}, \gamma \right\rangle \right) = 2 \cdot \left(\mathbb{Z} + \frac{1}{2} \cdot \langle \psi_{\beta}, \gamma \rangle \right) = \begin{cases} 2 \cdot \mathbb{Z} & \text{if } \beta \neq \gamma \\ 2 \cdot \mathbb{Z} + 1 & \text{if } \beta = \gamma, \end{cases}$$

because $\pi(\varphi_{\beta}) = \pi(1/2 \cdot \psi_{\beta})$ and the family $(\psi_{\beta})_{\beta \in B}$ is dual to B .

Applying the algebraic Lemma 7.6 below shows that there exists a cointegral basis B' of $H_k(C_*; \mathbb{F})$ dual to the family $(2 \cdot \varphi_{\beta})_{\beta \in B}$. Moreover, $2 \cdot f_{\beta}$ is a cocycle representing $2 \cdot \varphi_{\beta}$ with

$$|2 \cdot f_{\beta}|_{\infty} \leq 2 \cdot (L_k^{\mathbb{F}}(C_*) + \varepsilon).$$

Renaming the family and bases thus proves the first claim. In the following, we assume that B , $(\varphi_{\beta})_{\beta \in B}$, and $(f_{\beta})_{\beta \in B}$ are chosen as in Claim 1.

Ad 2. Let Σ_k be the marked basis of C_k and let $c = \sum_{\sigma \in \Sigma_k} \sigma \otimes \lambda_{\sigma} \in C_k \otimes_{\mathbb{Z}} \mathbb{F}$ be a cycle. By construction of s as in Claim 2, we have

$$\begin{aligned} s(c) &= \sum_{\sigma \in \Sigma_k} \lambda_{\sigma} \cdot \sum_{\beta \in B} f_{\beta}(\sigma) \cdot \beta = \sum_{\beta \in B} \left(\sum_{\sigma \in \Sigma_k} \lambda_{\sigma} \cdot f_{\beta}(\sigma) \right) \cdot \beta \\ &= \sum_{\beta \in B} \langle [f_{\beta}], [c] \rangle \cdot \beta = \sum_{\beta \in B} \langle \varphi_{\beta}, [c] \rangle \cdot \beta. \end{aligned}$$

Because $(\varphi_{\beta})_{\beta \in B}$ is dual to B , this simplifies to $s(c) = [c]$. Thus, s is a chain extension.

Ad 3. By definition, $\|s\|^B = \max_{\sigma \in \Sigma_k} |s(\sigma \otimes 1)|_1^B$. For each $\sigma \in \Sigma_k$, we obtain

$$\begin{aligned} |s(\sigma \otimes 1)|_1^B &= \left| \sum_{\beta \in B} f_{\beta}(\sigma) \cdot \beta \right|_1^B = \sum_{\beta \in B} |f_{\beta}(\sigma)| \leq \sum_{\beta \in B} |f_{\beta}|_{\infty} \\ &\leq \#B \cdot 2 \cdot (L_k^{\mathbb{F}}(C_*) + \varepsilon) = 2 \cdot \dim_{\mathbb{F}} H_k(C_*; \mathbb{F}) \cdot (L_k^{\mathbb{F}}(C_*) + \varepsilon). \quad \square \end{aligned}$$

In particular, if $\dim_{\mathbb{F}} H_k(C_*; \mathbb{F}) = 1$, then $2 \cdot L_k^{\mathbb{F}}(C_*) = G_k^{\mathbb{F}}(C_*)$.

Lemma 7.6. *Let $n \in \mathbb{N}_{>0}$ and let $(\varphi_j: \mathbb{F}^n \rightarrow \mathbb{F})_{j \in \{1, \dots, n\}}$ be $\mathbb{Z}$ -linear maps with the following property:*

$$\forall_{j,k \in \{1, \dots, n\}} \quad \varphi_j(e_k) \in \begin{cases} 2 \cdot \mathbb{Z} & \text{if } j \neq k \\ 2 \cdot \mathbb{Z} + 1 & \text{if } j = k. \end{cases}$$

Then, there exists a cointegral basis B of $\mathbb{F}^n$ such that $(\varphi_j)_{j \in \{1, \dots, n\}}$ is dual to B . Here, cointegral means that $\mathbb{Z}^n \subset \mathbb{Z}B$.

Proof. We consider the matrix

$$A := (\varphi_j(e_k))_{j,k \in \{1, \dots, n\}}.$$

In view of our hypothesis, this matrix has integer coefficients, whose diagonal entries are odd and whose off-diagonal entries are even. By the Leibniz expansion formula, $\det A$ is an odd integer. In particular, $\det A \neq 0$. Therefore, there exists a matrix $\bar{A} \in M_{n \times n}(\mathbb{Q})$ that is inverse to A . Let B be the family of columns of $\bar{A}$. By invertibility, B is an $\mathbb{F}$ -basis of $\mathbb{F}^n$ and $(\varphi_j)_{j \in \{1, \dots, n\}}$ is dual to B .

It remains to show that B is cointegral. Integrality of A and $\bar{A} \cdot A = I_n$ witness that each e_k lies in $\mathbb{Z}B$. So, B is cointegral. $\square$

Corollary 7.7 (characterisation of vanishing). *Let C_* be a marked free $\mathbb{Z}$ -chain complex, let $k \in \mathbb{N}$, and let $\mathbb{F}$ be a subfield of $\mathbb{R}$. Let $H_k(C_*)$ be finitely generated. Then the following are equivalent:*

1. *We have $G_k^{\mathbb{F}}(C_*) = 0$.*
2. *We have $L_k^{\mathbb{F}}(C_*) = 0$.*
3. *We have $H_k(C_*; \mathbb{F}) \cong_{\mathbb{F}} 0$.*

Proof. The equivalence 1. $\iff$ 2. follows from the sandwich estimate in Proposition 7.5.

The implication 3. $\implies$ 2. is immediate from the definition.

For the implication 2. $\implies$ 3., we proceed as follows: Let $H_k(C_*; \mathbb{F}) \not\cong_{\mathbb{F}} 0$. We show that $L_k^{\mathbb{F}}(C_*) > 0$. Because $H_k(C_*)$ is finitely generated and $H_k(C_*; \mathbb{F}) \not\cong_{\mathbb{F}} 0$, the canonical map $\text{Hom}_{\mathbb{Z}}(H_k(C_*), \mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(H_k(C_*), \mathbb{F})$ is not surjective. By the universal coefficient theorem (and divisibility of $\mathbb{F}$), therefore, the change-of-coefficients map $H^k(C_*; \mathbb{Z}) \rightarrow H^k(C_*; \mathbb{F})$ is not surjective. In view of the long exact coefficient sequence, the reduction map $\pi: H^k(C_*; \mathbb{F}) \rightarrow H^k(C_*; \mathbb{F}/\mathbb{Z})$ is non-trivial.

Let $\varphi \in \text{im } \pi$ be non-trivial. Then, by the universal coefficient theorem (and divisibility of $\mathbb{F}/\mathbb{Z}$), there exists a class $\alpha \in H_k(C_*)$ with $\langle \varphi, \alpha \rangle \neq 0$ in $\mathbb{F}/\mathbb{Z}$. Rescaling φ , we may assume that $\langle \varphi, \alpha \rangle$ is represented by $1/2$.

Let $c \in C_k$ be a cycle representing α . We show that $L_k^{\mathbb{F}}(C_*, \varphi) \geq 1/(2 \cdot |c|_1)$: Every cocycle $f \in \text{Hom}_{\mathbb{Z}}(C_k, \mathbb{F})$ representing a π -lift of φ satisfies $\langle [f], \alpha \rangle \in \mathbb{Z} + 1/2$ and so

$$|f|_{\infty} \cdot |c|_1 \geq |f(c)| = |\langle [f], \alpha \rangle| \geq \frac{1}{2}.$$

Hence, $L_k^{\mathbb{F}}(C_*) \geq L_k^{\mathbb{F}}(C_*, \varphi) \geq 1/(2 \cdot |c|_1) > 0$. $\square$

7.3 Universality

Gnarledness is universal among size invariants in the sense of Section 4.

Proposition 7.8 (universality of gnarledness). *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$ and let $k \in \mathbb{N}$. Then, $G_k^{\mathbb{F}}$ is universal among all size invariants over $\mathbb{F}$ in degree k :*

1. *Gnarledness $G_k^{\mathbb{F}}$ is a size invariant over $\mathbb{F}$ in degree k .*
2. *If $I: \mathbf{mCh} \rightarrow [0, \infty]$ is a size invariant over $\mathbb{F}$ in degree k and C_* is a marked free $\mathbb{Z}$ -chain complex, then*

$$I(C_*) \leq G_k^{\mathbb{F}}(C_*).$$

Proof. Ad 1. It is clear from the definition that gnarledness satisfies the normalisation property. We show monotonicity below (Proposition 7.9).

Ad 2. This is immediate from the chain extension estimate for size invariants (Proposition 4.3) and the definition of gnarledness. $\square$

Proposition 7.9 (monotonicity). *Let C_* and D_* be marked free $\mathbb{Z}$ -chain complexes, let $k \in \mathbb{N}$, and let $\mathbb{F}$ be a subfield of $\mathbb{R}$. Let $f_*: C_* \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow D_* \otimes_{\mathbb{Z}} \mathbb{F}$ be an $\mathbb{F}$ -chain map that induces an $\mathbb{F}$ -isomorphism $H_k(f_*): H_k(C_*; \mathbb{F}) \rightarrow H_k(D_*; \mathbb{F})$ and that satisfies the integrality condition $H_k(f_*)(H_k(C_*)_{\mathbb{F}}) \subset H_k(D_*)_{\mathbb{F}}$. Then, we have*

$$G_k^{\mathbb{F}}(C_*) \leq \|f_k\| \cdot G_k^{\mathbb{F}}(D_*),$$

where $\|f_k\|$ refers to the operator norm taken with respect to the ℓ^1 -norms associated with the marked bases on C_k and D_k , respectively.

Proof. This is a straightforward estimate, based on the observation in Lemma 2.4: Let $s: D_k \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow H_k(D_*; \mathbb{F})$ be a chain extension of D_* in degree k over $\mathbb{F}$ and let B be a cointegral basis of $H_k(D_*; \mathbb{F})$. Then

$$\tilde{s} := H_k(f_*)^{-1} \circ s \circ f_k: C_k \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow H_k(C_*; \mathbb{F})$$

is a chain extension of C_* in degree k over $\mathbb{F}$ (Lemma 2.4). We consider the family $\tilde{B} := H_k(f_*)^{-1}(B)$. Then $\tilde{B}$ is an $\mathbb{F}$ -basis of $H_k(C_*; \mathbb{F})$. The integrality condition and cointegrality of B ensure that $\tilde{B}$ is also cointegral: Indeed,

$$\begin{aligned} H_k(C_*)_{\mathbb{F}} &= H_k(f_*)^{-1} \circ H_k(f_*)(H_k(C_*)_{\mathbb{F}}) \subset H_k(f_*)^{-1}(H_k(D_*)_{\mathbb{F}}) \\ &\subset H_k(f_*)^{-1}(\mathbb{Z}B) \subset \mathbb{Z}(H_k(f_*)^{-1}(B)) \\ &= \mathbb{Z}\tilde{B}. \end{aligned}$$

By construction of $\tilde{B}$, we have $\|H_k(f_*)^{-1}\|^{B, \tilde{B}} \leq 1$. Therefore, we obtain

$$\begin{aligned} G_k^{\mathbb{F}}(C_*) &\leq \|\tilde{s}\|^{\tilde{B}} = \|H_k(f_*)^{-1} \circ s \circ f_k\|^{\tilde{B}} \\ &\leq \|H_k(f_*)^{-1}\|^{B, \tilde{B}} \cdot \|s\|^B \cdot \|f_k\| \leq 1 \cdot \|s\|^B \cdot \|f_k\|. \end{aligned}$$

Taking the infimum over B and s gives the claimed estimate. $\square$

As a consequence, we obtain via the sandwich estimate (Proposition 7.5) that the cohomological lifting volume is also (almost) universal. Universality can also be used to compare gnarledness over different coefficients (Corollary 8.5).

