THE ALGEBRAIC CHEAP REBUILDING PROPERTY

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Abstract. We present an axiomatic approach to combination theorems for various homological properties of groups and, more generally, of chain complexes. Examples of such properties include algebraic finiteness properties, ℓ^2-invisibility, ℓ^2-acyclicity, lower bounds for Novikov–Shubin invariants, and vanishing of homology growth.

We introduce an algebraic version of Abért–Bergeron–Frączyk–Gaboriau's cheap rebuilding property that implies vanishing of torsion homology growth and admits a combination theorem. As an application, we show that certain graphs of groups with amenable vertex groups and elementary amenable edge groups have vanishing torsion homology growth.

1. Introduction

Bootstrapping Theorems. Many homological properties of groups enjoy inheritance results, e.g., for graphs of groups (such as amalgamated products and HNN-extensions) and for group extensions. For a sequence $\mathsf{B}_* = (\mathsf{B}_n)_{n\in\mathbb{Z}}$ of classes of groups, the following are desirable:

(1) Let Γ be the fundamental group of a finite graph of groups. If all vertex groups lie in B_n and all edge groups lie in B_{n-1}, then $\Gamma \in \mathsf{B}_n$;
(2) Let $1 \to N \to \Gamma \to Q \to 1$ be a group extension. If $N \in \mathsf{B}_m$ for all $m \leq n$ and Q is of type F_n, then $\Gamma \in \mathsf{B}_n$.

Recall that a group Γ is *of type* F_n if there exists a model for the classifying space $K(\Gamma, 1)$ with finite n-skeleton. Statements (1) and (2) are very useful in practice, e.g., for inductive approaches. For instance, if the group of integers $\mathbb{Z}$ lies in B_m for all $m \leq n$, then it follows from (1) and (2) that all infinite elementary amenable groups of type FP_∞ lie in B_n. In fact, both (1) and (2) are special cases of the following more general inheritance result.

We introduce the notion of a *bootstrappable property* B_* *of groups* (Definition 3.4). It is a sequence $\mathsf{B}_* = (\mathsf{B}_n)_{n\in\mathbb{Z}}$ of classes of groups Γ for which the trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$ admits a projective resolution lying in a class of $\mathbb{Z}\Gamma$-chain complexes satisfying certain axioms. These axioms are easy to formulate and straightforward to verify in many situations.

Theorem 1.1 (Bootstrapping Theorem 3.6)**.** *Let* B_* *be a bootstrappable property of groups. Let* Γ *be a group and let* $n \in \mathbb{N}$*. Let* Ω *be a* Γ*-CW-complex such that the following hold:*

(i) Ω *is* $(n-1)$*-acyclic (i.e.,* $H_j(\Omega;\mathbb{Z}) \cong H_j(\mathrm{pt};\mathbb{Z})$ *for all* $j \leq n-1$*);*
(ii) *The quotient* $\Gamma\backslash\Omega^{(n)}$ *of the n-skeleton* $\Omega^{(n)}$ *is compact;*
(iii) *For every cell* σ *of* Ω *with* $\dim(\sigma) \leq n$*, the stabiliser* Γ_σ *lies in* $\mathsf{B}_{n-\dim(\sigma)}$*.*

Then $\Gamma \in \mathsf{B}_n$.

Date: September 8, 2024.

2020 *Mathematics Subject Classification.* 20J06, 20E26.

Key words and phrases. Combination theorems, (torsion) homology growth, ℓ^2-invariants, amenable groups.

Theorem 1.1 is called a Bootstrapping Theorem (or Combination Theorem) for the obvious reason: The property of the group Γ is obtained by "bootstrapping" from the properties of the stabilisers $(\Gamma_\sigma)_\sigma$. Examples of bootstrappable properties of groups include algebraic finiteness properties FP_n, ℓ^2-invisibility, ℓ^2-acyclicity, and lower bounds for Novikov–Shubin invariants. The Bootstrapping Theorems for the first three of these properties are well-known [Bro87, ST14, Jo07]. Our axiomatic treatment provides a unified point of view, highlighting the essential axioms that lead to a Bootstrapping Theorem. Moreover, while previous proofs involved spectral sequences, our proof is inductive on the level of chain complexes and enjoys a certain simplicity.

In case the bootstrappable property B_* satisfies an additional restriction axiom, it follows that the intersection $\bigcap_{m\leq n} \mathsf{B}_m$ of classes of groups is closed under commensurability and under passing from commensurated subgroups to overgroups (Corollary 3.11).

Torsion homology growth. In this introduction, we focus on bootstrappable properties related to the vanishing of (torsion) homology growth. Recall that a *residual chain* $\Lambda_* = (\Lambda_i)_{i\in\mathbb{N}}$ in a group Γ is a nested sequence $\Gamma = \Lambda_0 \geq \Lambda_1 \geq \cdots$, where each Λ_i is a finite index normal subgroup of Γ, such that $\bigcap_{i\in\mathbb{N}} \Lambda_i = \{1\}$. A group is *residually finite* if it admits a residual chain.

Definition 1.2. Let Γ be a residually finite group and let Λ_* be a residual chain in Γ. Let $j \in \mathbb{N}$ and let $\mathbb{F}$ be a field. Define

$$\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{F}) \coloneqq \limsup_{i\to\infty} \frac{\dim_{\mathbb{F}} H_j(\Lambda_i; \mathbb{F})}{[\Gamma : \Lambda_i]};$$

$$\widehat{t}_j(\Gamma, \Lambda_*) \coloneqq \limsup_{i\to\infty} \frac{\log \operatorname{tors} H_j(\Lambda_i; \mathbb{Z})}{[\Gamma : \Lambda_i]}.$$

Here tors denotes the cardinality of the torsion subgroup. If it is infinite, we set $\log \infty \coloneqq \infty$.

The motivation to study these invariants comes from Lück's Approximation Theorem which states that $\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{Q})$ agrees with the ℓ^2-Betti number $b_j^{(2)}(\Gamma)$, provided that Γ is of type FP_{j+1}. Conjecturally, $\widehat{t}_j(\Gamma, \Lambda_*)$ is related to ℓ^2-torsion. We refer to Lück's survey [Lüc16] for more background. We study the vanishing of these invariants and use the following non-standard notation.

For $n \in \mathbb{N}$, the class $\mathsf{H}_n(\mathbb{F})$ (resp. T_n) consists of all residually finite groups Γ satisfying $\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{F}) = 0$ (resp. $\widehat{t}_j(\Gamma, \Lambda_*) = 0$) for all residual chains Λ_* in Γ and all $j \leq n$. We also define the classes $\mathsf{H}_\infty(\mathbb{F}) \coloneqq \bigcap_{n\in\mathbb{Z}} \mathsf{H}_n(\mathbb{F})$ and $\mathsf{T}_\infty \coloneqq \bigcap_{n\in\mathbb{Z}} \mathsf{T}_n$.

While the rank of homology is bounded by the rank of the chain module, the situation is more complicated for torsion in homology. The main estimate is given by a result of Gabber: Let (X, ∂) be a chain complex consisting of finitely generated based free $\mathbb{Z}$-modules. Then

$$\log \operatorname{tors} H_j(X) \leq \operatorname{rk}(X_j) \cdot \log_+ \|\partial_{j+1}\|, \tag{1.1}$$

where $\|\partial_{j+1}\|$ denotes the operator-norm of the map $\partial_{j+1} \colon X_{j+1} \to X_j$ with respect to the ℓ^1-norms on X_{j+1} and X_j, and $\log_+ \coloneqq \max\{\log, 0\}$. (Gabber's original estimate [Sou99] is more refined and phrased for ℓ^2-norms.) The sequence $\mathsf{H}_*(\mathbb{F})$ is a bootstrappable property of residually finite groups, whereas T_* does not seem to be. The goal is to find bootstrappable properties of residually finite groups that are (degreewise) contained in T_*, based on the estimate (1.1).

Abért–Bergeron–Frączyk–Gaboriau [ABFG24] introduced the sequence of classes of residually finite groups satisfying the *(geometric) cheap rebuilding property* which is contained in $\mathsf{H}_*(\mathbb{F})$ and T_{*-1} and admits a Bootstrapping Theorem. Their definition is geometric and requires the existence of models for classifying spaces satisfying delicate estimates on the number of cells and norms of boundary maps, homotopy equivalences, and homotopies. The proof that the (geometric) cheap rebuilding property admits a Bootstrapping Theorem [ABFG24, Theorem F] relies on a quantitative version of Geoghegan's Rebuilding Lemma [Geo08]. The paper [ABFG24] was the main inspiration for our work and we owe much to its ingenuity.

We introduce the sequences CR_* and CWR_* of classes of residually finite groups satisfying the *algebraic cheap rebuilding property* and the *algebraic cheap weak rebuilding property*, respectively (Definition 4.18). These classes of groups are defined purely algebraically such that for all $n \in \mathbb{Z}$ and every field $\mathbb{F}$ the following inclusions hold:

$$\mathsf{CR}_n \subset \mathsf{CWR}_n \subset \mathsf{H}_n(\mathbb{F}) \cap \mathsf{T}_{n-1}.$$

We denote $\mathsf{CR}_\infty := \bigcap_{n\in\mathbb{Z}} \mathsf{CR}_n$ and $\mathsf{CWR}_\infty := \bigcap_{n\in\mathbb{Z}} \mathsf{CWR}_n$. We show that CR_* is a bootstrappable property of residually finite groups (Proposition 4.23). The key observation is that the mapping cone of homotopy retracts is again a homotopy retract, and that all maps can be made explicit allowing for norm estimates. Our approach is closely related to that of Okun–Schreve [OS24]. While the algebraic cheap rebuilding property is heavily inspired by the (geometric) cheap rebuilding property of Abért–Bergeron–Frączyk–Gaboriau [ABFG24], there is no obvious implication (Remark 2.11 and Remark 4.25).

The basic example of a group that lies in CR_∞ is the integers $\mathbb{Z}$ (Example 4.21). Then it follows from the Bootstrapping Theorem 1.1 for CR_* that all residually finite infinite elementary amenable groups of type FP_∞ lie in CR_∞ (Example 3.9). Hence, in order to show that a residually finite group Γ lies in CR_n, it suffices to exhibit a Γ-CW-complex as in Theorem 1.1 with suitable stabilisers (e.g., stabilisers containing appropriate elementary amenable normal subgroups). This strategy has previously been applied for the (geometric) cheap rebuilding property to special linear groups and mapping class groups [ABFG24]. Other classes of groups satisfying the (geometric) cheap rebuilding property include certain Artin groups [ABFG24], outer automorphism groups of free products of $\mathbb{Z}/2$ [GGH], mapping tori of polynomially growing automorphisms [AHK24, AGHK], and inner-amenable groups [Usc].

The sequence CWR_* does not seem to be a bootstrappable property of residually finite groups. Nevertheless, we obtain the following modified Bootstrapping Theorem for CWR_* under stronger assumptions on stabilisers of cells of dimension ≥ 1.

Theorem 1.3 (Theorem 4.24)**.** *Let Γ be a residually finite group and let $n \in \mathbb{N}$. Let Ω be a Γ-CW-complex such that the following hold:*

(i) *Ω is $(n-1)$-acyclic;*
(ii) *$\Gamma\backslash\Omega^{(n)}$ is compact;*
(iii) *For every vertex v of Ω, the stabiliser Γ_v lies in CWR_n;*
(iv) *For every cell σ of Ω with $\dim(\sigma) \in \{1,\dots,n\}$, the stabiliser Γ_σ lies in $\mathsf{CR}_{n-\dim(\sigma)}$.*

Then $\Gamma \in \mathsf{CWR}_n$.

We show that residually finite infinite amenable groups of type FP_∞ lie in CWR_∞ (Theorem 5.5). In particular, these groups lie in $\mathsf{H}_\infty(\mathbb{F})$ for every field $\mathbb{F}$ and in T_∞. The latter is a result of Kar–Kropholler–Nikolov [KKN17], whose proof we follow. We do not know if residually finite infinite amenable groups of type FP_∞ lie in CR_∞. As a special case, Theorem 1.3 applies to certain graphs of amenable groups:

Corollary 1.4 (Corollary 5.7)**.** *Let Γ be the fundamental group of a finite graph of groups that is residually finite and let $n \in \mathbb{N}$. If all vertex groups are infinite amenable of type FP_n and all edge groups are infinite elementary amenable of type FP_∞, then $\Gamma \in \mathsf{T}_{n-1}$.*

Outlook. As a companion to the present article, in future work we will develop dynamical versions of algebraic cheap rebuilding properties. This dynamical approach will go beyond the residually finite case and will provide estimates for (torsion) homology growth in terms of dynamical systems.

Contents

2. Preliminaries on chain complexes

We fix some notational conventions. For the following generalities, we work over an arbitrary ring that will be left implicit. A chain complex X consists of chain modules X_j and differentials $\partial_j \colon X_j \to X_{j-1}$ for all $j \in \mathbb{Z}$. Let $f, g \colon X \to Y$ be chain maps. For a chain homotopy $H \colon X_* \to Y_{*+1}$ between f and g, we write $H \colon f \simeq g$ to indicate that $\partial^Y_{j+1} \circ H_j + H_{j-1} \circ \partial^X_j = f_j - g_j$ for all $j \in \mathbb{Z}$.

Let X be a chain complex. We say that X is *concentrated in degrees* $\leq n$ if $X_j = 0$ for all $j > n$. A *weak equivalence* of chain complexes is a quasi-isomorphism (i.e., a chain map that induces an isomorphism on homology in all degrees).

The *suspension* of X is the chain complex ΣX with chain modules $(\Sigma X)_j := X_{j-1}$ and differentials $\partial^{\Sigma X}_j := -\partial^X_{j-1}$. The suspension $\Sigma f \colon \Sigma X \to \Sigma Y$ of a chain map $f \colon X \to Y$ is given by $(\Sigma f)_j := f_{j-1}$. We denote by $\Sigma^k X$ and $\Sigma^k f$ the k-fold suspension of X and f, respectively. We denote by Σ^{-1} the desuspension functor, which is inverse to Σ.

2.1. **Mapping cones.** We review the mapping cone construction for chain complexes and its functoriality.

A *homotopy commutative square* consists of chain maps

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\downarrow & \circlearrowleft_H & \downarrow{\scriptstyle b} \\ Z & \xrightarrow[g]{} & W \end{array} \tag{2.1}$$

together with a chain homotopy $H\colon g \circ a \simeq b \circ f$. If $H = 0$, then the square is (strictly) commutative.

Definition 2.1 (Mapping cone)**.** Let $f\colon X \to Y$ be a chain map. The *mapping cone* $\operatorname{Cone}(f)$ is the chain complex with chain modules $\operatorname{Cone}(f)_j := X_{j-1} \oplus Y_j$ and differentials $\partial_j\colon \operatorname{Cone}(f)_j \to \operatorname{Cone}(f)_{j-1}$ given by

$$\partial_j(x, y) := \big(-\partial^X_{j-1}(x), \partial^Y_j(y) + f_{j-1}(x)\big).$$

The mapping cone is functorial with respect to the chain map in the following sense: A homotopy commutative square as in diagram (2.1) induces a chain map

$$(a, b; H)\colon \operatorname{Cone}(f) \to \operatorname{Cone}(g)$$

given by

$$(a, b; H)_j(x, y) := \big(a_{j-1}(x), b_j(y) - H_{j-1}(x)\big).$$

For example, we have $\Sigma X \cong \operatorname{Cone}(X \to 0)$ and, more generally, $X \oplus Y \cong \operatorname{Cone}(0\colon \Sigma^{-1}X \to Y)$. We include the proof of the following basic lemma [Wei94] for completeness.

Lemma 2.2. *Let $f\colon X \to Y$ be a chain map. The following hold:*

(i) *There exists a short exact sequence of chain complexes*

$$0 \to Y \xrightarrow{\iota} \operatorname{Cone}(f) \xrightarrow{\pi} \Sigma X \to 0 \tag{2.2}$$

which splits degreewise;

(ii) *There exists a natural long exact sequence of homology groups*

$$\cdots \to H_j(X) \to H_j(Y) \to H_j\big(\operatorname{Cone}(f)\big) \to H_{j-1}(X) \to \cdots; \tag{2.3}$$

(iii) *Let a homotopy commutative square as in diagram* (2.1) *be given. Consider the diagram of mapping cone sequences*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & \operatorname{Cone}(f) \\ {\scriptstyle a}\downarrow & \circlearrowleft_H & \downarrow{\scriptstyle b} & \circlearrowleft_0 & \downarrow{\scriptstyle (a,b;H)} \\ Z & \xrightarrow[g]{} & W & \longrightarrow & \operatorname{Cone}(g) \end{array}$$

If the chain maps a and b are weak equivalences, then so is $(a, b; H)$;

(iv) *For every $n \in \mathbb{Z}$, there exists a subcomplex Y^n of Y that is concentrated in degrees $\geq n$ such that $\tau_{<n}Y := \operatorname{Cone}(Y^n \to Y)$ satisfies*

$$H_j(\tau_{<n}Y) \cong \begin{cases} 0 & \text{for } j \geq n; \\ H_j(Y) & \text{for } j \leq n-1. \end{cases}$$

In particular, $(\tau_{<n}Y)_j = Y_j$ for all $j \leq n$.

Proof. (i) Define the chain maps ι and π by $\iota(y) := (0, y)$ and $\pi(x, y) := x$. Then a degreewise split σ of ι is given by $\sigma(x, y) = y$.

(ii) Consider the long exact homology sequence associated to the short exact sequence (2.2) and apply the suspension isomorphism $H_j(\Sigma X) \cong H_{j-1}(X)$.

(iii) Consider the (strictly) commutative diagram with short exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & \operatorname{Cone}(f) & \longrightarrow & \Sigma X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle (a,b;H)} & & \downarrow{\scriptstyle \Sigma a} & & \\ 0 & \longrightarrow & W & \longrightarrow & \operatorname{Cone}(g) & \longrightarrow & \Sigma Z & \longrightarrow & 0 \end{array}$$

If the chain maps a (or equivalently, Σa) and b are weak equivalences, then so is $(a, b; H)$ by the naturality of the long exact sequence (2.3) and the Five Lemma.

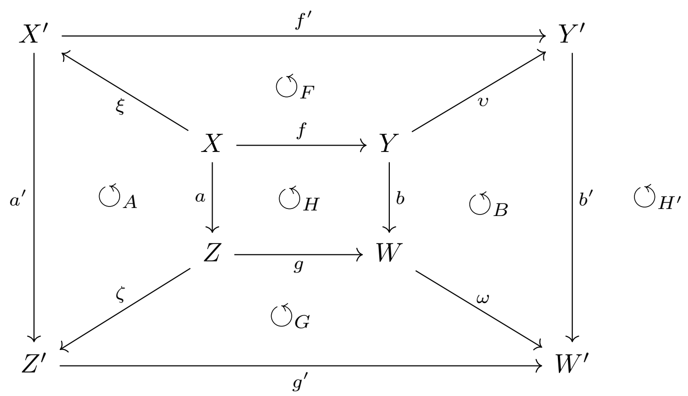
(iv) Take Y^n to be the subcomplex of Y given by

$$Y^n_j := \begin{cases} Y_j & \text{for } j \geq n+1; \\ \ker(\partial^Y_n) & \text{for } j = n; \\ 0 & \text{for } j \leq n-1. \end{cases}$$

It follows from the long exact sequence (2.3) that $\tau_{<n}Y = \operatorname{Cone}(Y^n \to Y)$ is as desired. ☐

We will use a "higher" functoriality of the mapping cone with respect to cubes that are coherently homotopy commutative. A *homotopy commutative cube* consists of six homotopy commutative squares

(2.4)



where

$$\begin{aligned} H &: g \circ a \simeq b \circ f; \\ H' &: g' \circ a' \simeq b' \circ f'; \\ A &: \zeta \circ a \simeq a' \circ \xi; \\ B &: \omega \circ b \simeq b' \circ \upsilon; \\ F &: f' \circ \xi \simeq \upsilon \circ f; \\ G &: g' \circ \zeta \simeq \omega \circ g; \end{aligned}$$

together with a map $\Phi\colon X_* \to W'_{*+2}$ such that

$$(2.5) \qquad \begin{aligned} &\partial^{W'}_{j+2} \circ \Phi_j - \Phi_{j-1} \circ \partial^X_j = \\ &\quad \omega_{j+1} \circ H_j - H'_j \circ \xi_j + B_j \circ f_j - g'_{j+1} \circ A_j + G_j \circ a_j - b'_{j+1} \circ F_j. \end{aligned}$$

The right hand side in equation (2.5) is the sum of all six maps $X_* \to W'_{*+1}$ in diagram (2.4), where opposite faces of the cube contribute with opposite signs. In this sense, the map Φ is a *filler* of the cube.