7.4 Comparison with geometric gnarledness

For the comparison with the geometric notion of gnarledness, we recall the original definition of gnarledness of simplicial complexes [CDMW18]:

Definition 7.10 (gnarledness of simplicial chain complexes). Let X be a finite simplicial complex and let $k \in \mathbb{N}$.

- A k -spanning tree of X is a k -dimensional subcomplex T of X with the following properties:
 - We have $T_{k-1} = X_{k-1}$.
 - The map $H_{k-1}(T; \mathbb{Q}) \rightarrow H_{k-1}(X; \mathbb{Q})$ induced by the inclusion is a $\mathbb{Q}$ -isomorphism.
 - We have $H_k(T; \mathbb{Q}) \cong_{\mathbb{Q}} 0$.
- Let T be a k -spanning tree of X . We write $B_k^{\mathbb{Q}}(T)$ for the set of all $\mathbb{Q}$ -bases B of $H_k(X, T; \mathbb{Q})$ that satisfy for all $\sigma \in X_k$ that $[\sigma] \in \mathbb{Z}B$. For $B \in B_k^{\mathbb{Q}}(T)$, we set

$$G_k(X, T, B) := \max_{\sigma \in X_k} |[\sigma]|_1^B.$$

- The *gnarledness* of X in degree k is defined as

$$G_k(X) := \min \{ G_k(X, T, B) \mid T \text{ is a } k\text{-spanning tree of } X, B \in B_k^{\mathbb{Q}}(T) \}.$$

In the definition of $G_k(X, T, B)$, because of $T_{k-1} = X_{k-1}$ indeed every simplex $\sigma \in X_k$ represents a relative cycle for (X, T) and so represents a class in $H_k^{\Delta}(X, T; \mathbb{Q})$.

Proposition 7.11. *Let X be a finite simplicial complex and let $k \in \mathbb{N}$. Then:*

$$G_k^{\mathbb{Q}}(C_*^{\Delta}(X)) \leq G_k(X).$$

Proof. Let T be a k -spanning tree of X and let $B \in B_k^{\mathbb{Q}}(T)$. We explain how to obtain a chain extension for $C_*^{\Delta}(X)$ from T , which will then witness the claimed estimate.

Because T is a k -spanning tree of X , the long exact pair sequence for (X, T) shows that the map

$$H_k^{\Delta}(X; \mathbb{Q}) \rightarrow H_k^{\Delta}(X, T; \mathbb{Q})$$

induced by the inclusion is a $\mathbb{Q}$ -isomorphism. Let $t: H_k^{\Delta}(X, T; \mathbb{Q}) \rightarrow H_k^{\Delta}(X; \mathbb{Q})$ be the inverse isomorphism and let $r: C_k^{\Delta}(X; \mathbb{Q}) \rightarrow H_k^{\Delta}(X, T; \mathbb{Q})$ be the homomorphism induced by the map $X_k \rightarrow H_k^{\Delta}(X, T; \mathbb{Q})$ given by $\sigma \rightarrow [\sigma]$. We consider

$$s := t \circ r: C_k^{\Delta}(X; \mathbb{Q}) \rightarrow H_k^{\Delta}(X; \mathbb{Q}) \quad \text{and} \quad \tilde{B} := t(B).$$

We proceed in the following steps:

1. The map s is a chain extension of $C_*^{\Delta}(X)$ in degree k over $\mathbb{Q}$, because: Let $c \in C_k^{\Delta}(X; \mathbb{Q})$ be a cycle. Then $t([c]) = [c]$ (because c is an actual cycle on X) and so

$$s(c) = t \circ r(c) = t([c]) = [c].$$

2. We have $\tilde{B} \in B_k^{\mathbb{Q}}(C_*^{\Delta}(X))$, because: By construction, $\tilde{B}$ is a $\mathbb{Q}$ -basis of $H_k^{\Delta}(X; \mathbb{Q})$. To show the cointegrality condition, let $\alpha \in H_k^{\Delta}(X; \mathbb{Z})_{\mathbb{Q}}$. Then α is represented by a cycle $c \in C_k^{\Delta}(X) = \mathbb{Z}X_k$. Because s is a chain extension, we obtain in $H_k^{\Delta}(X; \mathbb{Q})$:

$$\alpha = [c] = s(c) = t([c]) \in t(\mathbb{Z}B) \subset \mathbb{Z}t(B) = \mathbb{Z}\tilde{B}$$

Hence, $H_k^{\Delta}(X; \mathbb{Z})_{\mathbb{Q}} \subset \mathbb{Z}\tilde{B}$.

3. We have

$$\begin{aligned} G_k^{\mathbb{Q}}(C_*^{\Delta}(X)) &\leq G_k^{\mathbb{Q}}(C_*^{\Delta}(X), s) \leq \|s\|^{\tilde{B}} = \max_{\sigma \in X_k} |s(\sigma)|_1^{\tilde{B}} = \max_{\sigma \in X_k} \|\sigma\|_1^B \\ &= G_k(X, T, B). \end{aligned}$$

Finally, taking the minimum over all k -spanning trees T of X and all $B \in B_k^{\mathbb{Q}}(T)$, shows that $G_k^{\mathbb{Q}}(C_*^{\Delta}(X)) \leq G_k(X)$. $\square$

Remark 7.12. Let X be a finite simplicial complex, let $k \in \mathbb{N}$, and let s be a chain extension of $C_*^{\Delta}(X)$ in degree k over $\mathbb{Q}$. We may then define a k -dimensional subcomplex T of X by setting

$$\begin{aligned} T_{k-1} &:= X_{k-1} \\ T_k &:= \{\sigma \in X_k \mid s(\sigma) = 0 \in H_k^{\Delta}(X; \mathbb{Q})\}. \end{aligned}$$

By construction, the canonical map $H_{k-1}^{\Delta}(T; \mathbb{Q}) \rightarrow H_{k-1}^{\Delta}(X; \mathbb{Q})$ is surjective and $H_k^{\Delta}(T; \mathbb{Q}) \rightarrow H_k^{\Delta}(X; \mathbb{Q})$ is zero. However, in general, we cannot expect T to be an actual k -spanning tree of X .

An example where $G_k^{\mathbb{Q}}(C_*^{\Delta}(X))$ and $G_k(X)$ do not coincide is given below (Example 7.13), but for the purpose of cohomological filling estimates, we may work with the algebraic version $G_k^{\mathbb{Q}}(C_*^{\Delta}(X))$ instead of $G_k(X)$.

Moreover, $G_k^{\mathbb{Q}}(C_*^{\Delta}(X))$ satisfies monotonicity (Proposition 7.9); this applies, in particular, to simplicial maps that induce $\mathbb{Q}$ -homology isomorphisms. In contrast, for the gnarledness of simplicial complexes, it is not clear whether such a monotonicity property holds.

Example 7.13. Let $m \in \mathbb{N}_{\geq 3}$ and let X be the simplicial complex consisting of m edges chained together as a circle of length m . Let $C_* := C_*^{\Delta}(X)$ be the associated simplicial chain complex. Then $C_0 \cong_{\mathbb{Z}} \mathbb{Z}^m$ and $C_1 \cong_{\mathbb{Z}} \mathbb{Z}^m$, where the isomorphisms map the vertices and the (positively oriented) edges, respectively, to the standard basis of $\mathbb{Z}^m$.

On the one hand, by definition, $G_1(X) \in \mathbb{N}$ and clearly $G_1(X) \neq 0$. In particular, $G_1(X) \geq 1$.

On the other hand,

$$\begin{aligned} s: C_1 \otimes_{\mathbb{Z}} \mathbb{Q} &\longrightarrow H_1^{\Delta}(X; \mathbb{Q}) \\ e_j \otimes 1 &\longmapsto \frac{1}{m} \cdot [e_1 + \dots + e_m] \end{aligned}$$

is a chain extension of C_* in degree 1 over $\mathbb{Q}$ and $B := ([e_1 + \dots + e_m])$ is a cointegral basis of $H_1^{\Delta}(X; \mathbb{Q})$. Therefore,

$$G_1^{\mathbb{Q}}(C_*) \leq \|s\|^B = \frac{1}{m} < 1 \leq G_1(X).$$

Viewing this simplicial circle as an m -fold covering of the CW-structure of S^1 with a single 1-cell, this example is related to Example 7.4.

Geometric gnarledness is algorithmically computable (though it is not clear whether this can be done efficiently). In contrast, we do not know whether this is true for the algebraic gnarledness or the cohomological lifting volume of marked $\mathbb{Z}$ -chain complexes of finite type.

Proposition 7.14. *There exists an algorithm that*

- *given a finite simplicial complex X and $k \in \mathbb{N}$,*
- *computes $G_k(X)$.*

Proof. It suffices to prove the following two steps:

1. There exists an algorithm that
 - given a finite simplicial complex X and $k \in \mathbb{N}$,
 - computes the set of all k -spanning trees of X .
2. There exists an algorithm that
 - given a finite simplicial complex X , a $k \in \mathbb{N}$, a k -spanning tree T of X , and $K \in \mathbb{N}$
 - computes the (finite) set of all bases B in $B_k^{\mathbb{Q}}(T)$ with

$$\max_{\sigma \in X_k} |[\sigma]|_1^B \leq K.$$

Indeed, using these algorithms, we can compute $G_k(X)$ as the minimal number K such that there exists a k -spanning tree T for which the set computed by the second algorithm is non-empty.

Ad 1. Because X is finite, we can compute the finite set of all subcomplexes of X . For each subcomplex T of X , we can check algorithmically whether T is a k -spanning tree of X , as the simplicial homology functors over $\mathbb{Q}$ on finite simplicial complexes are algorithmically computable.

Ad 2. Given a finite simplicial complex X , a $k \in \mathbb{N}$, a k -spanning tree T of X , and $K \in \mathbb{N}$, we know, by the universal coefficient theorem, that $\{[\sigma] \mid \sigma \in X_k\}$ is a $\mathbb{Q}$ -generating set of $H_k(X, T; \mathbb{Q})$ and thus contains a $\mathbb{Q}$ -basis B of $H_k(X, T; \mathbb{Q})$.

We apply Lemma 7.15 below to the $\mathbb{Q}$ -isomorphism $f: \mathbb{Q}^{\dim_{\mathbb{Q}} H_k(X, T; \mathbb{Q})} \rightarrow H_k(X, T; \mathbb{Q})$ induced by B and the subset $\{f^{-1}([\sigma]) \mid \sigma \in X_k\}$ (which is algorithmically computable). Finally, we translate the result back via f . $\square$

Lemma 7.15. *There is an algorithm that*

- *given $d \in \mathbb{Q}$, a $\mathbb{Q}$ -generating set $\Sigma \subset \mathbb{Q}^d$ with $\{e_1, \dots, e_d\} \subset \Sigma$, and $K \in \mathbb{N}$*
- *computes the set*

$$\{B \mid B \text{ is a basis of } \mathbb{Q}^d \text{ with } \Sigma \subset \mathbb{Z}B \text{ and } \max_{\sigma \in \Sigma} |\sigma|_1^B \leq K\}.$$

Proof. For $d \in \mathbb{Q}$ and a $\mathbb{Q}$ -generating set $\Sigma \subset \mathbb{Q}^d$ with $\{e_1, \dots, e_d\} \subset \Sigma$, we have

$$\begin{aligned} & \{B \mid B \text{ is a basis of } \mathbb{Q}^d \text{ with } \Sigma \subset \mathbb{Z}B\} \\ &= \left\{ (A_{*1}, \dots, A_{*d}) \mid A \in \text{GL}_d(\mathbb{Q}), \forall_{\sigma \in \Sigma} \exists_{\lambda_1, \dots, \lambda_d \in \mathbb{Z}} \sigma = \sum_{j=1}^d \lambda_j \cdot A_{*j} \right\} \\ &= \left\{ (A_{*1}, \dots, A_{*d}) \mid A \in \text{GL}_d(\mathbb{Q}), A^{-1} \in \text{M}_{d \times d}(\mathbb{Z}), \forall_{\sigma \in \Sigma \setminus \{e_1, \dots, e_d\}} \exists_{\lambda_1, \dots, \lambda_d \in \mathbb{Z}} \sigma = \sum_{j=1}^d \lambda_j \cdot A_{*j} \right\} \\ &= \left\{ (A_{*1}^{-1}, \dots, A_{*d}^{-1}) \mid A \in \text{M}_{d \times d}(\mathbb{Z}), \det A \neq 0, \forall_{\sigma \in \Sigma \setminus \{e_1, \dots, e_d\}} \exists_{\lambda_1, \dots, \lambda_d \in \mathbb{Z}} \sigma = \sum_{j=1}^d \lambda_j \cdot A_{*j}^{-1} \right\}. \end{aligned}$$

Hence, given $K \in \mathbb{N}$, we obtain

$$\begin{aligned} & \{B \mid B \text{ is a basis of } \mathbb{Q}^d \text{ with } \Sigma \subset \mathbb{Z}B \text{ and } \max_{\sigma \in \Sigma} |\sigma|_1^B \leq K\} \\ &= \left\{ (A_{*1}^{-1}, \dots, A_{*d}^{-1}) \mid A \in \text{M}_{d \times d}(\mathbb{Z}), \det A \neq 0, \forall_{j \in \{1, \dots, d\}} |A_{*j}|_1 \leq K, \right. \\ & \quad \left. \forall_{\sigma \in \Sigma \setminus \{e_1, \dots, e_d\}} \exists_{\lambda_1, \dots, \lambda_d \in \{-K, \dots, K\}} \sigma = \sum_{j=1}^d \lambda_j \cdot A_{*j}^{-1} \wedge \sum_{j=1}^d |\lambda_j| \leq K \right\}, \end{aligned}$$

which is algorithmically computable due to the a priori bound on the A_{*j} . $\square$

7.5 Comparison with simplicial volume

We reformulate the duality principle for simplicial volume [Gro82] in terms of gnarledness of the singular chain complex:

Proposition 7.16. *Let C_* be a marked free $\mathbb{Z}$ -chain complex and let $n \in \mathbb{N}$ with $H_n(C_*; \mathbb{R}) \cong_{\mathbb{R}} \mathbb{R}$. Let $\mu_{\mathbb{Z}} \in H_n(C_*)$ be an element such that the induced class $\mu_{\mathbb{R}} \in H_n(C_*; \mathbb{R})$ is a $\mathbb{Z}$ -generator of $H_n(C_*)_{\mathbb{R}}$. Then*

$$G_n^{\mathbb{R}}(C_*) = \frac{1}{\|\mu_{\mathbb{R}}\|_1}.$$

Here, $\|\cdot\|_1$ denotes the ℓ^1 -semi-norm on $H_n(C_*; \mathbb{R})$ induced by the ℓ^1 -norm on the chain complex $C_* \otimes_{\mathbb{Z}} \mathbb{R}$ given by the marking of C_* . Moreover, we use the convention $1/0 = \infty$.

Proof. Let $\omega \in H^n(C_*; \mathbb{R})$ be the cohomology class dual to $\mu_{\mathbb{R}}$. By the duality principle [Gro82] (which holds in this generality), we have $\|\mu_{\mathbb{R}}\|_1 = 1/\|\omega\|_{\infty}$ (with $1/\infty = 0$). Therefore, it suffices to show that

$$G_n^{\mathbb{R}}(C_*) = \|\omega\|_{\infty}.$$

We write E_n for the set of all chain extensions of C_* over $\mathbb{R}$ in degree n . Then (when viewing ω as set of all cocycles in $\text{Hom}_{\mathbb{Z}}(C_n, \mathbb{R})$ dual to $\mu_{\mathbb{R}}$)

$$\begin{aligned} \Phi: E_n &\longrightarrow \omega \\ s &\longmapsto (c \mapsto \lambda \text{ with } s(c) = \lambda \cdot \mu_{\mathbb{R}}) \\ \Psi: \omega &\longrightarrow E_n \\ f &\longmapsto (c \mapsto f(c) \cdot \mu_{\mathbb{R}}) \end{aligned}$$

are well-defined mutually inverse bijections. Indeed, for a chain extension s , the chain extension condition ensures both that $\Phi(s)$ is a cocycle and that $\langle [\Phi(s)], \mu_{\mathbb{R}} \rangle = 1$, and vice versa.

If $B_0 := (\mu_{\mathbb{R}})$ denotes the cointegral basis of $H_n(C; \mathbb{R})$ given by $\mu_{\mathbb{R}}$, then

$$|\Phi(s)|_{\infty} = \sup_{\sigma \in \Sigma_n} |\Phi(s)(\sigma)| = \sup_{\sigma \in \Sigma_n} |s(\sigma)|_1^{B_0} = \|s\|^{B_0}$$

for all $s \in E_n$; here, Σ_n is the marked basis of C_n . Hence, $\|\omega\|_{\infty} = \inf_{s \in E_n} \|s\|^{B_0}$.

Finally, we show that other cointegral bases of $H_n(C_*; \mathbb{R})$ do not lead to better values: Let $B \in B_n^{\mathbb{R}}(C_*)$, i.e., there is a $b \in \mathbb{Z} \setminus \{0\}$ with $B = (1/b \cdot \mu_{\mathbb{R}})$. Then, for all $s \in E_n$, we obtain

$$\|s\|^B = |b| \cdot \|s\|^{B_0} \geq \|s\|^{B_0}.$$

Taking the infimum over all $B \in B_n^{\mathbb{R}}(C_*)$ shows that $\|\omega\|_{\infty} = G_n^{\mathbb{R}}(C_*)$. $\square$

Corollary 7.17. *Let M be an oriented closed connected n -manifold. Then*

$$G_n^{\mathbb{R}}(C_*(M; \mathbb{Z})) = \frac{1}{\|M\|}.$$

Here, $C_*(M; \mathbb{Z})$ is the singular chain complex of M , marked by the basis of singular simplices and $\|M\| := \|[M]_{\mathbb{R}}\|_1$ denotes the simplicial volume of M [Gro82].

Proof. We apply Proposition 7.16 to the singular chain complex $C_*(M; \mathbb{Z})$ and the integral fundamental class $[M]_{\mathbb{Z}} \in H_n(M; \mathbb{Z})$ of M . $\square$

Example 7.18. Thus, results on the spectrum of simplicial volume [HL21, HL24] show:

- Each rational number appears as gnarledness of singular chain complexes.
- Gnarledness of singular chain complexes can assume transcendental values.

Example 7.19. Let $d \in \mathbb{N}_{>0}$ and let $\overline{M} \rightarrow M$ be a d -sheeted covering of oriented closed connected n -manifolds. Then

$$2 \cdot L_n^{\mathbb{R}}(C_*(\overline{M}; \mathbb{Z})) = G_n^{\mathbb{R}}(C_*(\overline{M}; \mathbb{Z})) = \frac{1}{d} \cdot G_n^{\mathbb{R}}(C_*(M; \mathbb{Z})) = \frac{2}{d} \cdot L_n^{\mathbb{R}}(C_*(M; \mathbb{Z})).$$

Indeed, for the inner equality, we apply Corollary 7.17, which relates gnarledness to the simplicial volume, and the fact that simplicial volume is multiplicative under finite coverings [Gro82]. For the relation with the cohomological lifting volume, we use that the top-homology of oriented closed connected manifolds is one-dimensional and Proposition 7.5.

Corollary 7.20. *Let (S, r) be a one-relator presentation, where r is a non-trivial element of the commutator subgroup of $F(S)$ and let Γ be the corresponding one-relator group. If X is a model for the classifying space $B\Gamma$, then*

$$G_1^{\mathbb{R}}(C_*(X; \mathbb{Z})) = \frac{1}{\|\Gamma\|}.$$

Here, $C_*(X; \mathbb{Z})$ is the singular chain complex of X , marked by the basis of singular simplices and $\|\Gamma\|$ denotes the simplicial volume of Γ [HL22].

Proof. We apply Proposition 7.16 to the singular chain complex $C_*(X; \mathbb{Z})$ and the integral fundamental class of Γ in $H_1(X; \mathbb{Z})$. $\square$

8 Irrelevance of coefficients

We show that the cohomological lifting volume and gnarledness of chain complexes are independent of the chosen coefficient subfield of $\mathbb{R}$. In both cases, we use basic facts on approximation of reals through numbers in a subfield and on (co)integrality properties. For gnarledness, we in addition use base-change properties of size invariants and universality.

8.1 Cohomological lifting volume

Proposition 8.1. *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$, let $k \in \mathbb{N}$, and let C_* be a marked free $\mathbb{Z}$ -chain complex with $\text{rk}_{\mathbb{Z}} C_k < \infty$. Then*

$$L_k^{\mathbb{Q}}(C_*) = L_k^{\mathbb{F}}(C_*).$$

Proof. This can be viewed as a version of invariance of filling volumes under inclusions of dense subcomplexes. We carry out the argument in this concrete situation. We consider the commutative diagram

$$\begin{array}{ccc} H^k(C_*; \mathbb{Q}) & \longrightarrow & H^k(C_*; \mathbb{F}) \\ \downarrow \pi & & \downarrow \pi_{\mathbb{F}} \\ H^k(C_*; \mathbb{Q}/\mathbb{Z}) & \longrightarrow & H^k(C_*; \mathbb{F}/\mathbb{Z}) \end{array}$$

induced by the respective inclusions and projections on the coefficients.

We have $L_k^{\mathbb{Q}}(C_*) \leq L_k^{\mathbb{F}}(C_*)$: Let $\varphi \in H^k(C_*; \mathbb{Q}/\mathbb{Z})$ be in the image of π . It suffices to show that $L_k^{\mathbb{Q}}(C_*, \varphi) \leq L_k^{\mathbb{F}}(C_*)$. Let $\psi \in H^k(C_*; \mathbb{F}/\mathbb{Z})$ be the image of φ under coefficient change and let $\tilde{\psi} \in H^k(C_*; \mathbb{F})$ be a $\pi_{\mathbb{F}}$ -lift of ψ .

Let $g \in \text{Hom}_{\mathbb{Z}}(C_k, \mathbb{F})$ be a cocycle representing $\tilde{\psi}$. Then, there exists a cochain $h \in \text{Hom}_{\mathbb{Z}}(C_{k-1}, \mathbb{F})$ such that $g + \delta h$ is rational and $[g + \delta h]$ is a π -lift of φ . Given $\varepsilon \in \mathbb{R}_{>0}$, density of $\mathbb{Q}$ in $\mathbb{F}$ allows us to find a cochain $h_{\mathbb{Q}} \in \text{Hom}_{\mathbb{Z}}(C_{k-1}, \mathbb{Q})$ with $|h_{\mathbb{Q}} - h| < \varepsilon$. We consider

$$f := g + \delta h - \delta h_{\mathbb{Q}}.$$

By construction, f is a rational cocycle with $\pi([f]) = \varphi$ and

$$|f|_{\infty} \leq |g|_{\infty} + \|\delta^{k-1}\| \cdot |h - h_{\mathbb{Q}}|_{\infty} \leq |g|_{\infty} + \|\partial_k\| \cdot \varepsilon;$$

we know that $\|\partial_k\|$ is finite because $\text{rk}_{\mathbb{Z}} C_k < \infty$. Taking $\varepsilon \rightarrow 0$ and the infimum over all $\tilde{\psi}$ (and g , respectively) shows $L_k^{\mathbb{Q}}(C_*, \varphi) \leq L_k^{\mathbb{F}}(C_*, \psi) \leq L_k^{\mathbb{F}}(C_*)$.