Lemma 2.3. *Let a homotopy commutative cube as in diagram* (2.4) *be given. Then the following square is homotopy commutative*

$$\begin{array}{ccc} \operatorname{Cone}(f) & \xrightarrow{(\xi,\upsilon;F)} & \operatorname{Cone}(f') \\ {\scriptstyle (a,b;H)}\downarrow & \circlearrowleft_\Psi & \downarrow{\scriptstyle (a',b';H')} \\ \operatorname{Cone}(g) & \xrightarrow[(\zeta,\omega;G)]{} & \operatorname{Cone}(g') \end{array}$$

where $\Psi\colon \operatorname{Cone}(f)_* \to \operatorname{Cone}(g')_{*+1}$ *is defined as*

$$\Psi_j(x,y) := \bigl(-A_{j-1}(x), B_j(y) - \Phi_{j-1}(x)\bigr).$$

Proof. For $(x,y) \in X_{j-1} \oplus Y_j = \mathrm{Cone}(f)_j$, we have

$$\begin{aligned}
&\partial_{j+1}^{\mathrm{Cone}(g')} \circ \Psi_j(x,y) + \Psi_{j-1} \circ \partial_j^{\mathrm{Cone}(f)}(x,y) \\
&= \partial_{j+1}^{\mathrm{Cone}(g')}\big(-A_{j-1}(x), B_j(y) - \Phi_{j-1}(x)\big) + \Psi_{j-1}\big(-\partial_{j-1}^X(x), \partial_j^Y(y) + f_{j-1}(x)\big) \\
&= \big(\partial_j^{Z'} \circ A_{j-1}(x), \partial_{j+1}^{W'} \circ B_j(y) - \partial_{j+1}^{W'} \circ \Phi_{j-1}(x) - g'_j \circ A_{j-1}(x)\big) + \\
&\qquad \big(A_{j-2} \circ \partial_{j-1}^X(x), B_{j-1} \circ \partial_j^Y(y) + B_{j-1} \circ f_{j-1}(x) + \Phi_{j-2} \circ \partial_{j-1}^X(x)\big) \\
&= \big(\zeta_{j-1} \circ a_{j-1}(x) - a'_{j-1} \circ \xi_{j-1}(x), \omega_j \circ b_j(y) - b'_j \circ \upsilon_j(y)\big) + \\
&\qquad \big(0, -\omega_j \circ H_{j-1}(x) + H'_{j-1} \circ \xi_{j-1}(x) - G_{j-1} \circ a_{j-1}(x) + b'_j \circ F_{j-1}(x)\big) \\
&= \big(\zeta_{j-1} \circ a_{j-1}(x), \omega_j \circ b_j(y) - \omega_j \circ H_{j-1}(x) - G_{j-1} \circ a_{j-1}(x)\big) - \\
&\qquad \big(a'_{j-1} \circ \xi_{j-1}(x), b'_j \circ \upsilon_j(y) - b'_j \circ F_{j-1}(x) - H'_{j-1} \circ \xi_{j-1}(x)\big) \\
&= (\zeta, \omega; G)_j\big(a_{j-1}(x), b_j(y) - H_{j-1}(x)\big) - (a', b'; H')_j\big(\xi_{j-1}(x), \upsilon_j(y) - F_{j-1}(x)\big) \\
&= (\zeta, \omega; G)_j \circ (a, b; H)_j(x,y) - (a', b'; H')_j \circ (\xi, \upsilon; F)_j(x,y).
\end{aligned}$$

This proves the claim. □

2.2. **Quantitative homotopy retracts.** We introduce the key notion of rebuildings for chain complexes, which are homotopy retracts satisfying estimates on the ranks of chain modules and norms of chain maps. The reader who is interested only in the axiomatic Bootstrapping Theorem 3.6 (and not in torsion homology growth) can proceed directly to Section 2.3.

We work over the ring $\mathbb{Z}$. A free $\mathbb{Z}$-module endowed with a $\mathbb{Z}$-basis is called *based free*. The basis of a based free $\mathbb{Z}$-module M induces an ℓ^1-norm on M. For a map $f\colon M \to L$ between based free $\mathbb{Z}$-modules, we denote by $\|f\|$ the operator norm with respect to the ℓ^1-norms on M and L. A $\mathbb{Z}$-chain complex is *based free* if every chain module is based free.

A *homotopy retract* of chain complexes (X, X', ξ, ξ', Ξ) consists of chain complexes X and X', chain maps $\xi\colon X \to X'$ and $\xi'\colon X' \to X$, and a chain homotopy $\Xi\colon \mathrm{id}_X \simeq \xi' \circ \xi$. We will sometimes just write (X, X') instead of (X, X', ξ, ξ', Ξ), leaving the maps implicit.

Definition 2.4 (Rebuilding)**.** Let $n \in \mathbb{Z}$ and let X and X' be based free $\mathbb{Z}$-chain complexes such that X_j and X'_j are finitely generated for all $j \le n$. Let $T, \kappa \in \mathbb{R}_{\ge 1}$. We say that a homotopy retract (X, X', ξ, ξ', Ξ) is

- an *n-domination of X of quality* (T, κ) if for all $j \le n$
$$\mathrm{rk}_{\mathbb{Z}}(X'_j) \le \kappa T^{-1} \,\mathrm{rk}_{\mathbb{Z}}(X_j);$$
- a *weak n-rebuilding of X of quality* (T, κ) if for all $j \le n$
$$\mathrm{rk}_{\mathbb{Z}}(X'_j) \le \kappa T^{-1} \,\mathrm{rk}_{\mathbb{Z}}(X_j);$$
$$\|\partial_j^{X'}\|, \|\xi_j\| \le \exp(\kappa) T^{\kappa};$$
- an *n-rebuilding of X of quality* (T, κ) if for all $j \le n$
$$\mathrm{rk}_{\mathbb{Z}}(X'_j) \le \kappa T^{-1} \,\mathrm{rk}_{\mathbb{Z}}(X_j);$$
$$\|\partial_j^{X'}\|, \|\xi_j\|, \|\xi'_j\|, \|\Xi_j\| \le \exp(\kappa) T^{\kappa}.$$

Clearly, an n-rebuilding is in particular a weak n-rebuilding of the same quality, and a weak n-rebuilding is in particular an n-domination of the same quality. For $T' \le T$, an n-domination of quality (T, κ) is in particular of quality (T', κ). However, the analogous statement for (weak) n-rebuildings does not hold. For (weak) n-rebuildings, in the words of Abért–Bergeron–Frączyk–Gaboriau [ABFG24], the parameter T captures a tension between "having small ranks" and "maintaining tame norms". One wants to decrease the ranks linearly in T, while bounding the norms polynomially in T.

Remark 2.5. Definition 2.4 is designed in order to provide upper bounds on torsion in homology: Let (X, X', ξ, ξ', Ξ) be a weak n-rebuilding of quality (T, κ). Then $H_j(X)$ is a retract of $H_j(X')$ for all $j \in \mathbb{Z}$. Using Gabber's estimate (1.1), for all $j \leq n-1$, we have

$$\begin{aligned} \log \operatorname{tors} H_j(X) &\leq \log \operatorname{tors} H_j(X') \leq \operatorname{rk}_{\mathbb{Z}}(X'_j) \log_+ \|\partial_{j+1}^{X'}\| \\ &\leq \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}}(X_j) \kappa (1 + \log T). \end{aligned} \tag{2.6}$$

Now, suppose that X is a $\mathbb{Z}$-chain complex and there exists a uniform $\kappa \in \mathbb{R}_{\geq 1}$ such that for all $T \in \mathbb{R}_{\geq 1}$, X admits a weak n-rebuilding of quality (T, κ). Then the inequality (2.6) shows that $\log \operatorname{tors} H_j(X) = 0$ because $T^{-1} \log T \to 0$ as $T \to \infty$. Later we will employ an asymptotic version of this argument (Lemma 4.22). The other conditions in Definition 2.4 on the norms of ξ, ξ', and Ξ ensure that the norm of $\partial^{X'}$ stays controlled when taking mapping cones (Proposition 2.9).

The following two examples are algebraic versions of topological homotopy equivalences between circles of different length.

Example 2.6. For $d \in \mathbb{N}$, let $S^{[0,d]}$ be the chain complex with chain modules

$$S_j^{[0,d]} = \begin{cases} \bigoplus_{i=0}^{d-1} \mathbb{Z}\langle v_i \rangle & \text{for } j = 0; \\ \bigoplus_{i=0}^{d-1} \mathbb{Z}\langle e_i \rangle & \text{for } j = 1; \\ 0 & \text{otherwise}; \end{cases}$$

and differential $\partial_1(e_i) = v_{i+1} - v_i$ for $i \in \{0, \dots, d-1\}$ considered modulo d.

We construct an n-rebuilding $(S^{[0,d]}, S^{[0,1]})$ of quality $(d, 1)$. There is a homotopy retract $(S^{[0,d]}, S^{[0,1]}, \xi, \xi', \Xi)$, where the chain maps $\xi \colon S^{[0,d]} \to S^{[0,1]}$ and $\xi' \colon S^{[0,1]} \to S^{[0,d]}$ are given by

$$\begin{aligned} \xi_0(v_i) &= v_0 \quad \text{for all } i; \\ \xi_1(e_i) &= \begin{cases} e_0 & \text{if } i = 0; \\ 0 & \text{otherwise}; \end{cases} \\ \xi'_0(v_0) &= v_0; \\ \xi'_1(e_0) &= \sum_{i=0}^{d-1} e_i; \end{aligned}$$

and the chain homotopy $\Xi \colon \mathrm{id}_{S^{[0,d]}} \simeq \xi' \circ \xi$ is given by

$$\Xi_0(v_i) = \begin{cases} 0 & \text{if } i = 0; \\ -e_i - \dots - e_{d-1} & \text{if } i \in \{1, \dots, d-1\}. \end{cases}$$

For all $j \in \mathbb{Z}$, we have

$$\operatorname{rk}_{\mathbb{Z}}(S_j^{[0,1]}) \leq d^{-1} \operatorname{rk}_{\mathbb{Z}}(S_j^{[0,d]});$$

$$\|\partial_j^{S^{[0,1]}}\| \leq 0; \quad \|\xi_j\| \leq 1; \quad \|\xi'_j\| \leq d; \quad \|\Xi_j\| \leq d.$$

Hence, for all $n \in \mathbb{Z}$, the homotopy retract $(S^{[0,d]}, S^{[0,1]})$ is an n-rebuilding of quality $(d, 1)$. For $T \leq d$, the pair $(S^{[0,d]}, S^{[0,1]})$ provides a weak n-rebuilding of quality $(T, 1)$, but in general not an n-rebuilding of quality $(T, 1)$.

The following is an algebraic reformulation of [ABFG24, Lemma 10.10].

Example 2.7. For $d \in \mathbb{N}$ and $T \in \mathbb{R}_{\geq 1}$ with $T \leq d$, we construct an n-rebuilding of $S^{[0,d]}$ of quality $(T, 2)$. Choose a sequence of integers $0 = a_0 < a_1 < \cdots < a_m = d$ with

$$T/2 \leq a_{k+1} - a_k \leq T$$

for all $k \in \{0, \dots, m-1\}$. There is a homotopy retract $(S^{[0,d]}, S^{[0,m]}, \xi, \xi', \Xi)$, where the chain maps $\xi \colon S^{[0,d]} \to S^{[0,m]}$ and $\xi' \colon S^{[0,m]} \to S^{[0,d]}$ are given by

$$\begin{aligned} \xi_0(v_i) &= v_k \quad \text{if } i \in \{a_{k-1}+1, \dots, a_k\}; \\ \xi_1(e_i) &= \begin{cases} e_k & \text{if } i = a_k \text{ for some } k; \\ 0 & \text{otherwise;} \end{cases} \\ \xi'_0(v_i) &= v_{a_i}; \\ \xi'_1(e_i) &= e_{a_i} + \cdots + e_{a_{i+1}-1}; \end{aligned}$$

and the chain homotopy $\Xi \colon \mathrm{id}_{S^{[0,d]}} \simeq \xi' \circ \xi$ is given by

$$\Xi_0(v_i) = \begin{cases} 0 & \text{if } i = a_k \text{ for some } k; \\ -e_i - \cdots - e_{a_k - 1} & \text{if } i \in \{a_{k-1}+1, \dots, a_k - 1\}. \end{cases}$$

For all $j \in \mathbb{Z}$, we have

$$\mathrm{rk}_{\mathbb{Z}}(S_j^{[0,m]}) \leq 2T^{-1} \, \mathrm{rk}_{\mathbb{Z}}(S_j^{[0,d]});$$

$$\|\partial_j^{S^{[0,m]}}\| \leq 2; \quad \|\xi_j\| \leq 1; \quad \|\xi'_j\| \leq T; \quad \|\Xi_j\| \leq T.$$

Hence, for all $n \in \mathbb{Z}$, the homotopy retract $(S^{[0,d]}, S^{[0,m]})$ is an n-rebuilding of quality $(T, 2)$.

Directs sums of homotopy retracts are again homotopy retracts. The same is true for rebuildings and we specify the quality of the resulting rebuilding.

Lemma 2.8. *Let $\mathbf{X} = (X, X', \xi, \xi', \Xi)$ and $\mathbf{Y} = (Y, Y', \upsilon, \upsilon', \Upsilon)$ be homotopy retracts of $\mathbb{Z}$-chain complexes. Consider the tuple*

$$\mathbf{X} \oplus \mathbf{Y} := (X \oplus Y, X' \oplus Y', \xi \oplus \upsilon, \xi' \oplus \upsilon', \Xi \oplus \Upsilon).$$

Let $n \in \mathbb{Z}$, let $T, \kappa_{\mathbf{X}}, \kappa_{\mathbf{Y}} \in \mathbb{R}_{\geq 1}$, and set $\kappa := \max\{\kappa_{\mathbf{X}}, \kappa_{\mathbf{Y}}\}$. Then the following hold:

(i) *The tuple $\mathbf{X} \oplus \mathbf{Y}$ is a homotopy retract;*
(ii) *If $\mathbf{X}$ is an n-domination of quality $(T, \kappa_{\mathbf{X}})$ and $\mathbf{Y}$ is an n-domination of quality $(T, \kappa_{\mathbf{Y}})$, then $X \oplus Y$ is an n-domination of quality (T, κ);*
(iii) *If $\mathbf{X}$ is a weak n-rebuilding of quality $(T, \kappa_{\mathbf{X}})$ and $\mathbf{Y}$ is a weak n-rebuilding of quality $(T, \kappa_{\mathbf{Y}})$, then $X \oplus Y$ is a weak n-rebuilding of quality (T, κ);*
(iv) *If $\mathbf{X}$ is an n-rebuilding of quality $(T, \kappa_{\mathbf{X}})$ and $\mathbf{Y}$ is an n-rebuilding of quality $(T, \kappa_{\mathbf{Y}})$, then $X \oplus Y$ is an n-rebuilding of quality (T, κ).*

Proof. Part (i) is clear.

(ii) Suppose $\mathbf{X}$ and $\mathbf{Y}$ are n-dominations of quality $(T, \kappa_{\mathbf{X}})$ and $(T, \kappa_{\mathbf{Y}})$, respectively. For all $j \leq n$, we have

$$\mathrm{rk}_{\mathbb{Z}}(X'_j \oplus Y'_j) \leq \kappa_{\mathbf{X}} T^{-1} \, \mathrm{rk}_{\mathbb{Z}}(X_j) + \kappa_{\mathbf{Y}} T^{-1} \, \mathrm{rk}_{\mathbb{Z}}(Y_j) \leq \kappa T^{-1} \, \mathrm{rk}_{\mathbb{Z}}(X_j \oplus Y_j).$$

(iii) Suppose $\mathbf{X}$ and $\mathbf{Y}$ are weak n-rebuildings of quality $(T, \kappa_{\mathbf{X}})$ and $(T, \kappa_{\mathbf{Y}})$, respectively. Additionally to part (ii), for all $j \leq n$, we have

$$\|\partial_j^{X'} \oplus \partial_j^{Y'}\| \leq \max\{\|\partial_j^{X'}\|, \|\partial_j^{Y'}\|\} \leq \max\{\exp(\kappa_{\mathbf{X}}) T^{\kappa_{\mathbf{X}}}, \exp(\kappa_{\mathbf{Y}}) T^{\kappa_{\mathbf{Y}}}\} \leq \exp(\kappa) T^{\kappa}$$

and similarly $\|\xi_j \oplus \upsilon_j\| \leq \exp(\kappa) T^{\kappa}$.

(iv) Suppose $\mathbf{X}$ and $\mathbf{Y}$ are n-rebuildings of quality $(T, \kappa_{\mathbf{X}})$ and $(T, \kappa_{\mathbf{Y}})$, respectively. Additionally to part (iii), for all $j \leq n$, we have

$$\|\xi'_j \oplus \upsilon'_j\|, \|\Xi_j \oplus \Upsilon_j\| \leq \exp(\kappa) T^\kappa$$

by a similar computation. □

Mapping cones of homotopy retracts are again homotopy retracts. The same is true for rebuildings, which is our key stability result for rebuildings. A homotopy retract (X, X', ξ, ξ', Ξ) can be viewed as a homotopy commutative square of the form

$$\begin{array}{ccc} X & \xrightarrow{\ \xi\ } & X' \\ {\scriptstyle \mathrm{id}_X}\downarrow & \circlearrowleft_{\Xi} & \downarrow{\scriptstyle \xi'} \\ X & \xrightarrow[\mathrm{id}_X]{} & X \end{array}$$

Proposition 2.9 (Rebuilding of mapping cones). *Let* $\mathbf{X} = (X, X', \xi, \xi', \Xi)$ *and* $\mathbf{Y} = (Y, Y', \upsilon, \upsilon', \Upsilon)$ *be homotopy retracts of* $\mathbb{Z}$*-chain complexes. Let* $f \colon X \to Y$ *be a chain map and define the chain map* $f' := \upsilon \circ f \circ \xi' \colon X' \to Y'$*. Consider the tuple*

$$\mathbf{C}(f) := \bigl(\mathrm{Cone}(f), \mathrm{Cone}(f'), (\xi, \upsilon; -\upsilon \circ f \circ \Xi), (\xi', \upsilon'; \Upsilon \circ f \circ \xi'), \Psi\bigr),$$

where $\Psi \colon \mathrm{Cone}(f)_* \to \mathrm{Cone}(f)_{*+1}$ *is defined as*

$$\Psi_j(x, y) := \bigl(-\Xi_{j-1}(x), \Upsilon_j(y) + \Upsilon_j \circ f_j \circ \Xi_{j-1}(x)\bigr).$$

Let $n \in \mathbb{N}$*, let* $T, \kappa_{\mathbf{X}}, \kappa_{\mathbf{Y}} \in \mathbb{R}_{\geq 1}$*, and set*

$$\kappa := \kappa_{\mathbf{X}} + \kappa_{\mathbf{Y}} + \log 3 + \max\bigl\{\log_+ \|f_j\| \bigm| j \leq n\bigr\},$$

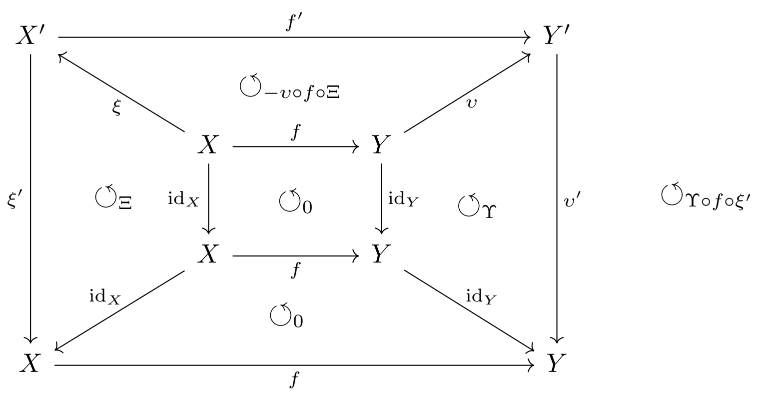
where $\log_+ := \max\{\log, 0\}$*. Then the following hold:*

(i) *The tuple* $\mathbf{C}(f)$ *is a homotopy retract;*
(ii) *If* $\mathbf{X}$ *is an* $(n-1)$*-domination of quality* $(T, \kappa_{\mathbf{X}})$ *and* $\mathbf{Y}$ *is an* n*-domination of quality* $(T, \kappa_{\mathbf{Y}})$*, then* $\mathbf{C}(f)$ *is an* n*-domination of quality* (T, κ)*;*
(iii) *If* $\mathbf{X}$ *is an* $(n-1)$*-rebuilding of quality* $(T, \kappa_{\mathbf{X}})$ *and* $\mathbf{Y}$ *is a weak* n*-rebuilding of quality* $(T, \kappa_{\mathbf{Y}})$*, then* $\mathbf{C}(f)$ *is a weak* n*-rebuilding of quality* (T, κ)*;*
(iv) *If* $\mathbf{X}$ *is an* $(n-1)$*-rebuilding of quality* $(T, \kappa_{\mathbf{X}})$ *and* $\mathbf{Y}$ *is an* n*-rebuilding of quality* $(T, \kappa_{\mathbf{Y}})$*, then* $\mathbf{C}(f)$ *is an* n*-rebuilding of quality* (T, κ)*.*

We point out that in part (iii), $\mathbf{X}$ is assumed to be an $(n-1)$-rebuilding, and not only a weak $(n-1)$-rebuilding. This will have important ramifications later (Proposition 4.23).