Conversely, we have $L_k^{\mathbb{F}}(C_*) \leq L_k^{\mathbb{Q}}(C_*)$: Let $\psi \in H^k(C_*; \mathbb{F}/\mathbb{Z})$ be in the image of $\pi_{\mathbb{F}}$, say $\psi = \pi_{\mathbb{F}}(\tilde{\psi})$ with $\tilde{\psi} \in H^k(C_*; \mathbb{F})$. It suffices to show that $L_k^{\mathbb{F}}(C_*, \psi) \leq L_k^{\mathbb{Q}}(C_*)$.

To this end, we first construct a class $\tilde{\varphi}$ in $H^k(C_*; \mathbb{Q})$ “sufficiently close to $\tilde{\psi}$ ”. Let B be a $\mathbb{Q}$ -basis of $H_k(C_*; \mathbb{Q})$ induced by a $\mathbb{Z}$ -basis of $H_k(C_*)_{\mathbb{Q}}$. Let $(\tilde{\varphi}_{\beta})_{\beta \in B}$ be the dual $\mathbb{Q}$ -basis of $H^k(C_*; \mathbb{Q})$ and let $(\tilde{\psi}_{\beta})_{\beta \in B}$ be the induced family in $H^k(C_*; \mathbb{F})$. Because $\text{rk}_{\mathbb{Z}} C_k$ is finite,

$$K := \max_{\beta \in B} \|\tilde{\psi}_{\beta}\|_{\infty}$$

is finite. By the universal coefficient theorem, $(\tilde{\psi}_\beta)_{\beta \in B}$ is an $\mathbb{F}$ -basis of $H^k(C_*; \mathbb{F})$, namely the basis dual to the $\mathbb{F}$ -basis of $H_k(C_*; \mathbb{F})$ induced by B . We write $\tilde{\psi} = \sum_{\beta \in B} \lambda_\beta \cdot \tilde{\psi}_\beta$ with $\lambda_\beta \in \mathbb{F}$. Given $\varepsilon \in \mathbb{R}_{>0}$, for all $\beta \in B$, there exist $\mu_\beta \in \mathbb{Q}$ with

$$|\mu_\beta - \lambda_\beta| < \varepsilon.$$

We consider the class

$$\varphi := \pi \left(\sum_{\beta \in B} \mu_\beta \cdot \tilde{\varphi}_\beta \right) \in H^k(C_*; \mathbb{Q}/\mathbb{Z}).$$

Let $\tilde{\varphi} \in H^k(C_*; \mathbb{Q})$ be any π -lift of φ and let $f \in \text{Hom}_{\mathbb{Z}}(C_k, \mathbb{Q})$ be a cocycle representing $\tilde{\varphi}$. We set

$$\tilde{\varrho} := [f] - \sum_{\beta \in B} (\mu_\beta - \lambda_\beta) \cdot \tilde{\psi}_\beta \in H^k(C_*; \mathbb{F}).$$

By construction, $\tilde{\varrho}$ is a $\pi_{\mathbb{F}}$ -lift of ψ , because

$$\begin{aligned} \pi_{\mathbb{F}}(\tilde{\varrho}) &= \pi_{\mathbb{F}}([f]) - \pi_{\mathbb{F}} \left(\sum_{\beta \in B} (\mu_\beta - \lambda_\beta) \cdot \tilde{\psi}_\beta \right) \\ &= \pi_{\mathbb{F}} \left(\sum_{\beta \in B} \mu_\beta \cdot \tilde{\psi}_\beta \right) - \pi_{\mathbb{F}} \left(\sum_{\beta \in B} (\mu_\beta - \lambda_\beta) \cdot \tilde{\psi}_\beta \right) = \pi_{\mathbb{F}} \left(\sum_{\beta \in B} \lambda_\beta \cdot \tilde{\psi}_\beta \right) \\ &= \pi_{\mathbb{F}}(\tilde{\psi}) = \psi, \end{aligned}$$

and we have

$$\|\tilde{\varrho}\|_\infty \leq |f|_\infty + \sum_{\beta \in B} |\mu_\beta - \lambda_\beta| \cdot \|\tilde{\psi}_\beta\|_\infty \leq |f|_\infty + \#B \cdot \varepsilon \cdot K.$$

Taking $\varepsilon \rightarrow 0$ and the infimum over all $\tilde{\varphi}$ (and f , respectively) shows $L_k^{\mathbb{F}}(C_*, \psi) \leq L_k^{\mathbb{Q}}(C_*, \varphi) \leq L_k^{\mathbb{Q}}(C_*)$. $\square$

8.2 Size invariants

Every size invariant over a subfield $\mathbb{F}$ is also a size invariant over $\mathbb{Q}$. Conversely, every size invariant over $\mathbb{Q}$ also satisfies monotonicity over $\mathbb{F}$ on domain chain complexes that satisfy a finiteness condition.

Proposition 8.2 (base change I). *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$, let $k \in \mathbb{N}$, and let $I: \text{mCh} \rightarrow [0, \infty]$ be a size invariant over $\mathbb{F}$ in degree k . Then I is also a size invariant over $\mathbb{Q}$ in degree k .*

Proof. We only need to check the monotonicity condition. If $f_*: C_* \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow D_* \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\mathbb{Q}$ -chain map inducing a $\mathbb{Q}$ -isomorphism $H_k(f_*): H_k(C_*; \mathbb{Q}) \rightarrow H_k(D_*; \mathbb{Q})$ satisfying $H_k(f_*)(H_k(C_*)_{\mathbb{Q}}) \subset H_k(C_*)_{\mathbb{Q}}$, then (under the canonical natural isomorphism $\cdot \otimes_{\mathbb{Z}} \mathbb{F} \cong \cdot \otimes_{\mathbb{Z}} \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{F}$) we may apply monotonicity over $\mathbb{F}$ to the base-changed chain map

$$f_*^{\mathbb{F}} := f_* \otimes_{\mathbb{Q}} \mathbb{F}: C_* \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow D_* \otimes_{\mathbb{Z}} \mathbb{F}$$

and obtain $I(C_*) \leq \|f_k^{\mathbb{F}}\| \cdot I(D_*) \leq \|f_k\| \cdot I(D_*)$, as desired. $\square$

Proposition 8.3 (base change II). *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$, let $k \in \mathbb{N}$, and let $I: \mathbf{mCh} \rightarrow [0, \infty]$ be a size invariant over $\mathbb{Q}$ in degree k . Let C_*, D_* in $\mathbf{mCh}$ with $\mathrm{rk}_{\mathbb{Z}} C_k < \infty$ and let $f_*: C_* \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow D_* \otimes_{\mathbb{Z}} \mathbb{F}$ be an $\mathbb{F}$ -chain map that induces an $\mathbb{F}$ -isomorphism $H_k(f_*): H_k(C_*; \mathbb{F}) \rightarrow H_k(D_*; \mathbb{F})$ and that satisfies the integrality condition $H_k(f_*)(H_k(C_*)_{\mathbb{F}}) \subset H_k(D_*)_{\mathbb{F}}$. Then:*

$$I(C_*) \leq \|f_k\| \cdot I(D_*).$$

Proof. Given $\varepsilon \in \mathbb{R}_{>0}$, we apply the approximation lemma (Lemma 8.4) below and obtain a chain map $g_*: C_* \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow D_* \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $H_k(g_*)$ is a $\mathbb{Q}$ -isomorphism with $H_k(g_*)(H_k(C_*)_{\mathbb{Q}}) \subset H_k(D_*)_{\mathbb{Q}}$ and $\|g_k\| \leq \|f_k\| + \varepsilon$. Therefore, monotonicity of I over $\mathbb{Q}$ gives us

$$I(C_*) \leq \|g_k\| \cdot I(D_*) \leq (\|f_k\| + \varepsilon) \cdot I(D_*).$$

Taking $\varepsilon \rightarrow 0$ completes the proof. $\square$

We may view the canonical maps $C_* \rightarrow C_* \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow C_* \otimes_{\mathbb{Z}} \mathbb{F}$ as inclusions of subcomplexes. Elements in $C_* \otimes_{\mathbb{Z}} \mathbb{F}$ coming from $C_* \otimes_{\mathbb{Z}} \mathbb{Q}$ will be called *rational*. Similarly, maps obtained via base change from $\mathbb{Q}$ to $\mathbb{F}$ or from $\mathbb{Z}$ to $\mathbb{F}$ will be called *defined over $\mathbb{Q}$* or *defined over $\mathbb{Z}$* , respectively.

Lemma 8.4 (rational approximation of chain maps). *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$, let $k \in \mathbb{N}$, let C_*, D_* be in $\mathbf{mCh}$ with $\mathrm{rk}_{\mathbb{Z}} C_k < \infty$, and let $f_*: C_* \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow D_* \otimes_{\mathbb{Z}} \mathbb{F}$ be an $\mathbb{F}$ -chain map with $H_k(f_*)(H_k(C_*)_{\mathbb{F}}) \subset H_k(D_*)_{\mathbb{F}}$. Then, for each $\varepsilon \in \mathbb{R}_{>0}$, there exists an $\mathbb{F}$ -linear chain map $g_*: C_* \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow D_* \otimes_{\mathbb{Z}} \mathbb{F}$ that is defined over $\mathbb{Q}$ with*

$$H_k(g_*) = H_k(f_*): H_k(C_*; \mathbb{F}) \rightarrow H_k(D_*; \mathbb{F}) \quad \text{and} \quad \|g_k\| \leq \|f_k\| + \varepsilon.$$

Proof. This is mainly a consequence of the fact that C_* is defined over $\mathbb{Z}$ and that $\mathbb{Q}$ is dense in $\mathbb{F}$. We construct g_* in degree k , $k-1$ and lower degrees, $k+1$ and higher degrees, in this order.

Degree k : Let $Z_k \subset C_k \otimes_{\mathbb{Z}} \mathbb{F}$ be the subspace of cycles. Because the boundary operator of C_* is defined over $\mathbb{Z}$, we can choose an $\mathbb{F}$ -basis B of Z_k consisting of rational chains. Let $p: C_k \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow Z_k$ be an $\mathbb{F}$ -linear projection that is defined over $\mathbb{Q}$. We construct g_k separately on Z_k and on $\bar{Z}_k := \ker p$:

- On Z_k : Let $c \in B$. By the integrality condition on $H_k(f_*)$, the homology class $[f_k(c)] \in H_k(D_*; \mathbb{F})$ is rational. Thus, there exists a rational cycle $z \in D_k \otimes_{\mathbb{Z}} \mathbb{F}$ and a chain $b \in D_{k+1} \otimes_{\mathbb{Z}} \mathbb{F}$ with

$$f_k(c) = z + \partial b.$$

Because $\mathbb{Q}$ is dense in $\mathbb{F}$, we may choose a rational chain $b' \in D_{k+1} \otimes_{\mathbb{Z}} \mathbb{F}$ with $|\partial(b' - b)|_1 \leq \varepsilon$. We set $g_k(c) := z + \partial b'$. By construction, $g_k(c)$ is a rational cycle with

$$[g_k(c)] = [z] = [f_k(c)] \quad \text{and} \quad |g_k(c) - f_k(c)|_1 \leq |\partial(b' - b)|_1 \leq \varepsilon.$$

We extend g_k linearly to Z_k . Hence, $g_k|_{Z_k}$ is defined over $\mathbb{Q}$.

- On $\overline{Z}_k$: Let $\overline{B}$ be an $\mathbb{F}$ -basis of $\overline{Z}_k$ consisting of rational elements and let $c \in \overline{B}$. Because $\mathbb{Q}$ is dense in $\mathbb{F}$, there exists a rational chain $g_k(c) \in D_k \otimes_{\mathbb{Z}} \mathbb{F}$ with

$$|g_k(c) - f_k(c)|_1 \leq \varepsilon.$$

We extend g_k linearly to $\overline{Z}_k$. Hence $g_k|_{\overline{Z}_k}$ is defined over $\mathbb{Q}$.

As $C_k \otimes_{\mathbb{Z}} \mathbb{F}$ is the (internal) direct sum of the rationally defined subspaces Z_k and $\overline{Z}_k$, we obtain a rationally defined $\mathbb{F}$ -linear map $g_k: C_k \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow D_k \otimes_{\mathbb{Z}} \mathbb{F}$. Moreover, for all $x \in C_k \otimes_{\mathbb{Z}} \mathbb{F}$, we have $p(x) \in Z_k$, $x - p(x) \in \overline{Z}_k$ and so

$$\begin{aligned} |g_k(x) - f_k(x)|_1 &\leq |g_k(x - p(x)) - f_k(x - p(x))|_1 + |g_k(p(x)) - f_k(p(x))|_1 \\ &\leq \varepsilon \cdot |x - p(x)|_1^{\overline{B}} + \varepsilon \cdot |p(x)|_1^B \\ &\leq \varepsilon \cdot \|\text{id} - p\|^{\overline{B}} \cdot |x|_1 + \varepsilon \cdot \|p\|^B \cdot |x|_1. \end{aligned}$$

We obtain $\|g_k\| \leq \|f_k\| + K \cdot \varepsilon$, where $K := \|\text{id} - p\|^{\overline{B}} + \|p\|^B$ is finite because $\text{rk}_{\mathbb{Z}} C_k < \infty$ and we take the operator norm with respect to $|\cdot|_1$ on the domain.

Degree $k-1$ and lower: By construction of g_k , we have $\ker \partial_k^C \subset \ker \partial_k^D \circ g_k$. Because the maps ∂_k^C and $\partial_k^D \circ g_k$ are defined over $\mathbb{Q}$, there exists a rationally defined $\mathbb{F}$ -linear map $g_{k-1}: C_{k-1} \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow D_{k-1} \otimes_{\mathbb{Z}} \mathbb{F}$ solving the mapping problem

$$g_{k-1} \circ \partial_k^C = \partial_k^D \circ g_k \quad \text{and} \quad \text{im } g_{k-1} \subset \text{im } \partial_k^D.$$

In particular, $\partial_{k-1}^D \circ g_{k-1} = 0$ and we define $g_{k-2}, \dots, g_0$ to be the zero maps.

Degree $k+1$ and higher: We construct an $\mathbb{F}$ -linear map $g_{k+1}: C_{k+1} \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow D_{k+1} \otimes_{\mathbb{Z}} \mathbb{F}$ with $\partial_{k+1} \circ g_{k+1} = g_k \circ \partial_{k+1}$ and $g_{k+1} \circ \partial_{k+2} = 0$ that is defined over $\mathbb{Q}$; to this end, we argue as in the second part of the degree k case: We define g_{k+1} as the zero map on $\text{im } \partial_{k+1}^C$. Because ∂_{k+1}^C is defined over $\mathbb{Z}$, the subspace $\text{im } \partial_{k+1}^C \subset C_{k+1} \otimes_{\mathbb{Z}} \mathbb{F}$ is rationally defined. We choose a rationally defined complementary $\mathbb{F}$ -subspace $W \subset C_{k+1} \otimes_{\mathbb{Z}} \mathbb{F}$ and an $\mathbb{F}$ -basis B of W consisting of rational chains. For $c \in B$, the chain $\partial c \in C_k \otimes_{\mathbb{Z}} \mathbb{F}$ is a cycle. By construction of g_k , there exists a chain $b \in D_{k+1} \otimes_{\mathbb{Z}} \mathbb{F}$ with

$$g_k(\partial c) = f_k(\partial c) + \partial b = \partial(f_{k+1}(c) + b).$$

Hence, the rational element $g_k(\partial c)$ lies in the image of ∂_{k+1}^D on $D_* \otimes_{\mathbb{Z}} \mathbb{F}$. Because the boundary operator of D_* is defined over $\mathbb{Z}$, there also exists a rational solution $d \in D_{k+1} \otimes_{\mathbb{Z}} \mathbb{F}$ with $g_k(\partial c) = \partial d$. We set $g_{k+1}(c) := d$ and define g_{k+1} on all of W through linear extension. By construction, g_{k+1} is defined over $\mathbb{Q}$ and $\partial \circ g_{k+1} = g_k \circ \partial$ as well as $g_{k+1} \circ \partial_{k+2} = 0$. We define $g_{k+2}, g_{k+3}, \dots$ to be the zero maps.

By construction, $g_*: C_* \otimes_{\mathbb{Z}} \mathbb{F} \rightarrow D_* \otimes_{\mathbb{Z}} \mathbb{F}$ is an $\mathbb{F}$ -chain map that is defined over $\mathbb{Q}$ and that satisfies $H_k(g_*) = H_k(f_*)$ as well as $\|g_k\| \leq \|f_k\| + K \cdot \varepsilon$. Rescaling ε completes the proof. $\square$

8.3 Gnarledness

The base change properties of size invariants and universality show that gnarledness over $\mathbb{F}$ and $\mathbb{Q}$ coincide:

Corollary 8.5. *Let $\mathbb{F}$ be a subfield of $\mathbb{R}$ and let $k \in \mathbb{N}$.*

1. If C_* is in $\mathbf{mCh}$, then $G_k^{\mathbb{F}}(C_*) \leq G_k^{\mathbb{Q}}(C_*)$.
2. If C_* is in $\mathbf{mCh}$ and $\mathrm{rk}_{\mathbb{Z}} C_k < \infty$, then $G_k^{\mathbb{Q}}(C_*) \leq G_k^{\mathbb{F}}(C_*)$.

Proof. We know that $G_k^{\mathbb{F}}$ and $G_k^{\mathbb{Q}}$ are size invariants in degree k over $\mathbb{F}$ and $\mathbb{Q}$, respectively (Proposition 7.8).

Ad 1. By base change, $G_k^{\mathbb{F}}$ is a size invariant over $\mathbb{Q}$ in degree k (Proposition 8.2). Therefore, universality of $G_k^{\mathbb{Q}}$ over $\mathbb{Q}$ gives $G_k^{\mathbb{F}} \leq G_k^{\mathbb{Q}}$ (Proposition 7.8).

Ad 2. Let C_* in $\mathbf{mCh}$ and $\mathrm{rk}_{\mathbb{Z}} C_k < \infty$. Then $G_k^{\mathbb{Q}}$ satisfies monotonicity over $\mathbb{F}$ in degree k for the domain complex C_* (Proposition 8.3). The proof of universality (Proposition 7.8) hence shows that $G_k^{\mathbb{F}}$ dominates $G_k^{\mathbb{Q}}$ on C_* , i.e., $G_k^{\mathbb{Q}}(C_*) \leq G_k^{\mathbb{F}}(C_*)$. $\square$

Alternatively, one can also show that base change gives a bijection between the cointegral bases of $H_k(\cdot; \mathbb{Q})$ and $H_k(\cdot; \mathbb{F})$, that chain extensions over $\mathbb{Q}$ lead to chain extensions over $\mathbb{F}$, and that chain extensions over $\mathbb{F}$ can be approximated by chain extensions over $\mathbb{Q}$.

9 Controlling the norm of products

We use chain extensions to construct bounded chain homotopy equivalences from a marked free $\mathbb{R}$ -chain complex C_* to its homology complex $H_*(C_*)$ with trivial boundary operators. In this context, boundedness refers to the semi-norm on $H_*(C_*)$ that is induced by the ℓ^1 -norm on C_* .

We use these chain homotopy equivalences to prove that the algebraic Künneth isomorphism is bilipschitz under a finiteness condition (Theorem 9.11).

We apply this result to the singular chain complex and singular homology with the canonical marked basis induced by the singular simplices. Under a finiteness condition, we then obtain norm-control for the topological Künneth isomorphism (Corollary 9.16) and a controlled behaviour of product maps (Corollary 9.18). These results allow us to classify elements with trivial norm in the singular homology of products of hyperbolic manifolds (Example 9.17).

9.1 Null spaces and reduced homology

In order to understand the semi-norm on homology of a normed chain complex, it is important to look at the “defect” of the semi-norm, i.e., the subspace of elements with trivial norm.

Definition 9.1 (null space). If $\|\cdot\|$ is a semi-norm on an $\mathbb{F}$ -vector space X , we call $N(X) := \{x \in X \mid \|x\| = 0\}$ the *null space* of the semi-norm $\|\cdot\|$.

If we want to specify the semi-norm of the space X explicitly, we write $N_{\|\cdot\|}(X)$ instead of $N(X)$.

We use the following notation in the context of normed chain complexes.

Notation 9.2. If C_* is a normed $\mathbb{F}$ -chain complex with induced semi-norm on $H_*(C_*)$ and $n \in \mathbb{N}$, we write

$$\begin{aligned} N_n(C_*) &:= N(H_n(C_*)), \\ \overline{H}_n(C_*) &:= H_n(C_*) / N_n(C_*). \end{aligned}$$

The latter is called the *reduced homology* of C_* .

Remark 9.3. The term *reduced homology* of a chain complex C_* of topological vector spaces usually refers to $\ker(\partial_n)/\overline{\text{im}(\partial_{n+1})}$. There is a canonical isometric isomorphism between $\overline{H}_n(C_*)$ and $\ker(\partial_n)/\overline{\text{im}(\partial_{n+1})}$.

For better readability, (semi-)norms on a chain complex will be written with single bars like $|\cdot|$ and the induced semi-norm on homology will have double bars like $\|\cdot\|$.

We collect basic results about semi-norms and null-spaces that will be needed in the upcoming sections.

Lemma 9.4. *Let X be a semi-normed $\mathbb{F}$ -vector space. Then the induced quotient semi-norm on $X/N(X)$ is a norm. Furthermore, the canonical projection $X \rightarrow X/N(X)$ is isometric with respect to the given semi-norm on X and its induced norm on $X/N(X)$.*

Proof. This is true by definition of the induced quotient norm. $\square$

Example 9.5 (truncated ℓ^1 -semi-norm). Let X be an $\mathbb{F}$ -vector space and let B be a basis of X . For a subfamily B' of B let

$$\|\cdot\|_1^{B,B'} : X \rightarrow \mathbb{R}_{\geq 0}$$

$$\sum_{b \in B} \lambda_b \cdot b \mapsto \sum_{b \in B'} |\lambda_b|.$$

Then, $\|\cdot\|_1^{B,B'}$ is a semi-norm on X , called the *truncated ℓ^1 -semi-norm on X associated to (B, B')* . The null space of X with respect to $\|\cdot\|_1^{B,B'}$ is the subspace of X spanned by $B \setminus B'$.

The equivalence of norms on finite dimensional $\mathbb{R}$ -vector spaces gives rise to an equivalence of semi-norms that have the same null space.