Proof. (i) The following cube is homotopy commutative:

(2.7)



with $\Phi_* := -\Upsilon_{*+1} \circ f_{*+1} \circ \Xi_* \colon X_* \to Y_{*+2}$. Indeed, one easily checks that the upper and outer squares are homotopy commutative. The other four squares are clearly homotopy commutative. We check that Φ is a filler of the cube:

$$\begin{aligned}
&0 - \Upsilon_j \circ f_j \circ \xi'_j \circ \xi_j + \Upsilon_j \circ f_j - f_{j+1} \circ \Xi_j + 0 + \upsilon'_{j+1} \circ \upsilon_{j+1} \circ f_{j+1} \circ \Xi_j \\
&= \Upsilon_j \circ f_j \circ (\mathrm{id}_{X_j} - \xi'_j \circ \xi_j) - (\mathrm{id}_{Y_{j+1}} - \upsilon'_{j+1} \circ \upsilon_{j+1}) \circ f_{j+1} \circ \Xi_j \\
&= \Upsilon_j \circ f_j \circ (\partial^X_{j+1} \circ \Xi_j + \Xi_{j-1} \circ \partial^X_j) - (\partial^Y_{j+2} \circ \Upsilon_{j+1} + \Upsilon_j \circ \partial^Y_{j+1}) \circ f_{j+1} \circ \Xi_j \\
&= \Upsilon_j \circ f_j \circ \Xi_{j-1} \circ \partial^X_j - \partial^Y_{j+2} \circ \Upsilon_{j+1} \circ f_{j+1} \circ \Xi_j \\
&= \partial^Y_{j+2} \circ \Phi_j - \Phi_{j-1} \circ \partial^X_j .
\end{aligned}$$

By Lemma 2.3, the homotopy commutative cube (2.7) induces a homotopy commutative square of mapping cones

$$\begin{array}{ccc}
\mathrm{Cone}(f) & \xrightarrow{(\xi,\upsilon;-\upsilon\circ f\circ\Xi)} & \mathrm{Cone}(f') \\
{\scriptstyle (\mathrm{id}_X,\mathrm{id}_Y;0)}\downarrow & \circlearrowleft_\Psi & \downarrow{\scriptstyle (\xi',\upsilon';\Upsilon\circ f\circ\xi')} \\
\mathrm{Cone}(f) & \xrightarrow[(\mathrm{id}_X,\mathrm{id}_Y;0)]{} & \mathrm{Cone}(f)
\end{array}$$

where

$$\Psi_j(x,y) = \big(-\Xi_{j-1}(x), \Upsilon_j(y) - \Phi_{j-1}(x)\big).$$

Hence $\mathbf{C}(f)$ is a homotopy retract of chain complexes.

(ii) Suppose that $\mathbf{X}$ is an $(n-1)$-domination of quality $(T, \kappa_{\mathbf{X}})$ and $\mathbf{Y}$ is an n-domination of quality $(T, \kappa_{\mathbf{Y}})$. For all $j \le n$, we have

$$\begin{aligned}
\mathrm{rk}_{\mathbb{Z}}\big(\mathrm{Cone}(f')_j\big) &= \mathrm{rk}_{\mathbb{Z}}(X'_{j-1}) + \mathrm{rk}_{\mathbb{Z}}(Y'_j) \\
&\le \kappa_{\mathbf{X}} T^{-1} \mathrm{rk}_{\mathbb{Z}}(X_{j-1}) + \kappa_{\mathbf{Y}} T^{-1} \mathrm{rk}_{\mathbb{Z}}(Y_j) \\
&\le \kappa T^{-1}\big(\mathrm{rk}_{\mathbb{Z}}(X_{j-1}) + \mathrm{rk}_{\mathbb{Z}}(Y_j)\big) \\
&= \kappa T^{-1} \mathrm{rk}_{\mathbb{Z}}\big(\mathrm{Cone}(f)_j\big).
\end{aligned}$$

(iii) Suppose that $\mathbf{X}$ is an $(n-1)$-rebuilding of quality $(T, \kappa_{\mathbf{X}})$ and $\mathbf{Y}$ is an n-weak rebuilding of quality $(T, \kappa_{\mathbf{Y}})$. Additionally to part (ii), for all $j \le n$, we have

$$\begin{aligned}
\|\partial_j^{\mathrm{Cone}(f')}\| &\le \|\partial^{X'}_{j-1}\| + \|\partial^{Y'}_j\| + \|\upsilon_{j-1}\| \cdot \|f_{j-1}\| \cdot \|\xi'_{j-1}\| \\
&\le \exp(\kappa_{\mathbf{X}}) T^{\kappa_{\mathbf{X}}} + \exp(\kappa_{\mathbf{Y}}) T^{\kappa_{\mathbf{Y}}} + \\
&\qquad \exp(\kappa_{\mathbf{Y}} + \log_+ \|f_{j-1}\| + \kappa_{\mathbf{X}}) T^{\kappa_{\mathbf{Y}} + \log_+ \|f_{j-1}\| + \kappa_{\mathbf{X}}} \\
&\le 3 \exp(\kappa_{\mathbf{X}} + \kappa_{\mathbf{Y}} + \log_+ \|f_{j-1}\|) T^{\kappa_{\mathbf{X}} + \kappa_{\mathbf{Y}} + \log_+ \|f_{j-1}\|} \\
&\le \exp(\kappa_{\mathbf{X}} + \kappa_{\mathbf{Y}} + \log_+ \|f_{j-1}\| + \log 3) T^{\kappa_{\mathbf{X}} + \kappa_{\mathbf{Y}} + \log_+ \|f_{j-1}\| + \log 3} \\
&\le \exp(\kappa) T^{\kappa}
\end{aligned}$$

and similarly

$$\|(\xi, \upsilon; -\upsilon \circ f \circ \Xi)_j\| \le \|\xi_{j-1}\| + \|\upsilon_j\| + \|\upsilon_j\| \cdot \|f_j\| \cdot \|\Xi_{j-1}\| \le \exp(\kappa) T^{\kappa}.$$

(iv) Suppose that $\mathbf{X}$ is an $(n-1)$-rebuilding of quality $(T, \kappa_{\mathbf{X}})$ and $\mathbf{Y}$ is an n-rebuilding of quality $(T, \kappa_{\mathbf{Y}})$. Additionally to part (iii), for all $j \le n$, we have

$$\begin{aligned}
\|(\xi', \upsilon'; \Upsilon \circ f \circ \xi')_j\| &\le \|\xi'_{j-1}\| + \|\upsilon'_j\| + \|\Upsilon_{j-1}\| \cdot \|f_{j-1}\| \cdot \|\xi'_{j-1}\| \le \exp(\kappa) T^{\kappa}; \\
\|\Psi_j\| &\le \|\Xi_{j-1}\| + \|\Upsilon_j\| + \|\Upsilon_j\| \cdot \|f_j\| \cdot \|\Xi_{j-1}\| \le \exp(\kappa) T^{\kappa};
\end{aligned}$$

by similar computations. $\square$

While direct sums can be viewed as mapping cones of the zero map, we formulated Lemma 2.8 separately from Proposition 2.9 for the optimality of the constant κ.

The composition of homotopy retracts is again a homotopy retract. The same is true for dominations and weak rebuildings. We will not require the corresponding result of rebuildings, though a similar statement (involving a degree shift) also holds. Lemma 2.10 is an algebraic version of [ABFG24, Lemma 6.3].

Lemma 2.10. *Let* $\mathbf{X} = (X, X', \xi, \xi', \Xi)$ *and* $\mathbf{Y} = (X', X'', \upsilon, \upsilon', \Upsilon)$ *be homotopy retracts of* $\mathbb{Z}$*-chain complexes. Consider the tuple*

$$\mathbf{Y} \circ \mathbf{X} := (X, X'', \upsilon \circ \xi, \xi' \circ \upsilon', \Xi + \xi' \circ \Upsilon \circ \xi).$$

Let $n \in \mathbb{N}$*, let* $T, S, \kappa_{\mathbf{X}}, \kappa_{\mathbf{Y}} \in \mathbb{R}_{\geq 1}$*, and set* $\kappa := 2\kappa_{\mathbf{Y}}\kappa_{\mathbf{X}}$*. Then the following hold:*

(i) *The tuple* $\mathbf{Y} \circ \mathbf{X}$ *is a homotopy retract;*
(ii) *If* $\mathbf{X}$ *is an* n*-domination of quality* $(T, \kappa_{\mathbf{X}})$ *and* $\mathbf{Y}$ *is an* n*-domination of quality* $(S, \kappa_{\mathbf{Y}})$*, then* $\mathbf{Y} \circ \mathbf{X}$ *is an* n*-domination of quality* (ST, κ)*;*
(iii) *If* $\mathbf{X}$ *is a weak* n*-rebuilding of quality* $(T, \kappa_{\mathbf{X}})$ *and* $\mathbf{Y}$ *is a weak* n*-rebuilding of quality* $(S, \kappa_{\mathbf{Y}})$*, then* $\mathbf{Y} \circ \mathbf{X}$ *is a weak* n*-rebuilding of quality* (ST, κ)*.*

Proof. (i) A straight-forward calculation shows that $\Xi + \xi' \circ \Upsilon \circ \xi$ indeed provides a chain homotopy between the chain maps id_X and $\xi' \circ \upsilon' \circ \upsilon \circ \xi$.

(ii) Suppose $\mathbf{X}$ and $\mathbf{Y}$ are n-dominations of quality $(T, \kappa_{\mathbf{X}})$ and $(T, \kappa_{\mathbf{Y}})$, respectively. For all $j \leq n$, we have

$$\mathrm{rk}_{\mathbb{Z}}(X''_j) \leq \kappa_{\mathbf{Y}} S^{-1} \,\mathrm{rk}_{\mathbb{Z}}(X'_j) \leq \kappa_{\mathbf{Y}} S^{-1} \kappa_{\mathbf{X}} T^{-1} \,\mathrm{rk}_{\mathbb{Z}}(X_j) \leq \kappa (ST)^{-1} \,\mathrm{rk}_{\mathbb{Z}}(X_j).$$

(iii) Suppose $\mathbf{X}$ and $\mathbf{Y}$ are weak n-rebuildings of quality $(T, \kappa_{\mathbf{X}})$ and $(T, \kappa_{\mathbf{Y}})$, respectively. Additionally to part (ii), for all $j \leq n$, we have

$$\begin{aligned} \|\partial_j^{X''}\| &\leq \exp(\kappa_{\mathbf{Y}}) T^{\kappa_{\mathbf{Y}}} \leq \exp(\kappa) T^{\kappa}; \\ \|\upsilon_j \circ \xi_j\| &\leq \|\upsilon_j\| \cdot \|\xi_j\| \leq \exp(\kappa_{\mathbf{Y}}) S^{\kappa_{\mathbf{Y}}} \exp(\kappa_{\mathbf{X}}) T^{\kappa_{\mathbf{X}}} \leq \exp(\kappa)(ST)^{\kappa}. \end{aligned}$$

Hence $\mathbf{Y} \circ \mathbf{X}$ is a weak n-rebuilding of quality (ST, κ). □

Remark 2.11 (Geometric rebuilding)**.** Definition 2.4 of n-rebuildings for chain complexes is an algebraic version of Abért–Bergeron–Frączyk–Gaboriau's geometric definition of n-rebuildings for CW-complexes [ABFG24, Definition 1 and Definition 2]. However, we point out the main differences: We require a homotopy retract in all degrees, as opposed to a truncated homotopy equivalence. We work with the ℓ^1-norm on based free $\mathbb{Z}$-modules, instead of the ℓ^2-norm, because it simplifies some calculations. We also ask for control on the norm of the homotopy Ξ_j in degrees $j \leq n$, and not only in degrees $j \leq n-1$, because it is needed in the proof of Proposition 2.9. Due to these differences, there is no obvious implication between the algebraic and geometric notions. We chose to formulate Definition 2.4 as above because it simplifies the proofs and still covers the main Example 2.7.

2.3. **Projective replacements.** We work over an arbitrary ring that will be left implicit. Let X be a chain complex and let $k \in \mathbb{Z}$. The *k-skeleton* of X is the chain subcomplex $X^{(k)}$ with chain modules

$$X_j^{(k)} := \begin{cases} 0 & \text{if } j \geq k+1; \\ X_j & \text{if } j \leq k; \end{cases}$$

and differentials

$$\partial_j^{X^{(k)}} := \begin{cases} 0 & \text{if } j \geq k+1; \\ \partial_j^X & \text{if } j \leq k. \end{cases}$$

If we regard the module X_k as a chain complex concentrated in degree zero, then $X^{(k)}$ is the mapping cone of the chain map $\Sigma^{k-1}X_k \to X^{(k-1)}$ given by ∂_k^X in degree $k-1$ and zero otherwise.

We recall a construction of projective replacements for chain complexes [Bro82b, Lemma 1.5]. Given a chain complex X and projective resolutions of its chain modules, it produces a *projective* chain complex $\widehat{X}$ that is weakly equivalent to X, obtained by iterated mapping cones from (suspensions of) the given projective resolutions.

Proposition 2.12 (Projective replacement)**.** *Let X be a chain complex that is concentrated in degrees ≥ 0. For each $j \geq 0$, let $P^j = (P_i^j)_{i\geq 0}$ be a projective resolution of the module X_j. Then there exists a projective chain complex $\widehat{X}$ together with a filtration $(\widehat{X}^{[k]})_{k\geq 0}$ and a chain map $q\colon \widehat{X} \to X$ such that for every $k \geq 0$ the following hold:*

(i) *$\widehat{X}^{[k]}$ is the mapping cone of a chain map $\Sigma^{k-1}P^k \to \widehat{X}^{[k-1]}$;*
(ii) *The restriction $q^k := q|_{\widehat{X}^{[k]}} \colon \widehat{X}^{[k]} \to X^{(k)}$ is a weak equivalence.*

In particular, the chain modules of $\widehat{X}$ are of the form

$$\widehat{X}_n = \bigoplus_{j+i=n} P_i^j$$

and $q\colon \widehat{X} \to X$ is a weak equivalence.

We call the chain map $q\colon \widehat{X} \to X$ a *projective replacement* of X with respect to the projective resolutions $(P^j)_j$.

Proof. We proceed by induction on $k \geq 0$. Set $\widehat{X}^{[0]} := P^0$ and let $q^0\colon P^0 \to X^{(0)}$ be given by the augmentation $P_0^0 \twoheadrightarrow X_0$ in degree 0. Since P^0 is a resolution of X_0, the chain map q^0 is a weak equivalence. For the inductive step, assume that the chain complex $\widehat{X}^{[k-1]}$ and a weak equivalence $q^{k-1}\colon \widehat{X}^{[k-1]} \to X^{(k-1)}$ have been constructed.

Consider the following diagram

$$\begin{array}{ccc} \Sigma^{k-1}P^k & & \widehat{X}^{[k-1]} \\ {\scriptstyle \Sigma^{k-1}\varepsilon^k}\big\downarrow & & \big\downarrow{\scriptstyle q^{k-1}} \\ \Sigma^{k-1}X_k & \xrightarrow{\partial_k^X} & X^{(k-1)} \end{array}$$

where $\varepsilon^k\colon P^k \to X_k$ is given by the augmentation $P_0^k \twoheadrightarrow X_k$ in degree 0. Since the chain complex $\Sigma^{k-1}P^k$ consists of projective modules and the chain map q^{k-1} is a weak equivalence, there exists a chain map $\widehat{\partial_k^X}\colon \Sigma^{k-1}P^k \to \widehat{X}^{[k-1]}$ making the square homotopy commutative [Bro82b, Lemma 1.1]. We set $\widehat{X}^{[k]} := \operatorname{Cone}(\widehat{\partial_k^X})$ and let $q^k\colon \widehat{X}^{[k]} \to X^{(k)} = \operatorname{Cone}(\partial_k^X)$ be the induced chain map on mapping cones. Thus we have a diagram of mapping cone sequences

$$\begin{array}{ccccc} \Sigma^{k-1}P^k & \xrightarrow{\widehat{\partial_k^X}} & \widehat{X}^{[k-1]} & \longrightarrow & \widehat{X}^{[k]} \\ {\scriptstyle \Sigma^{k-1}\varepsilon^k}\big\downarrow & & \big\downarrow{\scriptstyle q^{k-1}} & & \big\downarrow{\scriptstyle q^k} \\ \Sigma^{k-1}X_k & \xrightarrow{\partial_k^X} & X^{(k-1)} & \longrightarrow & X^{(k)} \end{array}$$

Since the chain maps $\Sigma^{k-1}\varepsilon^k$ and q^{k-1} are weak equivalences, so is the chain map q^k by Lemma 2.2 (iii).

Taking the limit, we set $\widehat{X} := \operatorname{colim}_k \widehat{X}^{[k]}$ and $q := \operatorname{colim}_k q^k$, which are as desired. By construction, we have

$$\widehat{X}_n^{[k]} = \bigoplus_{\substack{j+i=n,\\ 0\leq j\leq k}} P_i^j$$

and $\widehat{X}_n^{[k]} = \widehat{X}_n$ for $k \geq n$, and thus $\widehat{X}_n = \bigoplus_{j+i=n} P_i^j$. Since homology commutes with directed colimits and all maps $(q^k)_k$ are weak equivalences, the map q is a weak equivalence. □

3. Bootstrapping

The goal of this section is to prove the Bootstrapping Theorem 3.6. First, in Section 3.1 we work over a fixed ring (e.g., a group ring). Then, in Section 3.2 we pass to the global equivariant setting over varying group rings.

3.1. **Bootstrappable properties.** We propose a set of axioms on classes of chain complexes that ensure stability under projective replacements (Section 2.3). Various examples of classes satisfying these axioms will be discussed in Section 4. The classical example to have in mind is, for given $n \in \mathbb{N}$, the class of chain complexes that are finitely generated in all degrees $\leq n$ (Section 4.1).

Definition 3.1 (Bootstrappable property)**.** Let R be a ring. A *bootstrappable property of R-chain complexes* is a sequence $\mathsf{B}_* = (\mathsf{B}_n)_{n\in\mathbb{Z}}$ of classes of R-chain complexes that are closed under isomorphisms and satisfy the following axioms: Let $n \in \mathbb{N}$ and let X be an R-chain complex.

(B-deg) *Degree.* If X is concentrated in degrees ≥ 0, then $X \in \mathsf{B}_m$ for all $m < 0$;
(B-susp) *Suspension.* $X \in \mathsf{B}_n$ if and only if $\Sigma X \in \mathsf{B}_{n+1}$;
(B-cone) *Mapping cone.* Let $f\colon X \to Y$ be an R-chain map. If $X \in \mathsf{B}_{n-1}$ and $Y \in \mathsf{B}_n$, then $\operatorname{Cone}(f) \in \mathsf{B}_n$.

We regard axiom (B-cone) as the most important of the three, because in our examples axioms (B-deg) and (B-susp) will hold for trivial reasons. We record some immediate consequences of the axioms:

Lemma 3.2. *Let B_* be a bootstrappable property of R-chain complexes. Let $n \in \mathbb{Z}$ and let X and Y be R-chain complexes. Then the following hold:*

(i) *If X is concentrated in degrees $\geq n$, then $X \in \mathsf{B}_m$ for all $m < n$;*
(ii) *If $X, Y \in \mathsf{B}_n$, then $X \oplus Y \in \mathsf{B}_n$.*

Proof. (i) If X is concentrated in degrees $\geq n$, then $\Sigma^{-n}X$ is concentrated in degrees ≥ 0 and hence lies in B_m for all $m < 0$ by axiom (B-deg). Then $X \cong \Sigma^n\Sigma^{-n}X$ lies in B_m for all $m < n$ by axiom (B-susp).

(ii) Since $X \in \mathsf{B}_n$, the desuspension $\Sigma^{-1}X$ lies in B_{n-1} by axiom (B-susp). Since $Y \in \mathsf{B}_n$, then the direct sum $X \oplus Y \cong \operatorname{Cone}(0\colon \Sigma^{-1}X \to Y)$ lies in B_n by axiom (B-cone). □

The axioms in Definition 3.1 are conceived in order to be compatible with projective replacements (Proposition 2.12) in the following sense:

Proposition 3.3 (Bootstrapping for chain complexes)**.** *Let B_* be a bootstrappable property of R-chain complexes and let $n \in \mathbb{N}$. Let X be an R-chain complex that is concentrated in degrees ≥ 0. Suppose that for all $j \leq n$, the R-module X_j admits a projective resolution lying in B_{n-j}. Then there exists a projective R-chain complex $\overline{X} \in \mathsf{B}_n$ that is weakly equivalent to X.*

Proof. **Case 1:** Suppose that X is concentrated in degrees $\leq n$. For $j \leq n$, let P^j be a projective resolution of X_j lying in B_{n-j}. Proposition 2.12 yields a projective replacement $q\colon \widehat{X} \to X$ with respect to the resolutions $(P^j)_j$. The R-chain complex $\widehat{X}$ is equipped with a filtration

$$0 = \widehat{X}^{[-1]} \subset \widehat{X}^{[0]} \subset \cdots \subset \widehat{X}^{[n]} = \widehat{X}$$

such that for all $k \in \{1, \ldots, n\}$ there exists a mapping cone sequence

$$\Sigma^{k-1} P^k \to \widehat{X}^{[k-1]} \to \widehat{X}^{[k]}.$$

Since $\widehat{X}^{[0]} = P^0 \in \mathsf{B}_n$, it follows by induction on k from axioms (B-susp) and (B-cone) that $\widehat{X} \in \mathsf{B}_n$. Since the R-chain map q is a weak equivalence, the R-chain complex $\overline{X} := \widehat{X}$ is as desired.

Case 2: Let X be arbitrary (possibly of infinite length). For $j \leq n$, let P^j be a projective resolution of X_j lying in B_{n-j}. For $j > n$, let P^j be an arbitrary projective resolution of X_j. We write $X_{>n} := X/X^{(n)}$ for the quotient complex. Then X is the mapping cone of the R-chain map $\Sigma^{-1}(X_{>n}) \to X^{(n)}$ given by ∂^X_{n+1} in degree n and zero otherwise. Proposition 2.12 yields projective replacements $q^{(n)}\colon \widehat{X^{(n)}} \to X^{(n)}$ and $q_{>n}\colon \widehat{X_{>n}} \to X_{>n}$ with respect to the resolutions $(P^j)_j$. Consider the diagram

$$\begin{array}{ccc} \Sigma^{-1}(\widehat{X_{>n}}) & & \widehat{X^{(n)}} \\ {\scriptstyle \Sigma^{-1}(q_{>n})}\downarrow & & \downarrow{\scriptstyle q^{(n)}} \\ \Sigma^{-1}(X_{>n}) & \xrightarrow{\partial^X_{n+1}} & X^{(n)} \end{array}$$

Since the R-chain complex $\Sigma^{-1}(\widehat{X_{>n}})$ is projective and the R-chain map $q^{(n)}$ is a weak equivalence, there exists an R-chain map $\widehat{\partial^X_{n+1}}\colon \Sigma^{-1}(\widehat{X_{>n}}) \to \widehat{X^{(n)}}$ that lifts ∂^X_{n+1}, up to homotopy [Bro82b, Lemma 1.1]. We define $\overline{X} := \mathrm{Cone}(\widehat{\partial^X_{n+1}})$ and $q\colon \overline{X} \to X$ to be the induced map on mapping cones. We have constructed a diagram of mapping cone sequences

$$\begin{array}{ccccc} \Sigma^{-1}(\widehat{X_{>n}}) & \xrightarrow{\widehat{\partial^X_{n+1}}} & \widehat{X^{(n)}} & \longrightarrow & \overline{X} \\ {\scriptstyle \Sigma^{-1}(q_{>n})}\downarrow & & \downarrow{\scriptstyle q^{(n)}} & & \downarrow{\scriptstyle q} \\ \Sigma^{-1}(X_{>n}) & \xrightarrow{\partial^X_{n+1}} & X^{(n)} & \longrightarrow & X \end{array}$$

Since the R-chain complexes $\Sigma^{-1}(\widehat{X_{>n}})$ and $\widehat{X^{(n)}}$ are projective, so is $\overline{X}$. Since the R-chain maps $\Sigma^{-1}(q_{>n})$ and $q^{(n)}$ are weak equivalences, so is q by Lemma 2.2 (iii). By Case 1, $\widehat{X^{(n)}}$ lies in B_n. Since $\Sigma^{-1}(\widehat{X_{>n}})$ is concentrated in degrees $\geq n$, it lies in B_{n-1} by Lemma 3.2 (i). Hence $\overline{X}$ lies in B_n by axiom (B-cone). ☐

3.2. **Bootstrapping Theorem.** In order to relate bootstrappable properties of chain complexes over group rings for different groups, we demand a compatibility with the induction functor. An *is-class of groups* is a class of groups that is closed under isomorphisms and taking subgroups. We will use the question mark symbol ? to denote an is-class of groups (e.g., the class of all groups or the class of residually finite groups).

Definition 3.4 (Equivariantly bootstrappable property)**.** Let R be a ring and let ? be an is-class of groups. An *equivariantly bootstrappable property of* $R[?]$*-chain complexes* is a family $\mathsf{B}^?_* = (\mathsf{B}^\Gamma_n)_{\Gamma \in ?, n \in \mathbb{Z}}$, where B^Γ_n is a class of $R\Gamma$-chain complexes, such that for every group $\Gamma \in ?$, the sequence B_*^Γ is a bootstrappable property of $R\Gamma$-chain complexes, and for all $n \in \mathbb{Z}$ the following holds:

(B-ind) *Induction.* Let $\Gamma \in ?$, let Δ be a subgroup of Γ, and let X be an $R\Delta$-chain complex. If $X \in \mathsf{B}_n^\Delta$, then $\operatorname{ind}_\Delta^\Gamma X := R\Gamma \otimes_{R\Delta} X \in \mathsf{B}_n^\Gamma$.

Given an equivariantly bootstrappable property $\mathsf{B}_*^?$ of $R[?]$-chain complexes, for every $n \in \mathbb{N}$ we define the class B_n of groups $\Gamma \in ?$ for which the trivial $R\Gamma$-module R admits a projective resolution lying in B_n^Γ. The sequence B_* of classes of groups is called the *bootstrappable property of the is-class ? of groups over R associated to* $\mathsf{B}_*^?$. A *bootstrappable property of the is-class ? of groups over R* is a sequence B_* of classes of groups that is the bootstrappable property of the is-class ? over R associated to some equivariantly bootstrappable property $\mathsf{B}_*^?$ of $R[?]$-chain complexes. We denote the intersection $\mathsf{B}_\infty := \bigcap_{n\in\mathbb{Z}} \mathsf{B}_n$ of subclasses of the class of all groups.

Our axiomatic framework culminates in the Bootstrapping Theorem for groups.

Theorem 3.5. *Let B_* be a bootstrappable property of an is-class ? of groups over a ring R. Let $\Gamma \in ?$ and let $n \in \mathbb{N}$. Let X be an $R\Gamma$-chain complex such that the following hold:*

(i) *X is $(n-1)$-acyclic (i.e., $H_0(X) \cong R$ and $H_j(X) = 0$ for all $j \le n-1$);*
(ii) *For all $j \le n$, the $R\Gamma$-module X_j is isomorphic to a finite direct sum $\bigoplus_{\sigma\in S_j} R[\Gamma/\Gamma_\sigma]$, where Γ_σ is a subgroup of Γ lying in B_{n-j}.*

Then $\Gamma \in \mathsf{B}_n$.

Proof. Lemma 2.2 (iv) yields an $R\Gamma$-resolution $\tau_{<n}X$ of R with $(\tau_{<n}X)^{(n)} = X^{(n)}$. Replacing X by $\tau_{<n}X$, we may assume that X is an $R\Gamma$-resolution of R.

Let $\mathsf{B}_*^?$ be an equivariantly bootstrappable property of $R[?]$-chain complexes such that the bootstrappable property B_* of ? is associated to $\mathsf{B}_*^?$. For $j \le n$ and $\sigma \in S_j$, since the group Γ_σ lies in B_{n-j}, there exists a projective $R[\Gamma_\sigma]$-resolution P^σ of R lying in $\mathsf{B}_{n-j}^{\Gamma_\sigma}$. Then the projective $R\Gamma$-resolution $P^j := \bigoplus_{\sigma\in S_j} \operatorname{ind}_{\Gamma_\sigma}^\Gamma P^\sigma$ of X_j lies in B_{n-j}^Γ by axiom (B-ind) and Lemma 3.2 (ii). Finally, Proposition 3.3 yields a projective $R\Gamma$-chain complex $\overline{X} \in \mathsf{B}_n^\Gamma$ that is weakly equivalent to X. Hence $\overline{X}$ is a projective $R\Gamma$-resolution of R, witnessing that the group Γ lies in B_n. □

Suitable $R\Gamma$-chain complexes arise as the cellular chain complex of Γ-CW-complexes.

Theorem 3.6 (Bootstrapping for groups)**.** *Let B_* be a bootstrappable property of an is-class ? of groups over a ring R. Let $\Gamma \in ?$ and let $n \in \mathbb{N}$. Let Ω be a Γ-CW-complex such that the following hold:*

(i) *Ω is $(n-1)$-acyclic over R (i.e., $H_j(\Omega; R) \cong H_j(\mathrm{pt}; R)$ for all $j \le n-1$);*
(ii) *$\Gamma\backslash\Omega^{(n)}$ is compact;*
(iii) *For every cell σ of Ω with $\dim(\sigma) \le n$, the stabiliser Γ_σ lies in $\mathsf{B}_{n-\dim(\sigma)}$.*

Then $\Gamma \in \mathsf{B}_n$.

Proof. Let X be the cellular $R\Gamma$-chain complex of Ω. For $j \le n$, the $R\Gamma$-module X_j is of the form

$$X_j \cong \bigoplus_{\sigma\in S_j} R[\Gamma/\Gamma_\sigma],$$

where S_j is a finite set of representatives for the Γ-orbits of j-cells of Ω. Then Theorem 3.5 applies to the $R\Gamma$-chain complex X and yields the claim. □

As special cases of Theorem 3.5 and Theorem 3.6, we obtain inheritance results for classical group-theoretic constructions. Recall that a group Γ is *of type* $\mathsf{FP}_n(R)$ if the trivial $R\Gamma$-modules R admits a projective resolution P such that the $R\Gamma$-module P_j is finitely generated for all $j \leq n$. A group Γ is *of type* $\mathsf{FP}_\infty(R)$ if Γ is of type $\mathsf{FP}_n(R)$ for all $n \in \mathbb{Z}$. For $R = \mathbb{Z}$, we write FP_n and FP_∞ instead of $\mathsf{FP}_n(\mathbb{Z})$ and $\mathsf{FP}_\infty(\mathbb{Z})$, respectively.

Corollary 3.7. *Let B_* be a bootstrappable property of an is-class ? of groups over a ring R. For all $n \in \mathbb{Z}$, the following hold:*

(i) *(Graphs of groups). Let $\Gamma \in$? be the fundamental group of a finite graph of groups. If all vertex groups lie in B_n and all edge groups lie in B_{n-1}, then $\Gamma \in \mathsf{B}_n$;*
(ii) *(Extensions). Let $\Gamma \in$? be a group containing a normal subgroup N such that the quotient Γ/N is of type $\mathsf{FP}_n(R)$. (This is the case, e.g., if Γ is of type $\mathsf{FP}_n(R)$ and N is of type $\mathsf{FP}_{n-1}(R)$.) If $N \in \mathsf{B}_m$ for all $m \leq n$, then $\Gamma \in \mathsf{B}_n$;*
(iii) *(Finite extensions). Let $\Gamma \in$? and let N be a finite index normal subgroup of Γ. If $N \in \mathsf{B}_m$ for all $m \leq n$, then $\Gamma \in \mathsf{B}_n$;*
(iv) *(Finitely generated free abelian groups). Let $\Gamma \in$? be a non-trivial finitely generated free abelian group. If $\mathbb{Z} \in \mathsf{B}_m$ for all $m \leq n$, then $\Gamma \in \mathsf{B}_n$;*
(v) *(Groups containing a free abelian normal subgroup). Let $\Gamma \in$? be a group of type $\mathsf{FP}_n(R)$ containing a non-trivial finitely generated free abelian normal subgroup. If $\mathbb{Z} \in \mathsf{B}_m$ for all $m \leq n$, then $\Gamma \in \mathsf{B}_n$.*

Proof. (i) Apply Theorem 3.6 to the Bass–Serre tree.

(ii) Since the quotient $Q := \Gamma/N$ is of type $\mathsf{FP}_n(R)$, there exists a free RQ-resolution P of R such that the RQ-module P_j is finitely generated for all $j \leq n$. Then apply Theorem 3.5 to the $R\Gamma$-resolution $\operatorname{res}_{\Gamma \twoheadrightarrow Q} P$ of R.

(iii) This follows from part (ii), since finite groups are of type F_∞ and in particular of type $\mathsf{FP}_\infty(R)$.

(iv) This follows immediately from part (ii).

(v) This follows from parts (iv) and (ii). □

Part (i) of Corollary 3.7 applies in particular to amalgamated products and HNN-extensions. In parts (ii)–(v), since N (resp. $\mathbb{Z}$) lies in B_m for all $m \leq n$, tautologically it also lies in B_m for all $m \leq n-1$. Hence the conclusions are in fact that Γ lies in B_m for all $m \leq n$. In many of our examples of bootstrappable properties B_* of groups (Section 4), the class B_n is contained in B_m for all $m \leq n$ and the group $\mathbb{Z}$ lies in B_∞. However, in many examples the trivial group does not lie in B_0.

Part (v) applies to non-trivial finitely generated torsion-free nilpotent groups because these groups are of type F and have non-trivial centre. More generally, properties of the group $\mathbb{Z}$ can be bootstrapped to infinite polycyclic-by-finite groups (i.e., groups that admit a subnormal series whose factors are cyclic or finite; Example 3.8).

In each of the following examples, let B_* be a bootstrappable property of an is-class ? of groups over a ring R. Here it is crucial that the class ? of groups is an is-class.

Example 3.8 (Polycyclic-by-finite groups)**.** Let $\Gamma \in$? be an infinite polycyclic-by-finite group. If $\mathbb{Z} \in \mathsf{B}_m$ for all $m \leq n$, then $\Gamma \in \mathsf{B}_n$.

Indeed, Γ contains a finite index normal subgroup Λ that is poly-infinite-cyclic (i.e., Λ admits a subnormal series whose factors are infinite cyclic) [CSD21, Lemma 5.11]. By Corollary 3.7 (iii), it suffices to show that Λ lies in B_m for all $m \leq n$.

Let

$$\{1\} = \Lambda_0 \lhd \Lambda_1 \lhd \cdots \lhd \Lambda_k = \Lambda$$

be a subnormal series witnessing the structure of Λ as a poly-infinite-cyclic group. Since Λ_1 is isomorphic to $\mathbb{Z}$, the group Λ_1 lies in B_m for all $m \leq n$ by assumption. By induction on the length of the subnormal series, we may assume that Λ_{k-1} lies in B_m for all $m \leq n$. Since $\Lambda = \Lambda_k$ is an extension of Λ_{k-1} by $\mathbb{Z}$, Corollary 3.7 (ii) shows that Λ lies in B_m for all $m \leq n$.

In fact, properties of the group $\mathbb{Z}$ can be bootstrapped to infinite elementary amenable groups of type FP_∞. Recall that the class of *elementary amenable* groups is the smallest class of groups that contains all finite groups and all abelian groups, and that is closed under subgroups, quotients, extensions, and directed unions.

Example 3.9 (Elementary amenable groups)**.** Let $\Gamma \in ?$ be an infinite elementary amenable group of type FP_∞. If $\mathbb{Z} \in \mathsf{B}_m$ for all $m \leq n$, then $\Gamma \in \mathsf{B}_n$.

Indeed, a characterisation by Kropholler–Martínez-Pérez–Nucinkis [KMPN09, Theorem 1.1] yields that either

(1) Γ is an infinite polycyclic-by-finite group; or
(2) Γ contains a normal subgroup N such that:
   - N is a strictly ascending HNN-extension $H*_{H,t}$ over a finitely generated virtually nilpotent group H, and
   - Γ/N is a Euclidean crystallographic group.

In case (1), Γ lies in B_n by Example 3.8. In case (2), since Euclidean crystallographic groups are of type F_∞, it suffices to show that N lies in B_m for all $m \leq n$ by Corollary 3.7 (ii). Since the HNN-extension $N = H*_{H,t}$ is strictly ascending, the group H must be infinite. The infinite finitely generated virtually nilpotent group H is infinite polycyclic-by-finite and hence lies in B_m for all $m \leq n$ by Example 3.8. Then the HNN-extension N lies in B_m for all $m \leq n$ by Corollary 3.7 (i).

Properties of the group $\mathbb{Z}$ can also be bootstrapped to certain right-angled Artin groups that are iterated amalgamated products of free abelian groups. Recall that the *right-angled Artin group* $A_\mathcal{G}$ associated to a finite simplicial graph $\mathcal{G}$ has one generator for each vertex of $\mathcal{G}$ and the only relations are that two generators commute if their corresponding vertices are connected by an edge in $\mathcal{G}$.

Example 3.10 (Chordal right-angled Artin groups)**.** Let $\mathcal{G}$ be a non-empty connected finite simplicial graph that is chordal (i.e., $\mathcal{G}$ contains no cycles of length ≥ 4 as full subgraphs). Let $A_\mathcal{G}$ be the right-angled Artin group associated to $\mathcal{G}$ and suppose that $A_\mathcal{G} \in ?$. If $\mathbb{Z} \in \mathsf{B}_m$ for all $m \leq n$, then $A_\mathcal{G} \in \mathsf{B}_n$.

Indeed, it is a classical result about chordal graphs that either

(1) $\mathcal{G}$ is complete; or
(2) there exist full proper subgraphs $\mathcal{G}_1, \mathcal{G}_2$ of $\mathcal{G}$ such that $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ and $\mathcal{G}_0 := \mathcal{G}_1 \cap \mathcal{G}_2$ is complete.