Lemma 9.6. *Let $(X, \|\cdot\|)$ be a semi-normed $\mathbb{R}$ -vector space such that $X/N(X)$ is finite dimensional. Then there is a basis B and a subfamily B' of B such that $\|\cdot\|$ is equivalent to the truncated ℓ^1 -semi-norm on X associated to (B, B') .*

Proof. We choose an $\mathbb{R}$ -basis B_N of $N(X)$ and extend it to a basis B of X . Then, for $B' := B \setminus B_N$, we have $N := N(X) = N_{\|\cdot\|_1^{B,B'}}(X)$. Hence, the semi-norms induced by $\|\cdot\|$ and $\|\cdot\|_1^{B,B'}$ on the quotient space X/N are indeed norms by Lemma 9.4. Since this quotient space is finite dimensional, these induced norms are equivalent. As the projection map is isometric for both semi-norms and their induced quotient norms, we deduce that the semi-norms on X are also equivalent. $\square$

9.2 Boundedness of chain extensions with respect to the induced semi-norm on homology

We give pragmatic conditions under which for a marked free $\mathbb{Z}$ -chain complex C_* there is a chain homotopy equivalence between $C_* \otimes_{\mathbb{Z}} \mathbb{R}$ and $H_*(C_*; \mathbb{R})$ that is bounded in every degree. For a marked free chain complex C_* we will, for every $n \in \mathbb{N}$, denote by $|\cdot|_1$ the ℓ^1 -norm on C_n associated to the marked basis. The induced semi-norm on homology will be denoted by $\|\cdot\|_1$.

Proposition 9.7. *Let C_* be a marked free $\mathbb{R}$ -chain complex. Let $n \in \mathbb{N}$ and assume $\overline{H}_n(C_*)$ is finite dimensional. Let B be a basis of $H_n(C_*)$ and $B' \subset B$ such that the projection map $\pi: \ker \partial_n \rightarrow H_n(C_*)$ is bounded with respect to $|\cdot|_1^C$ on $\ker \partial_n \subset C_n$ and $\|\cdot\|_1^{B,B'}$ on $H_n(C_*)$. Then there is a chain extension of C_* in degree n that is bounded with respect to $|\cdot|_1$ and $\|\cdot\|_1^{B,B'}$.*

Proof. Since $\pi: \ker \partial_n \rightarrow H_n(C_*)$ is bounded with respect to $|\cdot|_1^C$ and $\|\cdot\|_1^{B,B'}$, the null space $N := N_{\|\cdot\|_1^{B,B'}}(H_n(C_*))$ contains $N_n(C_*)$. So there is a canonical projection

$$\overline{H}_n(C_*) = H_n(C_*)/N_n(C_*) \rightarrow H_n(C_*)/N.$$

Since $\overline{H}_n(C_*)$ is finite dimensional by assumption, this implies that $H_n(C_*)/N$ is finite dimensional and so B' is a finite subfamily of B .

In order to find an extension of π to C_n , we use the Hahn–Banach theorem to extend the coordinate functions of π with respect to the basis B . Let $b \in B$. We denote by ev_b the $\mathbb{R}$ -linear map given by

$$\begin{aligned} \text{ev}_b: H_n(C_*) &\rightarrow \mathbb{R} \\ B \ni b' &\mapsto \delta_{b,b'}. \end{aligned}$$

We can describe π via the coordinate maps ev_b since

$$\pi(x) = \sum_{b \in B} \text{ev}_b(\pi(x)) \cdot b$$

for all $x \in \ker(\partial_n)$.

For all $b \in B$, the map $\pi_b := \text{ev}_b \circ \pi$ satisfies

$$\|\pi_b\| = \|\text{ev}_b \circ \pi\| \leq \|\text{ev}_b\| \cdot \|\pi\| < \infty.$$

For the calculation of the operator norms we consider the norm $|\cdot|_1^C$ on $\ker \partial_n$, the semi-norm $\|\cdot\|_1^{B,B'}$ on $H_n(C_*)$, and the absolute value on $\mathbb{R}$.

For each of the finitely many elements $b \in B'$ we choose an extension $s_b: C_n \rightarrow \mathbb{R}$ of π_b with $\|s_b\| = \|\pi_b\|$. The existence of such extensions is guaranteed by the Hahn–Banach theorem.

In order to extend the maps π_b for the remaining basis elements $b \in B \setminus B'$, we choose a linear complement U of $\ker \partial_n$ in C_n and define

$$\begin{aligned} s_b: C_n = \ker \partial_n \oplus U &\rightarrow \mathbb{R} \\ (x, u) &\mapsto \pi_b(x). \end{aligned}$$

We will prove that the map

$$\begin{aligned} s: C_n &\rightarrow H_n(C_*) \\ c &\mapsto \sum_{b \in B} s_b(c) \cdot b \end{aligned}$$

is a well-defined chain extension of C_* in degree n :

To prove that the map is well-defined, we need to check that, for every $c \in C_n$, its image is a finite sum. This is true, as for $c \in \ker \partial_n$, we know that $s(c) = \pi(c)$ is a finite sum and if $c \in U$, then $s_b(c)$ is zero for all $b \in B \setminus B'$. As B'

is finite, $s(c)$ has only finitely many non-zero summands. With $C_n = \ker \partial_n \oplus U$, we conclude that s is well-defined.

Since every coordinate map s_b of s is $\mathbb{R}$ -linear, also the map s is $\mathbb{R}$ -linear and by construction it is a chain extension of π .

Furthermore, $\|s\|$ is finite, as for every $c \in C_n$ we have

$$\begin{aligned} \|s(c)\| &= \left\| \sum_{b \in B} s_b(c) \cdot b \right\|_1^{B, B'} \leq \sum_{b \in B'} |s_b(c)| \\ &\leq \sum_{b \in B'} \|s_b\| \cdot |c|_1 = \sum_{b \in B'} \|\pi_b\| \cdot |c|_1 \leq \#B' \cdot \|\pi\| \cdot |c|_1. \end{aligned}$$

Hence, $\|s\| \leq \#B' \cdot \|\pi\|$, which is finite, since B' is a finite family. $\square$

Corollary 9.8. *Let C_* be a marked free $\mathbb{Z}$ -chain complex and let $n \in \mathbb{N}$. If $\overline{H}_n(C_*; \mathbb{R})$ is finite dimensional, then there is a chain extension of C_* in degree n over $\mathbb{R}$ that is bounded with respect to the induced semi-norm on $H_n(C_*; \mathbb{R})$.*

Proof. Since $\overline{H}_n(C_*; \mathbb{R})$ is finite dimensional, the induced semi-norm $\|\cdot\|_1^C$ on $H_n(C_*; \mathbb{R})$ is equivalent to a truncated ℓ^1 -semi-norm on $H_n(C_*; \mathbb{R})$ by Lemma 9.6. Hence, the canonical projection $\pi: \ker \partial_n \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow H_n(C_*; \mathbb{R})$ that is bounded with respect to the canonical norm and semi-norm is also bounded with respect to this truncated ℓ^1 -semi-norm on homology. We conclude using Proposition 9.7 and using again the equivalence of the semi-norms on homology. $\square$

Corollary 9.9. *Let C_* be a free marked $\mathbb{Z}$ -chain complex such that $\overline{H}_n(C_*; \mathbb{R})$ is finite dimensional in every degree $n \in \mathbb{N}$. Then there is an $\mathbb{R}$ -chain homotopy equivalence $(C_*, \partial_*) \rightarrow (H_*(C_*), 0)$ that is bounded in every degree with respect to the induced semi-norm on homology.*

Proof. By Corollary 9.8, we obtain for every $n \in \mathbb{N}$ a chain extension s_n of C_* in degree n over $\mathbb{R}$ that is bounded with respect to the induced semi-norm on $H_n(C_*; \mathbb{R})$. The resulting chain map s_* is a chain homotopy equivalence [Bro94, Chapter 1, Theorem 8.4] as the $\mathbb{R}$ -chain complex $C_* \otimes_{\mathbb{Z}} \mathbb{R}$ is free in every degree and bounded below. $\square$

Example 9.10. Let X be a topological space such that $\overline{H}_n(X; \mathbb{R})$ is finite dimensional in every degree $n \in \mathbb{N}$. Then, by Corollary 9.9, there is a chain homotopy equivalence s_* between $C_*(X; \mathbb{R})$ and $H_*(X; \mathbb{R})$ endowed with the trivial boundary operator that is bounded in every degree with respect to the ℓ^1 -norm on $C_*(X; \mathbb{R})$ associated to the basis of singular simplices and the induced semi-norm on $H_*(X; \mathbb{R})$.

Instances of topological spaces X that fulfil these assumptions are closed manifolds and CW-complexes of finite type.

9.3 The Künneth isomorphism is bilipschitz

We consider the Künneth theorem, which describes the homology of a tensor product complex by the homology of the factors. The goal is to study how this isomorphism behaves with respect to norms. One key ingredient in this comparison is the existence of a bounded chain homotopy equivalence between an $\mathbb{R}$ -chain complex and its homology (Corollary 9.8).

Theorem 9.11 (Künneth isomorphism, normed version). *Let C_* and D_* be normed $\mathbb{R}$ chain complexes such that $\bar{H}_n(C_*; \mathbb{R})$ is finite dimensional in every degree $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, the Künneth isomorphism*

$$\varphi_n: \bigoplus_{p \in \mathbb{N}} H_p(C_*) \otimes_{\mathbb{R}} H_{n-p}(D_*) \rightarrow H_n(C_* \otimes_{\mathbb{R}} D_*)$$

$$[c] \otimes [d] \mapsto [c \otimes d]$$

is bilipschitz with respect to $\|\cdot\|_{\otimes}$ and $\|\cdot\|^{C \otimes D}$.

In order to define $\|\cdot\|_{\otimes}$ and $\|\cdot\|^{C \otimes D}$, we introduce (semi-)norms on tensor products.

Definition 9.12 (tensor (semi-)norm). Let X, Y be normed (resp. semi-normed) $\mathbb{R}$ -vector spaces. Then $X \otimes_{\mathbb{R}} Y$ carries a canonical norm (resp. semi-norm) $\|\cdot\|_{\otimes}$ that is given on $z \in X \otimes_{\mathbb{R}} Y$ by

$$\|z\|_{\otimes} := \inf \left\{ \sum_{i \in I} \|x_i\|_X \cdot \|y_i\|_Y \mid I \text{ a finite set, } \forall_{i \in I} x_i \in X \wedge y_i \in Y, \right.$$

$$\left. z = \sum_{i \in I} x_i \otimes y_i \right\}.$$

This (semi-)norm on a tensor product is often referred to as projective (tensor) norm to distinguish it from other tensor norms. Since we only consider this specific tensor norm, we simply call it tensor norm.

Notation 9.13. Let C_* and D_* be semi-normed $\mathbb{R}$ -chain complexes with norms $|\cdot|_C$ and $|\cdot|_D$, respectively, and let $n \in \mathbb{N}$.

- We denote by $\|\cdot\|_{\otimes}$ the semi-norm on $\bigoplus_{p \in \mathbb{N}} H_p(C_*) \otimes_{\mathbb{R}} H_{n-p}(D_*)$ given by the ℓ^1 -sum of the tensor semi-norms associated to $\|\cdot\|_C$ and $\|\cdot\|_D$ on each summand.
- We denote by $|\cdot|^{C \otimes D}$ the norm on

$$(C_* \otimes_{\mathbb{R}} D_*)_n = \bigoplus_{p \in \mathbb{N}} C_p \otimes_{\mathbb{R}} D_{n-p}$$

given by the ℓ^1 -sum of the tensor semi-norms associated to $|\cdot|_C$ and $|\cdot|_D$ on each summand. With $\|\cdot\|^{C \otimes D}$, we denote the induced semi-norm on the homology of the tensor product complex $C_* \otimes_{\mathbb{R}} D_*$.

- If C_* and D_* are marked free chain complexes, then $|\cdot|^{C \otimes D}$ is equivalent to the ℓ^1 -norm on $(C_* \otimes_{\mathbb{R}} D_*)_n$ given by the basis on the tensor product complex induced by the marked bases of the factors (Corollary 9.15). We denote it by $|\cdot|_1^{C \otimes D}$ and the induced semi-norm on homology by $\|\cdot\|_1^{C \otimes D}$.

We will often use the following description of the tensor (semi-)norm:

Lemma 9.14. *Let X be an $\mathbb{R}$ -vector space equipped with a truncated ℓ^1 -semi-norm associated to the basis B and a subfamily $B' \subset B$ and let Y be a (semi-)normed $\mathbb{R}$ -vector space.*

normed $\mathbb{R}$ -vector space. Then, for all $z \in X \otimes_{\mathbb{R}} Y$, the norm $\|z\|_{\otimes}$ is attained by a specific representative, namely

$$\|z\|_{\otimes} = \sum_{b \in B} \|b\|_1^{B, B'} \cdot \|z_b\|_Y = \sum_{b \in B'} \|z_b\|_Y,$$

where $z = \sum_{b \in B} b \otimes z_b$ with $z_b \in Y$. Note that every element in $X \otimes_{\mathbb{R}} Y$ has such a representative and that it is unique.

Proof. Let $z = \sum_{b \in B} b \otimes z_b \in X \otimes_{\mathbb{R}} Y$. By definition of the tensor semi-norm we have

$$\|z\|_{\otimes} \leq \sum_{b \in B} \|b\|_1^{B, B'} \cdot \|z_b\|_Y = \sum_{b \in B'} \|z_b\|_Y.$$

Conversely, let $z = \sum_{i \in I} x_i \otimes y_i$ be a tensor representative of z with $x_i = \sum_{b \in B} \lambda_{i,b} \cdot b$ for all $i \in I$. This means

$$z = \sum_{i \in I} x_i \otimes y_i = \sum_{i \in I} \sum_{b \in B} (\lambda_{i,b} \cdot b) \otimes y_i = \sum_{b \in B} b \otimes \left(\sum_{i \in I} \lambda_{i,b} \cdot y_i \right).$$

Therefore we have $z_b = \sum_{i \in I} \lambda_{i,b} \cdot y_i$ for all $b \in B$. We compute

$$\sum_{i \in I} \|x_i\|_X \cdot \|y_i\|_Y = \sum_{i \in I} \sum_{b \in B'} |\lambda_{i,b}| \cdot \|y_i\|_Y \geq \sum_{b \in B'} \left\| \sum_{i \in I} \lambda_{i,b} \cdot y_i \right\|_Y = \sum_{b \in B'} \|z_b\|_Y.$$

Taking the infimum over all tensor representatives of z yields

$$\|z\|_{\otimes} \geq \sum_{b \in B'} \|z_b\|_Y = \sum_{b \in B} \|b\|_1^{B, B'} \cdot \|z_b\|_Y. \quad \square$$

Example 9.15. Let X and Y be marked $\mathbb{R}$ -vector spaces with the corresponding ℓ^1 -norms. Then, by Lemma 9.14, the induced tensor norm on $X \otimes_{\mathbb{R}} Y$ coincides with the ℓ^1 -norm on $X \otimes_{\mathbb{R}} Y$ associated to the basis that is given by the marked bases of X and Y .

Proof of Theorem 9.11. We start by proving that the Künneth isomorphism φ_n in degree $n \in \mathbb{N}$ is bounded. Since $\|\cdot\|_{\otimes}$ is the ℓ^1 -sum of the tensor norms on the summands and only finitely many summands are non-zero, it suffices to show that φ_n is bounded on every summand. Let $p \in \{0, \dots, n\}$ and $x = \sum_{i \in I} \alpha_i \otimes \beta_i \in H_p(C_*) \otimes_{\mathbb{R}} H_{n-p}(D_*)$. For each $i \in I$, we choose a representing cycle c_i of α_i and d_i of β_i . Then

$$\begin{aligned} \|\varphi_n(x)\|^{C \otimes D} &= \left\| \left[\sum_{i \in I} c_i \otimes d_i \right] \right\|^{C \otimes D} \leq \left| \sum_{i \in I} c_i \otimes d_i \right|^{C \otimes D} \\ &\leq \sum_{i \in I} |c_i|_1^C \cdot |d_i|_1^D. \end{aligned}$$

Taking the infimum over all representing cycles and over all tensor representatives, we obtain

$$\|\varphi_n(x)\|^{C \otimes D} \leq \|x\|_{\otimes}.$$

In order to prove that the inverse of the Künneth isomorphism is bounded, we reduce to the case that C_* has trivial boundary operator and give an explicit description of such an inverse map in this case. Let $(H_*(C_*), 0)$ be the homology chain complex of C_* endowed with the semi-norm on homology of C_* . Then there is a chain homotopy equivalence $f_* : C_* \rightarrow H_*(C_*)$ that is bounded in every degree by Corollary 9.9.

By assumption, in every degree n , the space $H_n(C_*)/N(H_n(C_*))$ is finite dimensional. So by Lemma 9.6 we can choose a basis B_n of $H_n(C_*)$ and a subfamily B'_n of B_n such that the semi-norm on $H_n(C_*)$ is equivalent to the truncated ℓ^1 -semi norm $\|\cdot\|_1^{B_n, B'_n}$. We denote the homology complex endowed with this semi-norm by A_* and denote the truncated ℓ^1 -semi-norm on A_* by $|\cdot|_A$. By construction, the identity map is a chain homotopy equivalence between A_* and $H_*(C_*)$ bounded in every degree and the chain homotopy equivalence $f_* : C_* \rightarrow A_*$ is also bounded in every degree.

We have the following chain of isomorphisms that describes the inverse of the Künneth isomorphism for $C_* \otimes_{\mathbb{R}} D_*$:

$$\begin{aligned}
& H_n(C_* \otimes_{\mathbb{R}} D_*) \\
& \cong \downarrow H_n(f_* \otimes_{\mathbb{R}} \text{id}_{D_*}), \text{ bounded} \\
& H_n(A_* \otimes_{\mathbb{R}} D_*) \\
& \cong \downarrow \text{inverse of the Künneth isomorphism for } A_* \otimes_{\mathbb{R}} D_* \\
& \bigoplus_{p \in \mathbb{N}} H_p(A_*) \otimes_{\mathbb{R}} H_{n-p}(D_*) \\
& \cong \downarrow \text{bounded isomorphism by definition of } A_* \\
& \bigoplus_{p \in \mathbb{N}} H_p(C_*) \otimes_{\mathbb{R}} H_{n-p}(D_*)
\end{aligned}$$

So in order to prove the theorem, it remains to show that the inverse of the Künneth isomorphism

$$\begin{aligned}
\varphi_n : \bigoplus_{p \in \mathbb{N}} H_p(A_*) \otimes_{\mathbb{R}} H_{n-p}(D_*) & \rightarrow H_n(A_* \otimes_{\mathbb{R}} D_*) \\
[a] \otimes [d] & \mapsto [a \otimes d]
\end{aligned}$$

is bounded.

Claim. *The map*

$$\begin{aligned}
\psi_n : H_n(A_* \otimes_{\mathbb{R}} D_*) & \rightarrow \bigoplus_{p \in \mathbb{N}} H_p(A_*) \otimes_{\mathbb{R}} H_{n-p}(D_*) \\
\left[\sum_{p=0}^n \sum_{b \in B_p} b \otimes y_{p,b} \right] & \mapsto \sum_{p=0}^n \sum_{b \in B_p} [b] \otimes [y_{p,b}]
\end{aligned}$$

is well-defined, has bounded norm, and is inverse to the Künneth map.

Proof of Claim. In degree $n \in \mathbb{N}$, every n -chain of $A_* \otimes_{\mathbb{R}} D_*$ has a unique representative of the form $\sum_{p=0}^n \sum_{b \in B_p} b \otimes y_{p,b}$ with $y_{p,b} \in D_{n-p}$. Furthermore, if $\sum_{p=0}^n \sum_{b \in B_p} b \otimes y_{p,b}$ is a cycle, then, for every $p \in \{1, \dots, n\}$, the chain $\sum_{b \in B_p} b \otimes \partial^D y_{p,b} = 0$ is trivial, since ∂^A is trivial. By uniqueness of such a

representative, we conclude that every $\partial^D y_{p,b}$ is zero and $y_{p,b}$ is a cycle. Hence, the expression on the right hand side of the definition of ψ_n indeed describes classes in $\bigoplus_{p \in \mathbb{N}} H_p(A_*) \otimes_{\mathbb{R}} H_{n-p}(D_*)$.

If two cycles $x = \sum_{p=0}^n \sum_{b \in B_p} b \otimes x_{p,b}$ and $y = \sum_{p=0}^n \sum_{b \in B_p} b \otimes y_{p,b}$ represent the same class in $H_n(A_* \otimes_{\mathbb{R}} D_*)$, then we obtain a boundary

$$x - y = \sum_{p=0}^n \sum_{b \in B_p} b \otimes (x_{p,b} - y_{p,b}).$$

Since the boundary operator of A_* is zero, this means that for all $p \in \{0, \dots, n\}$ and $b \in B_p$ the element $x_{p,b} - y_{p,b}$ is a boundary in D_{n-p} . Hence, we have $[x_{p,b}] = [y_{p,b}]$. All in all we conclude that ψ_n is well-defined.

One can easily see that the composition $\varphi_n \circ \psi_n$ is the identity for every $n \in \mathbb{N}$. For proving that also $\psi_n \circ \varphi_n$ is identity, let $a \in A_p$ and $d \in D_{n-p}$ be cycles with $p \in \{0, \dots, n\}$. Let $(\lambda_b)_{b \in B_p}$ be a family in $\mathbb{R}$ such that $a = \sum_{b \in B_p} \lambda_b \cdot b$. We compute

$$\begin{aligned} \psi_n \circ \varphi_n([a] \otimes [d]) &= \psi_n([a \otimes d]) = \psi_n\left(\left[\sum_{b \in B_p} b \otimes (\lambda_b \cdot d)\right]\right) \\ &= \sum_{b \in B_p} [b] \otimes [\lambda_b \cdot d] = \sum_{b \in B_p} [\lambda_b \cdot b] \otimes [d] = [a] \otimes [d]. \end{aligned}$$

Hence, $\psi_n \circ \varphi_n$ is identity, as it is identity on a generating set. Put together, this means that ψ_n is the inverse of φ_n .

It remains to show that ψ_n is bounded. We observe that the quotient semi-norm $\|\cdot\|_A$ on $H_*(A_*)$ in degree $n \in \mathbb{N}$ is the truncated ℓ^1 -semi-norm associated to the basis $[B_n] := ([b])_{b \in B_n}$ and the subfamily $[B'_n] := ([b'])_{b' \in B'_n}$ as the boundary operator of A_* is zero. By Lemma 9.14, we can compute the semi-norm $\|\cdot\|_{\otimes}$ of an element in $H_p(A_*) \otimes_{\mathbb{R}} H_{n-p}(D_*)$ (for $p \in \{1, \dots, n\}$) by looking at its unique representative given by the bases $[B_p]$ of $H_p(A_*)$, i.e., we have

$$\left\| \sum_{b \in B_p} [b] \otimes [y_{p,b}] \right\|_{\otimes} = \sum_{b \in B'_p} \|[y_{p,b}]\|_D.$$

Analogously, let $\alpha \in H_n(A_* \otimes_{\mathbb{R}} D_*)$. Again by Lemma 9.14, we observe that

$$\|\alpha\|^{A \otimes D} = \inf \left\{ \sum_{b \in B'_p} |y_{p,b}|_D \left| \sum_{p=0}^n \sum_{b \in B_p} b \otimes y_{p,b} \text{ with } y_{p,b} \in D_{n-p} \text{ is a representative of } \alpha \right. \right\}.$$

For such a representative $\sum_{p=0}^n \sum_{b \in B_p} b \otimes y_{p,b}$ of α we compute

$$\|\psi_n(\alpha)\|_{\otimes} = \left\| \sum_{p=0}^n \sum_{b \in B_p} [b] \otimes [y_{p,b}] \right\|_{\otimes} = \sum_{p=0}^n \sum_{b \in B'_p} \|[y_{p,b}]\|_D \leq \sum_{b \in B'_p} |y_{p,b}|_D.$$

Taking the infimum over all such representatives of α yields the boundedness of ψ_n . $\square$

We now proved that the Künneth isomorphism and its inverse are bounded. This means that the Künneth isomorphism is bilipschitz. $\square$

Corollary 9.16. *Let X, Y be topological spaces such that X has finite dimensional reduced homology in each degree, e.g. X is a closed manifold or a CW-complex of finite type. Then, for each $n \in \mathbb{N}$, the topological Künneth isomorphism*

$$\bigoplus_{p \in \mathbb{N}} H_p(X; \mathbb{R}) \otimes_{\mathbb{R}} H_{n-p}(Y; \mathbb{R}) \rightarrow H_n(X \times Y; \mathbb{R})$$

$$\alpha \otimes \beta \mapsto \alpha \times \beta$$

is bilipschitz.