In case (1), the group $A_\mathcal{G}$ is non-trivial finitely generated free abelian and hence lies in B_n by Corollary 3.7 (iv). In case (2), the group $A_\mathcal{G}$ splits as an amalgamated product $A_\mathcal{G} \cong A_{\mathcal{G}_1} *_{A_{\mathcal{G}_0}} A_{\mathcal{G}_2}$. Since $\mathcal{G}$ is connected, the graph $\mathcal{G}_0$ is non-empty. Then the group $A_{\mathcal{G}_0}$ is non-trivial finitely generated free abelian and hence lies in B_{n-1} by Corollary 3.7 (iv). By induction on the number of vertices of $\mathcal{G}$, we may assume that $A_{\mathcal{G}_1}$ and $A_{\mathcal{G}_2}$ lie in B_n. Hence $A_\mathcal{G}$ lies in B_n by Corollary 3.7 (i).

An equivariantly bootstrappable property $\mathsf{B}^?_*$ of $R[?]$-chain complexes may satisfy the following additional axiom for all $n \in \mathbb{Z}$:

(B-res) *Restriction to finite index subgroups.* Let $\Gamma \in ?$, let Δ be a finite index subgroup of Γ, and let X be an $R\Gamma$-chain complex. If $X \in \mathsf{B}^\Gamma_n$, then $\operatorname{res}^\Gamma_\Delta X \in \mathsf{B}^\Delta_n$.

Corollary 3.11. *Let R be a ring and let $?$ be an is-class of groups. Let B_* be the bootstrappable property of $?$ over R associated to an equivariantly bootstrappable property $\mathsf{B}^?_*$ of $R[?]$-chain complexes satisfying axiom* (B-res). *For all $n \in \mathbb{Z}$, the following hold:*

(i) *(Finite index subgroups). Let $\Gamma \in ?$ and let Δ be a finite index subgroup of Γ. If $\Gamma \in \mathsf{B}_n$, then $\Delta \in \mathsf{B}_n$;*
(ii) *(Finite index overgroups). Let $\Gamma \in ?$ and let Δ be a finite index subgroup of Γ. If $\Delta \in \mathsf{B}_m$ for all $m \leq n$, then $\Gamma \in \mathsf{B}_n$;*
(iii) *(Commensurated subgroups). Let $\Gamma \in ?$ and let Λ be a commensurated subgroup of Γ (i.e., for all $\gamma \in \Gamma$, the intersection $\Lambda \cap \gamma\Lambda\gamma^{-1}$ has finite index in Λ and in $\gamma\Lambda\gamma^{-1}$). Suppose that Γ is of type F_n and Λ is of type F_{n-1}. If $\Lambda \in \mathsf{B}_m$ for all $m \leq n$, then $\Gamma \in \mathsf{B}_n$.*

It follows from parts (i) and (ii) that the intersection $\bigcap_{m \leq n} \mathsf{B}_m$ of classes of groups is closed under commensurability.

Proof. (i) If Γ lies in B_n, there exists a projective $R\Gamma$-resolution P of R lying in B^Γ_n. Then the restriction $\operatorname{res}^\Gamma_\Delta P$ is a projective $R\Delta$-resolution of R lying in B^Δ_n by axiom (B-res), witnessing that Δ lies in B_n.

(ii) Suppose that Δ lies in B_m for all $m \leq n$. The normal core $\operatorname{Core}_\Gamma(\Delta) := \bigcap_{\gamma \in \Gamma} \gamma\Delta\gamma^{-1}$ of Δ in Γ is a finite index normal subgroup of Γ. Since $\operatorname{Core}_\Gamma(\Delta)$ has finite index in Δ, we have $\operatorname{Core}_\Gamma(\Delta) \in \mathsf{B}_m$ for all $m \leq n$ by part (i). Hence Γ lies in B_n by Corollary 3.7 (iii).

(iii) We consider the Schlichting completion G of Γ relative to Λ, that is, G is the closure of the image of the translation action $\tau\colon \Gamma \to \operatorname{Sym}(\Gamma/\Lambda)$ with respect to the topology of pointwise convergence. Then G is a locally compact totally disconnected group and the closure U of $\tau(\Lambda)$ is a compact-open subgroup. Furthermore, we have $U \cap \tau(\Gamma) = \tau(\Lambda)$ [BS, Section 2]. Since Γ is of type F_n and Λ is of type F_{n-1}, the group G is of type F_n in the sense that there is a contractible G-CW-complex Ω with compact-open stabilisers such that the n-skeleton is cocompact [BS, Definition 3.1, Theorem 1.2]. Every compact-open subgroup of G is commensurable with U. Therefore the stabilisers of the Γ-CW-complex $\operatorname{res}_\tau \Omega$ are commensurable with Λ and hence lie in B_m for all $m \leq n$ by parts (i) and (ii). By Theorem 3.6, the group Γ lies in B_n. □

Remark 3.12. Other examples of groups that admit appropriate actions to which the Bootstrapping Theorem 3.6 can be applied include special linear groups, mapping class groups, chain commuting groups, certain Artin groups [ABFG24], outer automorphism groups of free products of $\mathbb{Z}/2$ [GGH], mapping tori of polynomially growing automorphisms [AHK24, AGHK], and inner-amenable groups [Usc].

Remark 3.13. The construction in the proof of Theorem 3.5 is an algebraic version of a topological "blow-up" construction by Lück [Lüc00, MPSS20] and Geoghegan [Geo08]. Let Ω be a Γ-CW-complex and let classifying spaces $(E\Gamma_\sigma)_\sigma$ for its cell stabilisers be given. Then there exists a *free* Γ-CW-complex $\widehat{\Omega}$ that is (non-equivariantly) homotopy equivalent to Ω and, roughly speaking, obtained from Ω by replacing each Γ-orbit of cells $\Gamma/\Gamma_\sigma \times \sigma$ with the free Γ-CW-complex $\operatorname{ind}^\Gamma_{\Gamma_\sigma} E\Gamma_\sigma \times \sigma$.

4. Examples of bootstrappable properties

We discuss several interesting equivariantly bootstrappable properties of $R[?]$-chain complexes and relations between them. Besides the unified approach, the main novelty is the algebraic cheap rebuilding property (Section 4.4).

Let $?$ be an is-class of groups. Given a family $\mathsf{B}^?_* = (\mathsf{B}^\Gamma_n)_{\Gamma \in ?, n \in \mathbb{Z}}$ of classes of $R[?]$-chain complexes, for every $n \in \mathbb{Z}$ we define the class B_n of groups $\Gamma \in ?$ for which there exists a projective $R\Gamma$-resolution of R lying in B_n^Γ. We also define the class of groups $\mathsf{B}_\infty := \bigcap_{n\in\mathbb{Z}} \mathsf{B}_n$.

4.1. **Algebraic finiteness properties.** For the basics on algebraic finiteness properties, we refer to Brown's book [Bro82a, Chapter VIII]. We work over an arbitrary ring R.

Definition 4.1. Let Γ be a group and let $n \in \mathbb{Z}$. The class $\mathsf{FG}_n^\Gamma(R)$ consists of all $R\Gamma$-chain complexes X satisfying for all $j \leq n$ that the $R\Gamma$-module X_j is finitely generated.

Then a group Γ lies in $\mathsf{FG}_n(R)$ if and only if Γ is of type $\mathsf{FP}_n(R)$. For example, it is easy to see that the group $\mathbb{Z}$ and finite groups lie in $\mathsf{FG}_\infty(R)$.

Lemma 4.2. *Let ? be the is-class of all groups. The family $\mathsf{FG}_*^?(R)$ is an equivariantly bootstrappable property of $R[?]$-chain complexes. Moreover, the family $\mathsf{FG}_*^?(R)$ satisfies axiom* (B-res).

Proof. First, for every group Γ, the sequence $\mathsf{FG}_*^\Gamma(R)$ is a bootstrappable property of $R\Gamma$-chain complexes because axioms (B-deg), (B-susp), and (B-cone) clearly hold.

Second, the family $\mathsf{FG}_*^?(R)$ satisfies axiom (B-ind). Indeed, let Δ be a subgroup of Γ and let M be an $R\Delta$-module. If the $R\Delta$-module M is finitely generated, then the $R\Gamma$-modules $\operatorname{ind}_\Delta^\Gamma M$ is finitely generated. Applying this fact degreewise to the chain modules of chain complexes shows that axiom (B-ind) holds.

Third, the family $\mathsf{FG}_*^?(R)$ satisfies axiom (B-res). Indeed, let Δ be a finite index subgroup of Γ and let M be an $R\Gamma$-module. If the $R\Gamma$-module M is finitely generated, then the $R\Delta$-module $\operatorname{res}_\Delta^\Gamma M$ is finitely generated. Applying this fact degreewise to the chain modules of chain complexes shows that axiom (B-res) holds. □

We obtain the Bootstrapping Theorem 3.6 for $\mathsf{FG}_*(R)$, which is a classical result [Bro87, Proposition 1.1].

4.2. **ℓ^2-Invariants.** For background on ℓ^2-invariants we refer to Lück's book [Lüc02]. We will show that vanishing of ℓ^2-homology, vanishing of ℓ^2-Betti numbers, and lower bounds for Novikov–Shubin invariants are bootstrappable properties. All cases follow from the general Lemma 4.3 using that the respective property is preserved by submodules, quotients, extensions, induction, and restriction to finite index subgroups. Throughout this section, we work over the ring $R = \mathbb{Z}$. For a group Γ, we denote by $\mathcal{N}\Gamma$ the group von Neumann algebra.

Lemma 4.3. *Let Γ be a group and let $j \in \mathbb{Z}$. The following hold:*

(i) *Let $f\colon X \to Y$ be a $\mathbb{Z}\Gamma$-chain map. There is a short exact sequence of $\mathcal{N}\Gamma$-modules*

$$0 \to M \to H_j\big(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} \operatorname{Cone}(f)\big) \to L \to 0,$$

where M is a quotient module of $H_j(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} Y)$ and L is a submodule of $H_{j-1}(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} X)$;

(ii) *Let Δ be a subgroup of Γ and let X be a $\mathbb{Z}\Delta$-chain complex. There is an isomorphism of $\mathcal{N}\Gamma$-modules*

$$H_j(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} \operatorname{ind}_\Delta^\Gamma X) \cong \mathcal{N}\Gamma \otimes_{\mathcal{N}\Delta} H_j(\mathcal{N}\Delta \otimes_{\mathbb{Z}\Delta} X);$$

(iii) *Let Δ be a finite index subgroup of Γ and let X be a $\mathbb{Z}\Gamma$-chain complex. There is an isomorphism of $\mathcal{N}\Delta$-modules*

$$H_j(\mathcal{N}\Delta \otimes_{\mathbb{Z}\Delta} \operatorname{res}_\Delta^\Gamma X) \cong \operatorname{res}_{\mathcal{N}\Delta}^{\mathcal{N}\Gamma} H_j(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} X).$$

Proof. (i) By Lemma 2.2 (i), we have a short exact sequence of $\mathbb{Z}\Gamma$-chain complexes

$$0 \to Y \to \mathrm{Cone}(f) \to \Sigma X \to 0$$

which splits degreewise over $\mathbb{Z}\Gamma$. Hence the induced sequence of $\mathcal{N}\Gamma$-chain complexes

$$0 \to \mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} Y \to \mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} \mathrm{Cone}(f) \to \mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} \Sigma X \to 0$$

is (degreewise split) exact. Let $\delta_*\colon H_*(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} \Sigma X) \to H_{*-1}(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} Y)$ denote the connecting homomorphism of the associated long exact homology sequence. For every $j \in \mathbb{Z}$, we obtain an induced short exact sequence of $\mathcal{N}\Gamma$-modules

$$0 \to \operatorname{coker} \delta_{j+1} \to H_j\big(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} \mathrm{Cone}(f)\big) \to \ker \delta_j \to 0.$$

In particular, $\operatorname{coker} \delta_{j+1}$ is a quotient of $H_j(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} Y)$ and $\ker \delta_j$ is a submodule of $H_j(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} \Sigma X) \cong H_{j-1}(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} X)$.

(ii) There are canonical and natural isomorphisms of $\mathcal{N}\Gamma$-chain complexes

$$\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} \mathrm{ind}_\Delta^\Gamma X = \mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} X \cong \mathcal{N}\Gamma \otimes_{\mathcal{N}\Delta} \mathcal{N}\Delta \otimes_{\mathbb{Z}\Delta} X.$$

Since $\mathcal{N}\Gamma$ is flat over $\mathcal{N}\Delta$ [Lüc02, Theorem 6.29], we obtain an isomorphism of $\mathcal{N}\Gamma$-modules

$$H_j(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} \mathrm{ind}_\Delta^\Gamma X) \cong \mathcal{N}\Gamma \otimes_{\mathcal{N}\Delta} H_j(\mathcal{N}\Delta \otimes_{\mathbb{Z}\Delta} X).$$

(iii) Since Δ has finite index in Γ, there is an isomorphism of $\mathcal{N}\Delta$-$\mathbb{Z}\Gamma$-bimodules [LRS99, Proof of Theorem 3.7 (1)]

$$\mathcal{N}\Delta \otimes_{\mathbb{Z}\Delta} \mathrm{res}_\Delta^\Gamma \mathbb{Z}\Gamma \cong \mathrm{res}_{\mathcal{N}\Delta}^{\mathcal{N}\Gamma} \mathcal{N}\Gamma.$$

Then we have isomorphisms of $\mathcal{N}\Delta$-modules

$$\mathcal{N}\Delta \otimes_{\mathbb{Z}\Delta} \mathrm{res}_\Delta^\Gamma X \cong \mathcal{N}\Delta \otimes_{\mathbb{Z}\Delta} \mathrm{res}_\Delta^\Gamma \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma} X \cong \mathrm{res}_{\mathcal{N}\Delta}^{\mathcal{N}\Gamma} \mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} X$$

and hence

$$H_j(\mathcal{N}\Delta \otimes_{\mathbb{Z}\Delta} \mathrm{res}_\Delta^\Gamma X) \cong \mathrm{res}_{\mathcal{N}\Delta}^{\mathcal{N}\Gamma} H_j(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} X).$$

This finishes the proof. □

4.2.1. *ℓ^2-Invisibility.* For an overview on ℓ^2-invisibility and its relevance for the Zero-in-the-spectrum Conjecture, we refer to [Lüc02, Chapter 12].

Definition 4.4. Let Γ be a group and let $n \in \mathbb{Z}$. The class I_n^Γ consists of all $\mathbb{Z}\Gamma$-chain complexes X satisfying for all $j \leq n$ that

$$H_j(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} X) = 0.$$

Then a group Γ lies in I_n if and only if $H_j(\Gamma; \mathcal{N}\Gamma) = 0$ for all $j \leq n$. Moreover, a group Γ of type F_∞ lies in I_∞ if and only if $H_j(\Gamma; \ell^2\Gamma) = 0$ for all $j \in \mathbb{Z}$ [Lüc02, Lemma 12.3]. The class I_0 is the class of non-amenable groups [Lüc02, Lemma 12.11 (4)]. To obtain groups in I_n, there is a product formula: If $\Gamma_1 \in \mathsf{I}_{n_1}$ and $\Gamma_2 \in \mathsf{I}_{n_2}$, then $\Gamma_1 \times \Gamma_2 \in \mathsf{I}_{n_1+n_2+1}$ [Lüc02, Lemma 12.11 (3)]. The groups in I_∞ are said to be *ℓ^2-invisible*.

Proposition 4.5. *Let ? be the is-class of all groups. The family $\mathsf{I}_*^?$ is an equivariantly bootstrappable property of $\mathbb{Z}[?]$-chain complexes. Moreover, the family $\mathsf{I}_*^?$ satisfies axiom* (B-res)*.*

Proof. First, for every group Γ, the sequence I_*^Γ is a bootstrappable property of $\mathbb{Z}\Gamma$-chain complexes. Indeed, the axioms (B-deg) and (B-susp) clearly hold and axiom (B-cone) follows from Lemma 4.3 (i). Second, the family $\mathsf{I}_*^?$ satisfies axiom (B-ind) which follows from Lemma 4.3 (ii). Third, the family $\mathsf{I}_*^?$ satisfies axiom (B-res) which follows from Lemma 4.3 (iii). □

We obtain the Bootstrapping Theorem 3.6 for I_*, which is implicit in the work of Sauer–Thumann [ST14, Proof of Theorem 1.1]. This result has been used to provide examples of ℓ^2-invisible groups of type F_∞, given by certain local similarity groups [ST14]. It remains unknown whether there exists an ℓ^2-invisible group of type F.

4.2.2. *ℓ^2-Acyclicity.* For an overview on ℓ^2-acyclicity and its plentiful applications, we refer to [Lüc02, Section 7.1].

Definition 4.6. Let Γ be a group and let $n \in \mathbb{Z}$. The class A_n^Γ consists of all $\mathbb{Z}\Gamma$-chain complexes X satisfying for all $j \leq n$ that

$$\dim_{\mathcal{N}\Gamma} H_j(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} X) = 0.$$

Here $\dim_{\mathcal{N}\Gamma}$ is the von Neumann dimension over $\mathcal{N}\Gamma$.

Then a group Γ lies in A_n if and only if $b_j^{(2)}(\Gamma) = 0$ for all $j \leq n$. For example, the class A_0 is the class of infinite groups [Lüc02, Theorem 6.54 (8)]. All infinite amenable groups lie in A_∞ [Lüc02, Corollary 6.75]. The groups in A_∞ are said to be *ℓ^2-acyclic*. Clearly, the class I_n is contained in A_n and this inclusion is strict, as witnessed by infinite amenable groups.

Proposition 4.7. *Let ? be the is-class of all groups. The family $\mathsf{A}_*^?$ is an equivariantly bootstrappable property of $\mathbb{Z}[?]$-chain complexes. Moreover, the family $\mathsf{A}_*^?$ satisfies axiom* (B-res).

Proof. First, for every group Γ, the sequence A_*^Γ is a bootstrappable property of $\mathbb{Z}\Gamma$-chain complexes. Indeed, the axioms (B-deg) and (B-susp) clearly hold. Axiom (B-cone) follows from Lemma 4.3 (i) using that $\dim_{\mathcal{N}\Gamma}$ is additive with respect to short exact sequences [Lüc02, Theorem 1.12 (2)]. Second, the family $\mathsf{A}_*^?$ satisfies axiom (B-ind). This follows from Lemma 4.3 (ii) using that the von Neumann dimension is preserved by the induction functor [Lüc02, Lemma 1.24 (2)]. Third, the family $\mathsf{A}_*^?$ satisfies axiom (B-res). This follows from Lemma 4.3 (iii) using that the von Neumann dimension is multiplicative with respect to restriction to finite index subgroups [Lüc02, Theorem 1.12 (6)]. □

We obtain the Bootstrapping Theorem 3.6 for A_*, which was proved (in a slightly weaker form) by Jo [Jo07, Theorem 3.5].

4.2.3. *Novikov–Shubin invariants and capacity.* The Novikov–Shubin invariants are spectral invariants that measure the difference between ℓ^2-acyclicity and ℓ^2-invisibility. For an introduction to Novikov–Shubin invariants, we refer to Lück's book [Lüc02, Chapter 2]. In the following, we will use the general setup developed by Lück–Reich–Schick [LRS99]. In particular, we will phrase the properties and arguments in terms of capacity, which is reciprocal to the Novikov–Shubin invariants.

Capacity and Novikov–Shubin invariants take values in the extended ranges of numbers

$$[\![0, \infty] := \{0^-\} \sqcup [0, \infty] \quad \text{and} \quad [0, \infty]\!] := [0, \infty] \sqcup \{\infty^+\},$$

respectively. Here $[\![0, \infty]\!] := \{0^-\} \sqcup [0, \infty] \sqcup \{\infty^+\}$ carries the ordering that extends the usual ordering on $[0, \infty]$ by $0^- < 0$ and $\infty < \infty^+$. Moreover, arithmetic operations are extended as expected and

$$\frac{1}{0^-} := \infty^+ \quad \text{and} \quad \frac{1}{\infty^+} := 0^-.$$

Definition 4.8. Let Γ be a group, let $n \in \mathbb{Z}$, and let $\kappa \in [\![0,\infty]\!]$.