Proof. As X has finite dimensional reduced homology, the singular chain complexes $C_*(X; \mathbb{R})$ and $C_*(Y; \mathbb{R})$ together with the ℓ^1 -norms given by the bases of singular simplices fulfil the assumptions of Theorem 9.11. So the algebraic Künneth isomorphism for $C_*(X; \mathbb{R})$ and $C_*(Y; \mathbb{R})$ is bilipschitz and we reduce the claim to proving that, for every $n \in \mathbb{N}$, the isomorphism

$$H_n(C_*(X; \mathbb{R}) \otimes_{\mathbb{R}} C_*(Y; \mathbb{R})) \rightarrow H_n(X \times Y; \mathbb{R})$$

induced by the Eilenberg–Zilber map is bilipschitz. Looking at explicit descriptions of the Eilenberg–Zilber map and its inverse [Dol72, Exercises VI.12.26], we see that both are bounded on the level of chains. This passes down to the level of homology with the induced semi-norm (Remark 3.7). $\square$

Example 9.17. Let M be an oriented, closed, connected hyperbolic manifold. The surjectivity of the comparison map $H_b^n(M; \mathbb{R}) \rightarrow H^*(M; \mathbb{R})$ for $n = 0$ and $n \geq 2$ shows that a class in the singular homology $H_n(M; \mathbb{R})$ of M has trivial norm if and only if the class is trivial or of degree $n = 1$ [Gro87, 8.3T].

Let N be another oriented, closed, connected hyperbolic manifold. By the norm control of the topological Künneth isomorphisms (Corollary 9.16) we see that a class in $H_n(M \times N; \mathbb{R})$ has trivial norm if and only if it is in the image of $H_1(M; \mathbb{R}) \otimes_{\mathbb{R}} H_{n-1}(N; \mathbb{R}) \oplus H_{n-1}(M; \mathbb{R}) \otimes_{\mathbb{R}} H_1(N; \mathbb{R})$ under the Künneth isomorphism.

Note that this can also be shown using duality results for the computation of the ℓ^1 -norm and the comparison maps for the product space $M \times N$.

The result about the norm control of the Künneth theorem can also be used to control the norm of the product of bilipschitz chain maps.

Corollary 9.18. *Let X, X', Y, Y' be topological spaces such that X and X' have finite dimensional reduced homology. Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be continuous maps. Let $n \in \mathbb{N}$ such that $H_p(f; \mathbb{R})$ and $H_q(g; \mathbb{R})$ are bilipschitz with respect to the ℓ^1 -semi-norm for all $p, q \in \{0, \dots, n\}$. Then the induced map $H_n(f \times g; \mathbb{R}): H_n(X \times Y; \mathbb{R}) \rightarrow H_n(X' \times Y'; \mathbb{R})$ is also bilipschitz.*

Proof. Since X and X' have finite dimensional reduced homology, we can apply that the topological Künneth isomorphism is bilipschitz, Corollary 9.16, to reduce the problem to showing that the map

$$\bigoplus_{p \in \mathbb{N}} H_p(f; \mathbb{R}) \otimes_{\mathbb{R}} H_{n-p}(g; \mathbb{R}) : \bigoplus_{p \in \mathbb{N}} H_p(X; \mathbb{R}) \otimes_{\mathbb{R}} H_{n-p}(Y; \mathbb{R}) \rightarrow \bigoplus_{p \in \mathbb{N}} H_p(X'; \mathbb{R}) \otimes_{\mathbb{R}} H_{n-p}(Y'; \mathbb{R})$$

is bilipschitz. As there are only finitely many non-trivial summands, it suffices to prove that for every $p \leq n$ the map $H_p(f; \mathbb{R}) \otimes_{\mathbb{R}} H_{n-p}(g; \mathbb{R})$ is bilipschitz.

As $H_p(f; \mathbb{R})$ and $H_{n-p}(g; \mathbb{R})$ are bilipschitz by assumption this follows from the following lemma. $\square$

Lemma 9.19. *Let X, X', Y, Y' be semi-normed $\mathbb{R}$ -vector spaces such that $X'/N(X')$ is finite dimensional. Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be bilipschitz linear maps. Then also $f \otimes_{\mathbb{R}} g: X \otimes_{\mathbb{R}} Y \rightarrow X' \otimes_{\mathbb{R}} Y'$ is bilipschitz.*

Proof. Let c_f and c_g be bilipschitz constants for f and g , respectively. Then, for $z \in X \otimes_{\mathbb{R}} Y$, we have $\|f \otimes_{\mathbb{R}} g(z)\|_{\otimes} \leq c_f \cdot c_g \|z\|_{\otimes}$: For every representative $\sum_{i \in I} x_i \otimes y_i$ of z , we compute

$$\begin{aligned} \|f \otimes_{\mathbb{R}} g(z)\|_{\otimes} &= \left\| \sum_{i \in I} f(x_i) \otimes g(y_i) \right\|_{\otimes} \leq \sum_{i \in I} \|f(x_i)\|_{X'} \cdot \|g(y_i)\|_{Y'} \\ &\leq \sum_{i \in I} c_f \cdot c_g \cdot \|x_i\|_X \cdot \|y_i\|_Y. \end{aligned}$$

Taking the infimum over all representatives of z gives the desired inequality.

For the lower bound, recall that the space $X'/N(X')$ is finite dimensional by assumption. Hence, all semi-norms on X' respecting the null space $N(X')$ are equivalent by Lemma 9.6. So without loss of generality we can assume that the semi-norm on X' is a truncated ℓ^1 -semi-norm associated to a basis B and a subfamily $B' \subset B$ compatible with $\text{im}(f)$ and $N(X')$ in the following way: Let B'' be a basis of $N(X') \cap \text{im}(f)$. We extend B'' on the one hand to a basis B_N of $N(X')$ and on the other hand to a basis B_f of $\text{im}(f)$. As $B_N \cap B_f = B''$ is a basis of $N(X') \cap \text{im}(f)$, we know that $B_N \cup B_f$ is a basis of $N(X') \cup \text{im}(f)$. Hence, we can extend $B_N \cup B_f$ to a basis B of X' and define $B' := B \setminus B_N$. By construction, B has both B_N and B_f as subfamily. For $b \in B_f$ we fix a preimage $x_b \in X$. For an element $z \in X \otimes_{\mathbb{R}} Y$ there is a family $(y_b)_{b \in B_f}$ in Y such that

$$z = \sum_{b \in B_f} x_b \otimes y_b \mod \text{im}(\ker(f) \otimes_{\mathbb{R}} Y \rightarrow X \otimes_{\mathbb{R}} Y).$$

So we obtain

$$f \otimes_{\mathbb{R}} g(z) = \sum_{b \in B_f} f(x_b) \otimes g(y_b) = \sum_{b \in B_f} b \otimes g(y_b). \quad (1)$$

Since elements in $\ker(f)$ and hence elements in $\text{im}(\ker(f) \otimes_{\mathbb{R}} Y \rightarrow X \otimes_{\mathbb{R}} Y)$ have norm zero, we obtain

$$\begin{aligned} \|z\|_{\otimes} &\leq \sum_{b \in B_f} \|x_b\|_X \cdot \|y_b\|_Y \leq c_f \cdot c_g \cdot \sum_{b \in B_f} \|f(x_b)\|_{X'} \cdot \|g(y_b)\|_{Y'} \\ &= c_f \cdot c_g \cdot \sum_{b \in B_f} \|b\|_{X'} \cdot \|g(y_b)\|_{Y'}. \end{aligned}$$

As we chose the norm on X' to be the ℓ^1 -semi-norm associated to B and B' and as B_f is a subfamily of B , we obtain

$$\|z\|_{\otimes} \leq c_f \cdot c_g \cdot \sum_{b \in B_f} \|b\|_X \cdot \|g(y_b)\|_Y = c_f \cdot c_g \cdot \|f(z)\|_{\otimes}$$

using Lemma 9.14 and Equation 1. Thus, $f \otimes_{\mathbb{R}} g$ is bilipschitz. $\square$

Replacing “bilipschitz” by “isometric” in the previous results gives rise to the following question.

Question 9.20. Let X, X', Y, Y' be topological spaces. Let $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ be continuous maps. Let $n \in \mathbb{N}$ such that $H_p(f; \mathbb{R})$ and $H_q(g; \mathbb{R})$ are isometric with respect to the ℓ^1 -semi-norm for all $p, q \in \{0, \dots, n\}$. Does it follow that $H_n(f \times g; \mathbb{R}): H_n(X \times Y; \mathbb{R}) \rightarrow H_n(X' \times Y'; \mathbb{R})$ is isometric with respect to the ℓ^1 -semi-norm?

In our proof of Corollary 9.18 we use that the Künneth isomorphism is bilipschitz (Theorem 9.11). Using our techniques, there is no control over the bilipschitz constant of the Künneth isomorphism. So this approach is not suited to address the question above.

Declaration

We have not used any LLM tools in the research for or writing of this article.

References

- [BK08] Michelle Bucher-Karlsson. The simplicial volume of closed manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$. *J. Topol.*, 1(3):584–602, 2008. Cited on page: 3
- [Bro94] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original. Cited on page: 5, 36
- [BS26] Uri Bader and Roman Sauer. Uniform waist inequalities in codimension two for manifolds with Kazhdan fundamental group. *Geom. Funct. Anal.*, 36:981–997, 2026. Cited on page: 1, 3, 12, 14
- [CDMW18] Gregory R. Chambers, Dominic Dotterrer, Fedor Manin, and Shmuel Weinberger. Quantitative null-cobordism. *J. Amer. Math. Soc.*, 31(4):1165–1203, 2018. With an appendix by F. Manin and S. Weinberger. Cited on page: 1, 2, 11, 17, 18, 24
- [Dol72] Albrecht Dold. *Lectures on Algebraic Topology*, volume Band 200 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York-Berlin, 1972. Cited on page: 41
- [Gro82] Michael Gromov. Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.*, (56):5–99 (1983), 1982. Cited on page: 1, 27, 28
- [Gro87] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987. Cited on page: 41
- [HL21] Nicolaus Heuer and Clara Löh. The spectrum of simplicial volume. *Invent. Math.*, 223(1):103–148, 2021. Cited on page: 3, 28
- [HL22] Nicolaus Heuer and Clara Löh. Simplicial volume of one-relator groups and stable commutator length. *Algebr. Geom. Topol.*, 22(4):1615–1661, 2022. Cited on page: 28
- [HL24] Nicolaus Heuer and Clara Löh. Transcendental simplicial volumes. *Ann. Inst. Fourier (Grenoble)*, 74(2):763–781, 2024. Cited on page: 28

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