- The class CM_n^Γ consists of all $\mathbb{Z}\Gamma$-chain complexes X satisfying for all $j \leq n$ that the $\mathcal{N}\Gamma$-module $H_j(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} X)$ is cofinal-measurable [LRS99, Definition 2.1];
- The class $\mathsf{C}(\leq \kappa)_n^\Gamma$ consists of all $\mathbb{Z}\Gamma$-chain complexes $X \in \mathsf{CM}_n^\Gamma$ satisfying for all $j \leq n$ that

$$c_{\mathcal{N}\Gamma}\big(H_j(\mathcal{N}\Gamma \otimes_{\mathbb{Z}\Gamma} X)\big) \leq \kappa.$$

  Here $c_{\mathcal{N}\Gamma}$ is the capacity of $\mathcal{N}\Gamma$-modules [LRS99, Section 2], i.e., the reciprocal of the Novikov–Shubin invariant;
- The class $\mathsf{C}(< \kappa)_n^\Gamma$ is defined analogously to $\mathsf{C}(\leq \kappa)_n^\Gamma$ by replacing the inequality with a strict inequality.

All cofinal-measurable modules have von Neumann dimension 0. In particular, CM_n is a subclass of A_n. Every ℓ^2-acyclic group of type FP_∞ lies in CM_∞. A group Γ of type F_∞ lies in $\mathsf{C}(< \infty)_n$ if and only if Γ is ℓ^2-acyclic and the Novikov–Shubin invariants of Γ satisfy $\alpha_j(\Gamma) > 0$ for all $j \leq n$ (in the indexing of Novikov–Shubin invariants of boundary operators, not of Laplacians). For some calculations of the capacity of groups, see [LRS99, Section 3]. By construction of the capacity/Novikov–Shubin invariants, for every group Γ, we have $\mathsf{I}_n^\Gamma = \mathsf{C}(\leq 0^-)_n^\Gamma$.

Proposition 4.9. *Let ? be the is-class of all groups. The following hold:*

(i) *The family $\mathsf{CM}_*^?$ is an equivariantly bootstrappable property of $\mathbb{Z}[?]$-chain complexes and satisfies axiom* (B-res)*;*

(ii) *The families $\mathsf{C}(\leq 0^-)_*^?$, $\mathsf{C}(\leq 0)_*^?$, and $\mathsf{C}(< \infty)_*^?$ are equivariantly bootstrappable properties of $\mathbb{Z}[?]$-chain complexes and satisfy axiom* (B-res).

Proof. (i) First, for every group Γ, the sequence CM_*^Γ is a bootstrappable property of $\mathbb{Z}\Gamma$-chain complexes. Indeed, the axioms (B-deg) and (B-susp) clearly hold. Axiom (B-cone) follows from Lemma 4.3 (i) using that for $\mathcal{N}\Gamma$-modules, being cofinal-measurable is stable under taking submodules, quotients, and extensions [LRS99, Lemma 2.11]. Second, the family $\mathsf{CM}_*^?$ satisfies axiom (B-ind). This follows from Lemma 4.3 (ii) using that being cofinal-measurable is preserved by the induction functor [LRS99, Lemma 2.12 (1)]. Third, the family $\mathsf{CM}_*^?$ satisfies axiom (B-res). This follows from Lemma 4.3 (iii) using that being cofinal-measurable is preserved by restriction to finite index subgroups [LRS99, Lemma 2.12 (3)].

(ii) Let $\kappa \in \{0^-, 0\}$. We will only treat the family $\mathsf{C}(\leq \kappa)_*^?$. The family $\mathsf{C}(< \infty)_*^?$ can be handled analogously.

First, for every group Γ, the sequence $\mathsf{C}(\leq \kappa)_*^\Gamma$ is a bootstrappable property of $\mathbb{Z}\Gamma$-chain complexes. Indeed, the axioms (B-deg) and (B-susp) clearly hold. Axiom (B-cone) follows from Lemma 4.3 (i) using that for cofinal-measurable $\mathcal{N}\Gamma$-modules, capacity is monotone for submodules and quotients, and subadditive with respect to short exact sequences [LRS99, Theorem 2.7 (1)]. Here we use that our choices of κ satisfy $\kappa + \kappa \leq \kappa$. Second, the family $\mathsf{C}(\leq \kappa)_*^?$ satisfies axiom (B-ind). This follows from Lemma 4.3 (ii) using that capacity is preserved by the induction functor [LRS99, Lemma 2.12 (1)]. Third, the family $\mathsf{C}(\kappa)_*^?$ satisfies axiom (B-res). This follows from Lemma 4.3 (iii) using that capacity is preserved by restriction to finite index subgroups [LRS99, Lemma 2.12 (3)]. □

We obtain corresponding instances of the Bootstrapping Theorem 3.6 for each of the families $\mathsf{C}(\leq 0^-)_*$, $\mathsf{C}(\leq 0)_*$, and $\mathsf{C}(< \infty)_*$. The family $\mathsf{C}(\leq 0^-)_*$ recovers the case of I_*. Translated to Novikov–Shubin invariants, these three cases correspond to the constraints of having cofinal-measurable ℓ^2-homology whose Novikov–Shubin invariants are

- equal to ∞^+;
- equal to ∞ or ∞^+;
- positive;

respectively. It would be interesting if also the constraint "the Novikov–Shubin invariants are ≥ 1" could be proved to be (equivariantly) bootstrappable. However, the known inheritance results for Novikov–Shubin invariants of short exact sequences are not strong enough to handle this case.

Remark 4.10. It was conjectured by Lott–Lück [LL95, Conjecture 7.2] that, for every group Γ, the Novikov–Shubin invariants of every $\mathbb{Z}\Gamma$-chain complex of finitely generated free $\mathbb{Z}\Gamma$-modules are positive. Grabowski found a counterexample by constructing a solvable group Γ, which is not of type F_∞, and an element $c \in \mathbb{Z}\Gamma$ with vanishing Novikov–Shubin invariant [Gra15]. The corresponding chain complex is concentrated in two degrees with the differential being multiplication by c. However, the following weakening of Lott–Lück's conjecture remains open: Are the Novikov–Shubin invariants of a group of type F_∞ positive?

We provide some evidence: Since the group $\mathbb{Z}$ lies in $\mathsf{C}(<\infty)_\infty$ [LRS99, Theorem 3.7], it follows from the Bootstrapping Theorem 3.6 for $\mathsf{C}(<\infty)_*$ that infinite elementary amenable groups of type FP_∞ lie in $\mathsf{C}(<\infty)_\infty$ (Example 3.9). In particular, the Novikov–Shubin invariants of these groups are positive.

Remark 4.11. In Definition 4.4, Definition 4.6, and Definition 4.8 we consider vanishing resp. upper bounds in all degrees $\leq n$. In particular, for every sequence $\mathsf{B}_* \in \{\mathsf{I}_*, \mathsf{A}_*, \mathsf{CM}_*, \mathsf{C}(\leq \kappa)_*, \mathsf{C}(<\kappa)_*\}$, the class B_n is contained in B_m for all $m \leq n$. Instead, one may also consider the vanishing of ℓ^2-homology, vanishing of von Neumann dimension, and upper bounds for capacity in a fixed degree n only. One obtains corresponding equivariantly bootstrappable properties of $\mathbb{Z}[?]$-chain complexes by the same proofs as Proposition 4.5, Proposition 4.7, and Proposition 4.9.

4.3. **Vanishing of (torsion) homology growth.** For an overview on (torsion) homology growth, we refer to Lück's survey [Lüc16]. We work over the ring $R = \mathbb{Z}$.

Let Γ be a group and let Λ be a subgroup of Γ. We consider the functor $(-)_\Lambda$ that associates to a $\mathbb{Z}\Gamma$-module M the $\mathbb{Z}$-module $M_\Lambda := \mathbb{Z} \otimes_{\mathbb{Z}\Lambda} \operatorname{res}^\Gamma_\Lambda M$ of Λ-coinvariants. A $\mathbb{Z}\Gamma$-basis of a free $\mathbb{Z}\Gamma$-module M induces a $\mathbb{Z}$-basis of the free $\mathbb{Z}$-module M_Λ. We also denote by $(-)_\Lambda$ the induced functor of chain complexes that associates to a $\mathbb{Z}\Gamma$-chain complex X the $\mathbb{Z}$-chain complex

$$X_\Lambda := \mathbb{Z} \otimes_{\mathbb{Z}\Lambda} \operatorname{res}^\Gamma_\Lambda X$$

of Λ-coinvariants. We record some basic properties:

Lemma 4.12. *Let Γ be a group. The following hold:*

(i) *Let Λ be a finite index normal subgroup of Γ. Let Δ be a subgroup of Γ and let M be a $\mathbb{Z}\Delta$-module. Then there is an isomorphism of $\mathbb{Z}$-modules*

$$(\operatorname{ind}^\Gamma_\Delta M)_\Lambda \cong \bigoplus_{\frac{[\Gamma:\Lambda]}{[\Delta:\Lambda\cap\Delta]}} M_{\Lambda\cap\Delta}$$

which is natural in M;

(ii) *Let Λ be a finite index normal subgroup of Γ. Let Δ be a subgroup of Γ and let X be a $\mathbb{Z}\Delta$-chain complex. Then there is a natural isomorphism of $\mathbb{Z}$-chain complexes*

$$(\operatorname{ind}^\Gamma_\Delta X)_\Lambda \cong \bigoplus_{\frac{[\Gamma:\Lambda]}{[\Delta:\Lambda\cap\Delta]}} X_{\Lambda\cap\Delta};$$

(iii) *Let $f\colon M \to L$ be a $\mathbb{Z}\Gamma$-chain map between based free $\mathbb{Z}\Gamma$-modules. Let Λ be a subgroup of Γ. Then the induced map $f_\Lambda\colon M_\Lambda \to L_\Lambda$ of based free $\mathbb{Z}$-modules satisfies $\|f_\Lambda\| \le \|f\|$.*

Proof. (i) We have natural isomorphisms of $\mathbb{Z}$-modules

$$\begin{aligned}(\operatorname{ind}_\Delta^\Gamma M)_\Lambda &\cong \mathbb{Z}[\Lambda\backslash\Gamma] \otimes_{\mathbb{Z}\Gamma} (\operatorname{ind}_\Delta^\Gamma M) \cong \mathbb{Z}[\Lambda\backslash\Gamma] \otimes_{\mathbb{Z}\Gamma} (\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Delta} M)\\ &\cong \mathbb{Z}[\Lambda\backslash\Gamma] \otimes_{\mathbb{Z}\Delta} M \cong \bigoplus_{\frac{[\Gamma:\Lambda]}{[\Delta:\Lambda\cap\Delta]}} \mathbb{Z}[\Lambda\cap\Delta\backslash\Delta] \otimes_{\mathbb{Z}\Delta} M \cong \bigoplus_{\frac{[\Gamma:\Lambda]}{[\Delta:\Lambda\cap\Delta]}} M_{\Lambda\cap\Delta}.\end{aligned}$$

For the second to last isomorphism it is crucial that Λ is normal in Γ.

(ii) This follows directly from part (i).

(iii) Consider the diagram of based free $\mathbb{Z}$-modules

$$\begin{array}{ccc} M & \xrightarrow{f} & L \\ {\scriptstyle p_M}\downarrow & & \downarrow{\scriptstyle p_L} \\ M_\Lambda & \xrightarrow[f_\Lambda]{} & L_\Lambda \end{array}$$

where p_M and p_L are the obvious projections. For $m \in M_\Lambda$, there exists $\widetilde{m} \in M$ with $p_M(\widetilde{m}) = m$ and $|\widetilde{m}|_1 = |m|_1$. Then we have

$$|f_\Lambda(m)|_1 = |f_\Lambda \circ p_M(\widetilde{m})|_1 = |p_L \circ f(\widetilde{m})|_1 \le \|p_L\| \cdot \|f\| \cdot |\widetilde{m}|_1 \le \|f\| \cdot |m|_1$$

and hence $\|f_\Lambda\| \le \|f\|$. □

In the following, we consider the is-class ? of residually finite groups.

Definition 4.13. Let Γ be a residually finite group, let $n \in \mathbb{Z}$, and let $\mathbb{F}$ be a field. The class $\mathsf{H}_n^\Gamma(\mathbb{F})$ consists of all $\mathbb{Z}\Gamma$-chain complexes X satisfying for all residual chains Λ_* in Γ and all $j \le n$ that the Betti number gradient

$$\widehat{b}_j(X, \Lambda_*; \mathbb{F}) := \limsup_{i\to\infty} \frac{\dim_{\mathbb{F}} H_j(\mathbb{F} \otimes_{\mathbb{Z}} X_{\Lambda_i})}{[\Gamma : \Lambda_i]}$$

is equal to 0.

Then the class $\mathsf{H}_n(\mathbb{F})$ consists of all residually finite groups Γ satisfying that $\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{F}) = 0$ for all residual chains Λ_* in Γ and all $j \le n$ (Definition 1.2). Clearly, the class $\mathsf{H}_0(\mathbb{F})$ is the class of residually finite infinite groups. It is easy to see that the group $\mathbb{Z}$ lies in $\mathsf{H}_\infty(\mathbb{F})$. More generally, residually finite infinite amenable groups of type FP_n lie in $\mathsf{H}_n(\mathbb{F})$ (Corollary 5.6).

For a group Γ of type FP_n, the Universal Coefficient Theorem shows that if Γ lies in $\mathsf{H}_n(\mathbb{F})$ for some field $\mathbb{F}$, then Γ lies in $\mathsf{H}_n(\mathbb{Q})$. If Γ is residually finite and of type FP_{j+1}, then $\widehat{b}_j(\Gamma, \Lambda_*; \mathbb{Q})$ coincides with the ℓ^2-Betti number $b_j^{(2)}(\Gamma)$ by Lück's Approximation Theorem [Lüc94]. In particular, a residually finite group Γ of type FP_{n+1} lies in $\mathsf{H}_n(\mathbb{Q})$ if and only if Γ lies in A_n.

Lemma 4.14. *Let Γ be a residually finite group and let Λ_* be a residual chain in Γ. Let $\mathbb{F}$ be a field and let $j \in \mathbb{Z}$. The following hold:*

(i) *Let $f\colon X \to Y$ be a $\mathbb{Z}\Gamma$-chain map. Then*

$$\widehat{b}_j\big(\operatorname{Cone}(f), \Lambda_*; \mathbb{F}\big) \le \widehat{b}_j(Y, \Lambda_*; \mathbb{F}) + \widehat{b}_{j-1}(X, \Lambda_*; \mathbb{F});$$

(ii) *Let Δ be a subgroup of Γ and let X be a $\mathbb{Z}\Delta$-chain complex. Then*

$$\widehat{b}_j(\operatorname{ind}_\Delta^\Gamma X, \Lambda_*; \mathbb{F}) = \widehat{b}_j(X, \Lambda_* \cap \Delta; \mathbb{F}).$$

Proof. (i) The sequence of $\mathbb{Z}\Gamma$-chain complexes

$$0 \to Y \to \operatorname{Cone}(f) \to \Sigma X \to 0$$

splits degreewise over $\mathbb{Z}\Gamma$ by Lemma 2.2 (i). Then, for every subgroup Λ of Γ, the induced sequence of $\mathbb{Z}$-chain complexes

$$0 \to Y_\Lambda \to \operatorname{Cone}(f)_\Lambda \to (\Sigma X)_\Lambda \to 0$$

is (degreewise split) exact. By the long exact sequence in homology, for all $j \in \mathbb{Z}$, we have

$$\dim_{\mathbb{F}} H_j\bigl(\mathbb{F} \otimes_{\mathbb{Z}} \operatorname{Cone}(f)_\Lambda\bigr) \le \dim_{\mathbb{F}} H_j\bigl(\mathbb{F} \otimes_{\mathbb{Z}} Y_\Lambda\bigr) + \dim_{\mathbb{F}} H_j\bigl(\mathbb{F} \otimes_{\mathbb{Z}} (\Sigma X)_\Lambda\bigr).$$

Since $(\Sigma X)_\Lambda \cong \Sigma(X_\Lambda)$, the claim follows using the suspension isomorphism.

(ii) For a finite index normal subgroup Λ of Γ, we have an isomorphism of $\mathbb{Z}$-chain complexes

$$(\operatorname{ind}_\Delta^\Gamma X)_\Lambda \cong \bigoplus_{\frac{[\Gamma:\Lambda]}{[\Delta:\Lambda\cap\Delta]}} X_{\Lambda\cap\Delta}$$

by Lemma 4.12 (ii). By additivity of homology, for all $j \in \mathbb{Z}$, we have

$$\dim_{\mathbb{F}} H_j\bigl(\mathbb{F} \otimes_{\mathbb{Z}} (\operatorname{ind}_\Delta^\Gamma X)_\Lambda\bigr) = \frac{[\Gamma : \Lambda]}{[\Delta : \Lambda \cap \Delta]} \dim_{\mathbb{F}} H_j(\mathbb{F} \otimes_{\mathbb{Z}} X_{\Lambda\cap\Delta})$$

and the claim follows. □

Proposition 4.15. *Let ? be the is-class of residually finite groups and let $\mathbb{F}$ be a field. The family $\mathsf{H}_*^?(\mathbb{F})$ is an equivariantly bootstrappable property of $\mathbb{Z}[?]$-chain complexes.*

Proof. First, for every residually finite group Γ, the sequence $\mathsf{H}_*^\Gamma(\mathbb{F})$ is a bootstrappable property of $\mathbb{Z}\Gamma$-chain complexes. Indeed, the axioms (B-deg) and (B-susp) clearly hold and axiom (B-cone) follows from Lemma 4.14 (i). Second, the family $\mathsf{H}_*^?(\mathbb{F})$ satisfies axiom (B-ind), which follows from Lemma 4.14 (ii). □

We obtain the Bootstrapping Theorem 3.6 for $\mathsf{H}_*(\mathbb{F})$. The family $\mathsf{H}_*^?(\mathbb{F})$ does not seem to satisfy axiom (B-res) due to the non-transitivity of normal subgroups.

Remark 4.16. The topological interpretation of Definition 4.13 is as follows: Let Ω be a CW-complex with residually finite fundamental group Γ. Let Λ_* be a residual chain in Γ and denote by Ω_i the covering of Ω associated to the subgroup Λ_i of Γ. Let X be the cellular $\mathbb{Z}\Gamma$-chain complex of the universal covering $\widetilde{\Omega}$. Then X_{Λ_i} is isomorphic to the cellular $\mathbb{Z}$-chain complex of Ω_i and hence

$$\widehat{b}_j(X, \Lambda_*; \mathbb{F}) = \limsup_{i\to\infty} \frac{\dim_{\mathbb{F}} H_j(\Omega_i; \mathbb{F})}{[\Gamma : \Lambda_i]}.$$

Vanishing of $\widehat{b}_j(X, \Lambda_*; \mathbb{F})$ means that the j-th $\mathbb{F}$-Betti number of the coverings $(\Omega_i)_i$ grows sublinearly with respect to the index $[\Gamma : \Lambda_i]$ as $i \to \infty$.

Instead of the dimension of homology, we can also consider the cardinality of the torsion subgroup of homology. It is our primary interest to study the following classes:

Definition 4.17. Let Γ be a residually finite group and let $n \in \mathbb{Z}$. The class T_n^Γ consists of all $\mathbb{Z}\Gamma$-chain complexes X satisfying for all residual chains Λ_* in Γ and all $j \le n$ that the torsion homology gradient

$$\widehat{t}_j(X, \Lambda_*) := \limsup_{i\to\infty} \frac{\log \operatorname{tors} H_j(X_{\Lambda_i})}{[\Gamma : \Lambda_i]}.$$

is equal to 0. Here we use the convention $\log \infty := \infty$.

Then the class T_n consists of all residually finite groups Γ satisfying $\widehat{t}_j(\Gamma, \Lambda_*) = 0$ for all residual chains Λ_* in Γ and all $j \leq n$ (Definition 1.2). Clearly, the class T_0 is the class of all residually finite groups. It is easy to see that finite groups, free abelian groups, free groups, and surface groups lie in T_∞. Residually finite infinite amenable groups of type FP_n lie in T_{n-1} [KKN17, Corollary 2] (Corollary 5.6).

For a group Γ of type FP_n, the Universal Coefficient Theorem shows that if Γ lies in $\mathsf{H}_n(\mathbb{Q}) \cap \mathsf{T}_n$, then Γ lies in $\mathsf{H}_n(\mathbb{F})$ for every field $\mathbb{F}$.

The sequence T_*^Γ does not seem to be a bootstrappable property of $\mathbb{Z}\Gamma$-chain complexes, due to the fact that torsion does not behave well with taking quotients, which causes complications for axiom (B-cone). In the next section, we introduce the sequence CR_*^Γ which is designed to be (degreewise) contained in T_{*-1}^Γ and simultaneously to be a bootstrappable property of $\mathbb{Z}\Gamma$-chain complexes.

4.4. **Algebraic cheap rebuilding property.** We work over the ring $R = \mathbb{Z}$. Recall that for a subgroup Λ of Γ and a $\mathbb{Z}\Gamma$-chain complex X, we write $X_\Lambda := \mathbb{Z} \otimes_{\mathbb{Z}\Lambda} \operatorname{res}_\Lambda^\Gamma X$ for the $\mathbb{Z}$-chain complex of Λ-coinvariants. The following definition is the main novelty of the present article. It relies on the notion of rebuildings of chain complexes (Section 2.2). We consider the is-class ? of residually finite groups.

Definition 4.18. Let Γ be a residually finite group and let $n \in \mathbb{Z}$. The class CR_n^Γ (resp. CWR_n^Γ, CD_n^Γ) consists of all based free $\mathbb{Z}\Gamma$-chain complexes X lying in $\mathsf{FG}_n^\Gamma(\mathbb{Z})$ (Definition 4.1) that satisfy the following: there exists $\kappa \in \mathbb{R}_{\geq 1}$ such that for all $T \in \mathbb{R}_{\geq 1}$ and all residual chains Λ_* in Γ, there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, the $\mathbb{Z}$-chain complex X_{Λ_i} admits an n-rebuilding (resp. weak n-rebuilding, n-domination) of quality (T, κ) (Definition 2.4).

In symbols,

$$\exists_{\kappa \geq 1} \ \forall_{T \geq 1} \ \forall_{\Lambda_* \text{ res. chain in } \Gamma} \ \exists_{i_0 \in \mathbb{N}} \ \forall_{i \geq i_0} \ \exists_{n\text{-rebuilding of } X_{\Lambda_i} \text{ of quality } (T,\kappa)}$$

If the group Γ lies in CR_n (resp. CWR_n, CD_n), we say that Γ satisfies the *algebraic cheap n-rebuilding property* (resp. *algebraic cheap weak n-rebuilding property*, *algebraic cheap n-domination property*).

Recall from Remark 2.5 that (weak) rebuildings yield upper bounds and vanishing for torsion in homology. The algebraic cheap (weak) rebuilding property is designed to yield *asymptotic* vanishing for torsion in homology along every residual chain (Lemma 4.22).

Clearly, we have inclusions of classes $\mathsf{CR}_n^\Gamma \subset \mathsf{CWR}_n^\Gamma \subset \mathsf{CD}_n^\Gamma$ for every $n \in \mathbb{Z}$.

Lemma 4.19. *We have the following equalities of classes:*

$$\mathsf{CR}_0 = \mathsf{CWR}_0 = \mathsf{CD}_0 = \text{class of all residually finite infinite groups.}$$

Proof. The inclusions $\mathsf{CR}_0 \subset \mathsf{CWR}_0 \subset \mathsf{CD}_0$ are clear.

Supposing that Γ lies in CD_0, we show that Γ is infinite. Let X be a free $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}$ lying in CD_0^Γ witnessed by the constant κ. For every $T \in \mathbb{R}_{\geq 1}$ and every residual chain Λ_* in Γ, there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, there exists a 0-domination $(X_{\Lambda_i}, X'_{\Lambda_i})$ of quality (T, κ). That is,

$$\operatorname{rk}_\mathbb{Z}\big((X'_{\Lambda_i})_0\big) \leq \kappa T^{-1} \operatorname{rk}_\mathbb{Z}\big((X_{\Lambda_i})_0\big) = \kappa T^{-1} [\Gamma : \Lambda_i] \operatorname{rk}_{\mathbb{Z}\Gamma}(X_0).$$

Since $H_0(X'_{\Lambda_i})$ contains $H_0(X_{\Lambda_i}) \cong \mathbb{Z}$ as a retract, we have $\operatorname{rk}_\mathbb{Z}((X'_{\Lambda_i})_0) \geq 1$. Hence

$$\kappa^{-1} T \operatorname{rk}_{\mathbb{Z}\Gamma}(X_0)^{-1} \leq [\Gamma : \Lambda_i].$$

Since $T \in \mathbb{R}_{\geq 1}$ was arbitrary, we must have $\lim_{i \to \infty} [\Gamma : \Lambda_i] = \infty$. Hence Γ is infinite.

Supposing that Γ is residually finite infinite, we show that Γ lies in CR_0. Fix a based free $\mathbb{Z}\Gamma$-resolution X of $\mathbb{Z}$ with $X_0 = \mathbb{Z}\Gamma$, $X_1 = \bigoplus_\Gamma \mathbb{Z}\Gamma$, and $\partial_1^X(e_\gamma) = \gamma - 1$.

Here $e_\gamma \in X_1$ denotes the $\mathbb{Z}\Gamma$-basis element corresponding to $\gamma \in \Gamma$. We will show for every finite index subgroup Λ of Γ, that X_Λ admits a 0-rebuilding of quality $(T, 1)$ for all $T \leq [\Gamma : \Lambda]$. Then X lies in CR_0^Γ witnessed by the constant $\kappa = 1$ because for every $T \in \mathbb{R}_{\geq 1}$ and every residual chain Λ_* in Γ, there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, $[\Gamma : \Lambda_i] \geq T$ since Γ is infinite.

Now, let Λ be a finite index subgroup of Γ. Choose a set S of right-coset representatives for $\Lambda\backslash\Gamma$ with $1_\Gamma \in S$. Consider the isomorphism of $\mathbb{Z}\Lambda$-modules

$$\begin{aligned} \operatorname{res}_\Lambda^\Gamma \mathbb{Z}\Gamma &\cong \bigoplus_S \mathbb{Z}\Lambda \\ s &\leftarrow\!\shortmid e_s \\ \gamma &\mapsto \gamma t(\gamma)^{-1} e_{t(\gamma)} \end{aligned}$$

where e_s is the $\mathbb{Z}\Lambda$-basis element corresponding to $s \in S$ and $t(\gamma) \in S$ is such that $\Lambda\gamma = \Lambda t(\gamma)$. Under this isomorphism, the based free $\mathbb{Z}\Lambda$-resolution $\operatorname{res}_\Lambda^\Gamma X$ of $\mathbb{Z}$ is given in degrees 1 and 0 by

$$\cdots \to \bigoplus_{\Gamma\times S} \mathbb{Z}\Lambda \xrightarrow{\operatorname{res}_\Lambda^\Gamma \partial_1^X} \bigoplus_S \mathbb{Z}\Lambda,$$

where

$$\operatorname{res}_\Lambda^\Gamma \partial_1^X(e_{(\gamma,s)}) = s\gamma t(s\gamma)^{-1} e_{t(s\gamma)} - e_s.$$

Let Y be a based free $\mathbb{Z}\Lambda$-resolution of $\mathbb{Z}$ with $Y_0 = \mathbb{Z}\Lambda$. Then there exist mutually $\mathbb{Z}\Lambda$-chain homotopy inverse $\mathbb{Z}\Lambda$-chain maps $\xi \colon \operatorname{res}_\Lambda^\Gamma X \to Y$ with $\xi_0(e_s) = 1$ and $\xi' \colon Y \to \operatorname{res}_\Lambda^\Gamma X$ with $\xi'_0(1) = e_{1_\Gamma}$. The $\mathbb{Z}\Lambda$-chain homotopy $\Xi \colon \mathrm{id}_{\operatorname{res}_\Lambda^\Gamma X} \simeq \xi' \circ \xi$ is given in degree 0 by $\Xi_0(e_s) = -e_{(s^{-1},s)}$. By construction, we have

$$\begin{aligned} \operatorname{rk}_{\mathbb{Z}\Lambda}(Y_0) &\leq [\Gamma : \Lambda]^{-1} \operatorname{rk}_{\mathbb{Z}\Lambda}(\operatorname{res}_\Lambda^\Gamma X_0); \\ \|\partial_0^Y\|, \|\xi_0\|, \|\xi'_0\|, \|\Xi_0\| &\leq 1. \end{aligned}$$

Hence by the following Remark 4.20, the $\mathbb{Z}\Lambda$-homotopy equivalence between $\operatorname{res}_\Lambda^\Gamma X$ and Y descends to a 0-rebuilding (X_Λ, Y_Λ) of quality $(T, 1)$ for all $T \leq [\Gamma : \Lambda]$. □

Remark 4.20 (Equivariant rebuilding)**.** Let Γ be a group and let (X, X', ξ, ξ', Ξ) be a $\mathbb{Z}\Gamma$-homotopy retract of based free $\mathbb{Z}\Gamma$-chain complexes lying in $\mathsf{FG}_n^\Gamma(\mathbb{Z})$. There exist $T, \kappa \in \mathbb{R}_{\geq 1}$ such that for all $j \leq n$, we have

$$\begin{aligned} \operatorname{rk}_{\mathbb{Z}\Gamma}(X'_j) &\leq \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}\Gamma}(X_j); \\ \|\partial_j^{X'}\|, \|\xi_j\|, \|\xi'_j\|, \|\Xi_j\| &\leq \exp(\kappa) T^\kappa. \end{aligned}$$

Let Λ be a finite index subgroup of Γ. Then the $\mathbb{Z}\Gamma$-homotopy retract (X, X') descends to an n-rebuilding (X_Λ, X'_Λ) of quality (T, κ).

Indeed, for all $j \leq n$, we have

$$\operatorname{rk}_{\mathbb{Z}}\big((X'_\Lambda)_j\big) = [\Gamma : \Lambda] \operatorname{rk}_{\mathbb{Z}\Gamma}(X'_j) \leq [\Gamma : \Lambda] \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}\Gamma}(X_j) = \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}}\big((X_\Lambda)_j\big).$$

Since the functor $(-)_\Lambda$ does not increase the norm of maps by Lemma 4.12 (iii), for every $f \in \{\partial^{X'}, \xi, \xi', \Xi\}$ and all $j \leq n$, we have

$$\|(f_\Lambda)_j\| \leq \|f_j\| \leq \exp(\kappa) T^\kappa.$$

Similar statements hold for weak rebuildings and dominations.

The main example of a group for which we can show directly from the definition that it lies in CR_∞ is the group of integers.

Example 4.21. The group $\mathbb{Z}$ lies in CR_∞.

Indeed, let $t \in \mathbb{Z}$ be a generator. Consider the projective $\mathbb{Z}[\langle t\rangle]$-resolution X of $\mathbb{Z}$

$$0 \to X_1 = \mathbb{Z}[\langle t\rangle] \xrightarrow{\partial_1} X_0 = \mathbb{Z}[\langle t\rangle] \to \mathbb{Z} \to 0,$$

where $\partial_1(t) = t-1$. Set $\kappa \coloneqq 2$. Let $T \in \mathbb{R}_{\geq 1}$ and let Λ_* be a residual chain in Γ. Take $i_0 \in \mathbb{N}$ such that $[\Gamma : \Lambda_{i_0}] \geq T$. For all $i \geq i_0$, the subgroup Λ_i of $\langle t\rangle$ is of the form $\langle t^{d_i}\rangle$, where $d_i \coloneqq [\Gamma : \Lambda_i]$. Then the $\mathbb{Z}$-chain complex X_{Λ_i} can be identified with $S^{[0,d_i]}$ from Example 2.7, which admits an n-rebuilding of quality $(T, 2)$.

We will show in Section 5 that residually finite infinite amenable groups of type FP_∞ lie in CWR_∞. We do not know if all these groups lie in the subclass CR_∞. However, it will follow from the Bootstrapping Theorem 3.6 that residually finite infinite *elementary* amenable groups of type FP_∞ lie in CR_∞.

The sequences CD^Γ_* and CWR^Γ_*, which are defined on the level of chain complexes, are (degreewise) contained in the sequences $\mathsf{H}^\Gamma_*(\mathbb{F})$ and T^Γ_{*-1} (Definition 4.13 and Definition 4.17), respectively, which are defined on the level of homology. Our arguments are algebraic versions of [ABFG24, Theorem 10.20].

Lemma 4.22. *Let Γ be a residually finite group, let $n \in \mathbb{Z}$, and let X be a $\mathbb{Z}\Gamma$-chain complex. The following hold:*

(i) *If $X \in \mathsf{CD}^\Gamma_n$, then $X \in \mathsf{H}^\Gamma_n(\mathbb{F})$ for every field $\mathbb{F}$;*
(ii) *If $X \in \mathsf{CWR}^\Gamma_n$, then $X \in \mathsf{T}^\Gamma_{n-1}$;*
(iii) *If $X \in \mathsf{CWR}^\Gamma_n$ and $X \in \mathsf{FG}^\Gamma_{n+1}(\mathbb{Z})$, then $X \in \mathsf{T}^\Gamma_n$;*
(iv) *If the group Γ lies in CWR_n and is of type FP_{n+1}, then $\Gamma \in \mathsf{T}_n$.*

Proof. (i) Suppose that X lies in CD^Γ_n witnessed by the constant κ. Let Λ_* be a residual chain in Γ and let $j \leq n$. For all $T \in \mathbb{R}_{\geq 1}$, there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, the $\mathbb{Z}$-chain complex X_{Λ_i} admits an n-domination $(X_{\Lambda_i}, X'_{\Lambda_i})$ of quality (T, κ). That is, for all $j \leq n$, we have

$$\operatorname{rk}_\mathbb{Z}\bigl((X'_{\Lambda_i})_j\bigr) \leq \kappa T^{-1} \operatorname{rk}_\mathbb{Z}\bigl((X_{\Lambda_i})_j\bigr) \leq \kappa T^{-1} [\Gamma : \Lambda_i] \operatorname{rk}_{\mathbb{Z}\Gamma}(X_j).$$

Since $H_j(\mathbb{F} \otimes_\mathbb{Z} X_{\Lambda_i})$ is a retract of $H_j(\mathbb{F} \otimes_\mathbb{Z} X'_{\Lambda_i})$, we have

$$\dim_\mathbb{F} H_j(\mathbb{F} \otimes_\mathbb{Z} X_{\Lambda_i}) \leq \dim_\mathbb{F} H_j(\mathbb{F} \otimes_\mathbb{Z} X'_{\Lambda_i}) \leq \operatorname{rk}_\mathbb{Z}\bigl((X'_{\Lambda_i})_j\bigr).$$

Together, for all $j \leq n$, we have

$$\widehat{b}_j(X, \Lambda_*; \mathbb{F}) \leq \limsup_{i\to\infty} \frac{\kappa T^{-1} [\Gamma : \Lambda_i] \operatorname{rk}_{\mathbb{Z}\Gamma}(X_j)}{[\Gamma : \Lambda_i]} = \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}\Gamma}(X_j).$$

Taking $T \to \infty$, this shows that $\widehat{b}_j(X, \Lambda_*; \mathbb{F}) = 0$. Hence X lies in $\mathsf{H}^\Gamma_n(\mathbb{F})$.

(ii) Suppose that X lies in CWR^Γ_n witnessed by the constant κ. Let Λ_* be a residual chain in Γ and let $j \leq n$. For all $T \in \mathbb{R}_{\geq 1}$, there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, the $\mathbb{Z}$-chain complex X_{Λ_i} admits a weak n-rebuilding $(X_{\Lambda_i}, X'_{\Lambda_i})$ of quality (T, κ). Using Gabber's estimate (1.1), for all $j \leq n-1$, we have

$$\begin{aligned}\log \operatorname{tors} H_j(X'_{\Lambda_i}) &\leq \operatorname{rk}_\mathbb{Z}\bigl((X'_{\Lambda_i})_j\bigr) \log_+ \|\partial^{X'_{\Lambda_i}}_{j+1}\| \\ &\leq \kappa T^{-1} \operatorname{rk}_\mathbb{Z}\bigl((X_{\Lambda_i})_j\bigr) \kappa (1 + \log T) \\ &= \kappa^2 T^{-1} [\Gamma : \Lambda_i] \operatorname{rk}_{\mathbb{Z}\Gamma}(X_j)(1 + \log T).\end{aligned}$$

Since $H_j(X_{\Lambda_i})$ is a retract of $H_j(X'_{\Lambda_i})$, we have

$$\log \operatorname{tors} H_j(X_{\Lambda_i}) \leq \log \operatorname{tors} H_j(X'_{\Lambda_i}).$$

Together, for all $j \leq n-1$, we have

$$\begin{aligned} \widehat{t}_j(X, \Lambda_*) &\leq \limsup_{i \to \infty} \frac{\kappa^2 T^{-1} [\Gamma : \Lambda_i] \operatorname{rk}_{\mathbb{Z}\Gamma}(X_j)(1 + \log T)}{[\Gamma : \Lambda_i]} \\ &= \kappa^2 T^{-1} \operatorname{rk}_{\mathbb{Z}\Gamma}(X_j)(1 + \log T). \end{aligned}$$

Taking $T \to \infty$, this shows that $\widehat{t}_j(X, \Lambda_*) = 0$. Hence X lies in $\mathsf{T}^{\Gamma}_{n-1}$.

(iii) Suppose that X lies in CWR^{Γ}_n witnessed by the constant κ and that X lies in $\mathsf{FG}^{\Gamma}_{n+1}(\mathbb{Z})$. Let Λ_* be a residual chain in Γ. In view of part (ii), it remains to show that $\widehat{t}_n(X, \Lambda_*) = 0$.

For all $T \in \mathbb{R}_{\geq 1}$, there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, the $\mathbb{Z}$-chain complex X_{Λ_i} admits a weak n-rebuilding $(X_{\Lambda_i}, X'_{\Lambda_i}, \xi, \xi', \Xi)$ of quality (T, κ). In the following, let $i \geq i_0$. We have $\|\partial^{X_{\Lambda_i}}_{n+1}\| \leq \|\partial^X_{n+1}\|$ by Lemma 4.12 (iii) and $\|\partial^X_{n+1}\|$ is finite because X_{n+1} is finitely generated over $\mathbb{Z}\Gamma$. Define the $\mathbb{Z}$-chain complex X''_{Λ_i} with chain modules

$$(X''_{\Lambda_i})_j := \begin{cases} 0 & \text{if } j \geq n+2; \\ (X_{\Lambda_i})_{n+1} & \text{if } j = n+1; \\ (X'_{\Lambda_i})_j & \text{if } j \leq n; \end{cases}$$

and differentials

$$\partial^{X''_{\Lambda_i}}_j := \begin{cases} 0 & \text{if } j \geq n+2; \\ \xi_n \circ \partial^{X_{\Lambda_i}}_{n+1} & \text{if } j = n+1; \\ \partial^{X'_{\Lambda_i}}_j & \text{if } j \leq n. \end{cases}$$

Consider the partial chain maps $\overline{\xi}\colon X_{\Lambda_i} \to X''_{\Lambda_i}$ and $\overline{\xi'}\colon X''_{\Lambda_i} \to X_{\Lambda_i}$ defined up to degree $n+1$ as follows:

$$\begin{aligned} \overline{\xi}_j &:= \begin{cases} \mathrm{id}_{(X_{\Lambda_i})_{n+1}} & \text{if } j = n+1; \\ \xi_j & \text{if } j \leq n; \end{cases} \\ \overline{\xi'}_j &:= \begin{cases} \mathrm{id}_{(X_{\Lambda_i})_{n+1}} - \Xi_n \circ \partial^{X_{\Lambda_i}}_{n+1} & \text{if } j = n+1; \\ \xi'_j & \text{if } j \leq n. \end{cases} \end{aligned}$$

We have constructed

$$\begin{array}{ccccccc} (X_{\Lambda_i})_{n+1} & \xrightarrow{\partial^{X_{\Lambda_i}}_{n+1}} & (X_{\Lambda_i})_n & \xrightarrow{\partial^{X_{\Lambda_i}}_{n}} & (X_{\Lambda_i})_{n-1} & \longrightarrow & \cdots \\ {\scriptstyle \mathrm{id} - \Xi_n \circ \partial^{X_{\Lambda_i}}_{n+1}} \uparrow \downarrow {\scriptstyle \mathrm{id}} & & {\scriptstyle \xi'_n} \uparrow \downarrow {\scriptstyle \xi_n} & & {\scriptstyle \xi'_{n-1}} \uparrow \downarrow {\scriptstyle \xi_{n-1}} & & \\ (X_{\Lambda_i})_{n+1} & \xrightarrow[\xi_n \circ \partial^{X_{\Lambda_i}}_{n+1}]{} & (X'_{\Lambda_i})_n & \xrightarrow[\partial^{X'_{\Lambda_i}}_{n}]{} & (X'_{\Lambda_i})_{n-1} & \longrightarrow & \cdots \end{array}$$

The $\mathbb{Z}$-chain homotopy Ξ provides a partial chain homotopy between $\mathrm{id}_{X_{\Lambda_i}}$ and $\overline{\xi'} \circ \overline{\xi}$ up to degree n. In particular, $H_n(X_{\Lambda_i})$ is a retract of $H_n(X''_{\Lambda_i})$ and hence

$$\log \operatorname{tors} H_n(X_{\Lambda_i}) \leq \log \operatorname{tors} H_n(X''_{\Lambda_i}).$$

By construction, we have

$$\operatorname{rk}_{\mathbb{Z}}\bigl((X''_{\Lambda_i})_n\bigr) = \operatorname{rk}_{\mathbb{Z}}\bigl((X'_{\Lambda_i})_n\bigr) \leq \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}}\bigl((X_{\Lambda_i})_n\bigr) \leq \kappa T^{-1} [\Gamma : \Lambda_i] \operatorname{rk}_{\mathbb{Z}\Gamma}(X_n)$$

and

$$\begin{aligned} \log_+ \|\partial^{X''_{\Lambda_i}}_{n+1}\| &= \log_+ \|\xi_n \circ \partial^{X_{\Lambda_i}}_{n+1}\| \leq \log_+ \|\xi_n\| + \log_+ \|\partial^{X_{\Lambda_i}}_{n+1}\| \\ &\leq \kappa(1 + \log T) + \log_+ \|\partial^X_{n+1}\|. \end{aligned}$$

Together, using Gabber's estimate (1.1) we obtain

$$\log \operatorname{tors} H_n(X''_{\Lambda_i}) \le \kappa T^{-1}[\Gamma : \Lambda_i] \operatorname{rk}_{\mathbb{Z}\Gamma}(X_n)\big(\kappa(1+\log T) + \log_+ \|\partial^X_{n+1}\|\big).$$

Finally,

$$\begin{aligned}\widehat{t}_n(X,\Lambda_*) &\le \limsup_{i\to\infty} \frac{\log \operatorname{tors} H_n(X''_{\Lambda_i})}{[\Gamma : \Lambda_i]} \\ &\le \kappa T^{-1} \operatorname{rk}_{\mathbb{Z}\Gamma}(X_n)\big(\kappa(1+\log T) + \log_+ \|\partial^X_{n+1}\|\big).\end{aligned}$$

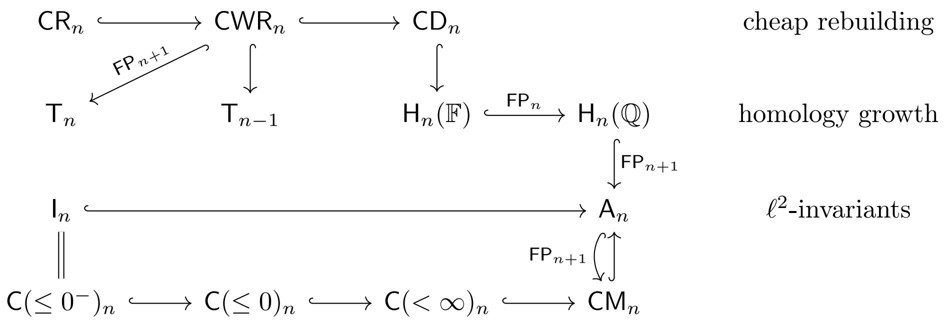
Since $\|\partial^X_{n+1}\| < \infty$, taking $T \to \infty$, this shows that $\widehat{t}_n(X,\Lambda_*) = 0$. Hence X lies in T^Γ_n.

(iv) Since Γ is of type FP_{n+1}, there exists a based free $\mathbb{Z}\Gamma$-resolution Y of $\mathbb{Z}$ lying in $\mathsf{FG}^\Gamma_{n+1}(\mathbb{Z})$. We will show that Y also lies in CWR^Γ_n. Then the claim follows from part (iii).

Since Γ lies in CWR_n, there exists a based free $\mathbb{Z}\Gamma$-resolution X of $\mathbb{Z}$ lying in CWR^Γ_n witnessed by the constant κ. We fix a $\mathbb{Z}\Gamma$-chain homotopy equivalence between Y and X given by the Fundamental Lemma of homological algebra. Since Y and X lie in $\mathsf{FG}^\Gamma_n(\mathbb{Z})$, there exists $\mu \in \mathbb{R}_{\ge 1}$ such that for every finite index subgroup Λ of Γ, the $\mathbb{Z}\Gamma$-chain homotopy equivalence between Y and X descends to a weak n-rebuilding (Y_Λ, X_Λ) of quality $(1,\mu)$ (Remark 4.20).

Let $T \in \mathbb{R}_{\ge 1}$ and let Λ_* be a residual chain in Γ. Since X lies in CWR^Γ_n, there exists $i_0 \in \mathbb{N}$ such that for all $i \ge i_0$, there exists a weak n-rebuilding $(X_{\Lambda_i}, X'_{\Lambda_i})$ of quality (T,κ). The composition of the weak n-rebuildings $(Y_{\Lambda_i}, X_{\Lambda_i})$ and $(X_{\Lambda_i}, X'_{\Lambda_i})$ is a weak n-rebuilding $(Y_{\Lambda_i}, X'_{\Lambda_i})$ of quality $(T, 2\kappa\mu)$ by Lemma 2.10 (iii). Hence Y lies in CWR^Γ_n witnessed by the constant $2\kappa\mu$. $\square$

We summarise the relations between the various classes above in a diagram. Let $n \in \mathbb{Z}$ and let $\mathbb{F}$ be a field. The following inclusions hold between classes of groups:



Here some inclusions hold when restricting to groups of type FP_{n+1} or FP_n, respectively, as indicated by the labels.

Finally, we show that the algebraic cheap rebuilding property is an equivariantly bootstrappable property. The main work has already been done in Proposition 2.9, where we showed that rebuildings are preserved under mapping cones.

Proposition 4.23. *Let ? be the is-class of residually finite groups. The following hold:*

(i) *The family $\mathsf{CR}^?_*$ is an equivariantly bootstrappable property of $\mathbb{Z}[?]$-chain complexes;*
(ii) *Let Γ be a residually finite group. The following hold:*
  - *The sequence CWR^Γ_* of classes of $\mathbb{Z}\Gamma$-chain complexes satisfies axioms* (B-deg) *and* (B-susp)*;*

- *Let* $f\colon X \to Y$ *be a* $\mathbb{Z}\Gamma$*-chain map. If* $X \in \mathsf{CR}^\Gamma_{n-1}$ *and* $Y \in \mathsf{CWR}^\Gamma_n$*, then* $\operatorname{Cone}(f) \in \mathsf{CWR}^\Gamma_n$*;*
- *The family* $\mathsf{CWR}^?_*$ *satisfies axiom* (B-ind)*;*

(iii) *The family* $\mathsf{CD}^?_*$ *is an equivariantly bootstrappable property of* $\mathbb{Z}[?]$*-chain complexes.*

Proof. (i) First, we show that for every residually finite group Γ, the sequence CR^Γ_* is a bootstrappable property of $\mathbb{Z}\Gamma$-chain complexes. Indeed, the axioms (B-deg) and (B-susp) clearly hold. For axiom (B-cone), let $f\colon X \to Y$ be a $\mathbb{Z}\Gamma$-chain map, where X lies in CR^Γ_{n-1} witnessed by the constant κ_X and Y lies in CR^Γ_n witnessed by the constant κ_Y. Let the constant $\kappa \in \mathbb{R}_{\geq 1}$ be as in Proposition 2.9, let $T \in \mathbb{R}_{\geq 1}$, and let Λ_* be a residual chain in Γ. There exist $i_0^X, i_0^Y \in \mathbb{N}$ such that for all $i \geq i_0 := \max\{i_0^X, i_0^Y\}$, the $\mathbb{Z}$-chain complex X_{Λ_i} admits an $(n-1)$-rebuilding of quality (T, κ_X) and Y_{Λ_i} admits an n-rebuilding of quality (T, κ_Y). Consider the $\mathbb{Z}$-chain map $f_{\Lambda_i}\colon X_{\Lambda_i} \to Y_{\Lambda_i}$. By Proposition 2.9, the $\mathbb{Z}$-chain complex $\operatorname{Cone}(f_{\Lambda_i})$ admits an n-rebuilding of quality (T, κ). Since $\operatorname{Cone}(f_{\Lambda_i}) \cong \operatorname{Cone}(f)_{\Lambda_i}$, this shows that the $\mathbb{Z}\Gamma$-chain complex $\operatorname{Cone}(f)$ lies in CR^Γ_n witnessed by the constant κ.

Second, we show that the family $\mathsf{CR}^?_*$ satisfies axiom (B-ind). Let Δ be a subgroup of Γ and let X be a $\mathbb{Z}\Delta$-chain complex lying in CR^Δ_n witnessed by the constant κ. Let $T \in \mathbb{R}_{\geq 1}$ and let Λ_* be a residual chain in Γ. There exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, the $\mathbb{Z}$-chain complex $X_{\Lambda_i \cap \Delta}$ admits an n-rebuilding of quality (T, κ). Lemma 4.12 (ii) provides an isomorphism of $\mathbb{Z}$-chain complexes

$$(\operatorname{ind}^\Gamma_\Delta X)_{\Lambda_i} \cong \bigoplus_{\frac{[\Gamma:\Lambda_i]}{[\Delta:\Lambda_i\cap\Delta]}} X_{\Lambda_i \cap \Delta}.$$

By Lemma 2.8, the $\mathbb{Z}$-chain complex $(\operatorname{ind}^\Gamma_\Delta X)_{\Lambda_i}$ admits an n-rebuilding of quality (T, κ). This shows that the $\mathbb{Z}\Gamma$-chain complex $\operatorname{ind}^\Gamma_\Delta X$ lies in CR^Γ_n witnessed by the constant κ.

Parts (ii) and (iii) follow from the same proof as part (i) by replacing "rebuilding" with "weak rebuilding" or "domination" in the appropriate places. □

We obtain the Bootstrapping Theorem 3.6 for CR_*, which is an algebraic version of [ABFG24, Theorem 10.9], and for CD_*. Since the group $\mathbb{Z}$ lies in CR_∞ (Example 4.21), it follows that residually finite infinite elementary amenable group of type FP_∞ lie in CR_∞ (Example 3.9).

Proposition 4.23 (ii) means that the family $\mathsf{CWR}^?_*$ is close to being an equivariantly bootstrappable property of $\mathbb{Z}[?]$-chain complexes. However, we can show axiom (B-cone) only for chain maps under stronger assumptions on the domain, which goes back to Proposition 2.9 (iii). Nevertheless, we obtain a modified Bootstrapping Theorem for CWR_* under stronger assumptions on stabilisers of cells of dimension ≥ 1. The proof is analogous to the proof of Theorem 3.6.

Theorem 4.24. *Let* Γ *be a residually finite group and let* $n \in \mathbb{N}$*. Let* Ω *be a* Γ*-CW-complex such that the following hold:*

(i) Ω *is* $(n-1)$*-acyclic;*
(ii) $\Gamma\backslash\Omega^{(n)}$ *is compact;*
(iii) *For every vertex* v *of* Ω*, the stabiliser* Γ_v *lies in* CWR_n*;*
(iv) *For every cell* σ *of* Ω *with* $\dim(\sigma) \in \{1, \dots, n\}$*, the stabiliser* Γ_σ *lies in* $\mathsf{CR}_{n-\dim(\sigma)}$*.*

Then $\Gamma \in \mathsf{CWR}_n$*.*

Remark 4.25 (Geometric cheap rebuilding property)**.** The algebraic cheap rebuilding property (Definition 4.18) is an algebraic version of Abért–Bergeron–Frączyk–Gaboriau's geometric cheap rebuilding property [ABFG24, Definition 10.6]. However, there is no obvious implication between the algebraic and geometric cheap rebuilding properties due to differences in the definitions of algebraic and geometric rebuildings (Remark 2.11).

Another significant difference is that in Definition 4.18 we consider only residual chains for simplicity, instead of more general Farber sequences and Farber neighbourhoods. The advantage is that axiom (B-ind) for $\mathsf{CR}^?_*$ is easy to show via Lemma 4.12 (ii). However, the family $\mathsf{CR}^?_*$ does not seem to satisfy axiom (B-res), due to the non-transitivity of normality. In particular, we are not able to show that the class CR_n of groups is closed under commensurability. On the other hand, the geometric cheap rebuilding property of Abért–Bergeron–Frączyk–Gaboriau is invariant under commensurability [ABFG24, Corollary 10.13 (1)]. It seems plausible that a strengthened version of $\mathsf{CR}^?_*$ involving Farber neighbourhoods analogous to [ABFG24, Definition 10.6] also satisfies axiom (B-res).

It is not known whether residually finite infinite amenable groups of type F_∞ satisfy the geometric cheap rebuilding property [ABFG24, Question 10.21]. However, it follows from [ABFG24, Section 10] using the argument in Example 3.9 that residually finite infinite *elementary* amenable groups of type FP_∞ satisfy the geometric cheap rebuilding property. Similarly, using the argument in Corollary 3.11 (iii), it follows that the geometric cheap rebuilding property passes from commensurated subgroups to overgroups.

In contrast to the geometric approach, the algebraic approach also allows us to formulate conditions such as $\mathsf{CD}^\Gamma_n(\mathbb{F})$ over general fields $\mathbb{F}$. These could lead to further examples of groups in $\mathsf{H}_n(\mathbb{F})$. Currently, we are not aware of concrete new examples of this type.

Remark 4.26. The algebraic cheap domination property (Definition 4.18) is related to Bridson–Kochloukova's geometric notion of "slowness" [BK17][Lüc16, Section 5.2]. Slow groups have vanishing homology growth. It is not known whether residually finite amenable groups are slow [BK17, Section 5].

5. Amenable groups

We show that amenable groups satisfy the algebraic cheap *weak* rebuilding property (Definition 4.18). The proof is inspired by the work of Kar–Kropholler–Nikolov [KKN17], using Følner sequences that are compatible with residual chains. We work over the ring $R = \mathbb{Z}$.

Lemma 5.1. *Let* $0 \to A \xrightarrow{f} X \xrightarrow{g} Y \to 0$ *be a short exact sequence of free* $\mathbb{Z}$*-chain complexes. If the inclusion* f *is nullhomotopic, then there exists a* $\mathbb{Z}$*-chain map* $h\colon Y \to X$ *such that* $\mathrm{id}_X \simeq h \circ g$.

Proof. The obvious $\mathbb{Z}$-chain map $q\colon \mathrm{Cone}(f) \to Y$ makes the following diagram commute

$$\begin{array}{ccc} X & \hookrightarrow & \mathrm{Cone}(f) \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle q} \\ & & Y \end{array}$$

The map q is a weak equivalence by the long exact sequences in homology and the Five Lemma. Since $\mathrm{Cone}(f)$ and Y are free $\mathbb{Z}$-chain complexes, the weak equivalence q is a chain homotopy equivalence. Let $r\colon Y \to \mathrm{Cone}(f)$ be a chain homotopy inverse of q. Then the composition $r \circ g$ is chain homotopic to the inclusion $X \hookrightarrow \operatorname{Cone}(f)$. Set $h\colon Y \to X$ to be the composition

$$h\colon Y \xrightarrow{r} \operatorname{Cone}(f) \xrightarrow{(\mathrm{id}_A,\mathrm{id}_X;H)} \operatorname{Cone}(0) \cong \Sigma A \oplus X \xrightarrow{p_2} X,$$

where H is a chain homotopy between the nullhomotopic inclusion f and the zero map 0, and p_2 is the projection to the second summand. By construction, the composition $h \circ g$ is homotopic to id_X. □

A *based short exact sequence* of based free $\mathbb{Z}$-modules is a short exact sequence $0 \to K \xrightarrow{f} M \xrightarrow{g} L \to 0$ of based free $\mathbb{Z}$-modules with bases I_K, I_M, and I_L, respectively, such that $f(I_K) \subset I_M$ and $g(I_M \setminus f(I_K)) = I_L$. We say that $f(K)$ is a *based submodule* of M. For based free $\mathbb{Z}$-chain complexes, we obtain corresponding degreewise notions of based subcomplexes and based short exact sequences.

Let X be a based free $\mathbb{Z}$-chain complex that is concentrated in degrees ≥ 0. Consider the *standard augmentation map* $\varepsilon\colon X_0 \to \mathbb{Z}$ that maps each $\mathbb{Z}$-basis element of X_0 to $1 \in \mathbb{Z}$. We say that X is *augmented over* $\mathbb{Z}$ if $\varepsilon \circ \partial_1^X = 0$. If X is augmented over $\mathbb{Z}$, we define the *augmented chain complex* X^ε *associated to* X as the based free $\mathbb{Z}$-chain complex with chain modules

$$X_j^\varepsilon := \begin{cases} X_j & \text{if } j \geq 0; \\ \mathbb{Z} & \text{if } j = -1; \end{cases}$$

and differentials

$$\partial_j^{X^\varepsilon} := \begin{cases} \partial_j^X & \text{if } j \geq 1; \\ \varepsilon & \text{if } j = 0. \end{cases}$$

If X is augmented over $\mathbb{Z}$, then so is every based subcomplex of X. We say that the inclusion $f\colon A \to X$ of a based subcomplex is *augmentedly nullhomotopic* if the $\mathbb{Z}$-chain map $f^\varepsilon\colon A^\varepsilon \to X^\varepsilon$ given by

$$f_j^\varepsilon := \begin{cases} f_j & \text{if } j \geq 0; \\ \mathrm{id}_\mathbb{Z} & \text{if } j = -1; \end{cases}$$

is nullhomotopic. In this case we also say that A is *augmentedly contractible in* X.

For a $\mathbb{Z}$-chain complex Y that is concentrated in degrees ≥ 0, we denote by Y^+ the $\mathbb{Z}$-chain complex with chain modules

$$Y_j^+ := \begin{cases} Y_j & \text{if } j \geq 1; \\ Y_0 \oplus \mathbb{Z} & \text{if } j = 0; \end{cases}$$

and differentials

$$\partial_j^{Y^+} := \begin{cases} \partial_j^Y & \text{if } j \geq 2; \\ (\partial_1^Y, 0) & \text{if } j = 1. \end{cases}$$

Lemma 5.2. *Let $0 \to A \xrightarrow{f} X \xrightarrow{g} Y \to 0$ be a based short exact sequence of based free $\mathbb{Z}$-chain complexes that are concentrated in degrees ≥ 0. If X is augmented over $\mathbb{Z}$ and the inclusion f is augmentedly nullhomotopic, then there exist $\mathbb{Z}$-chain maps $g^+\colon X \to Y^+$ and $h^+\colon Y^+ \to X$ such that $\mathrm{id}_X \simeq h^+ \circ g^+$.*

Moreover, let $n \in \mathbb{N}$ and suppose that for all $j \leq n$ the $\mathbb{Z}$-module X_j is finitely generated. Denote

$$T' := \min\Big\{ \frac{\mathrm{rk}_\mathbb{Z}(X_j)}{\mathrm{rk}_\mathbb{Z}(Y_j^+)} \;\Big|\; 0 \leq j \leq n \Big\}, \quad \kappa := \max\Big\{1, \max\big\{\log_+ \|\partial_j^X\| \;\big|\; 0 \leq j \leq n\big\}\Big\}.$$

Then, for all $T \in \mathbb{R}_{\geq 1}$ with $T \leq T'$, there exists a weak n-rebuilding (X, Y^+) of quality (T, κ).

Proof. We have a short exact sequence of free $\mathbb{Z}$-chain complexes

$$0 \to A^\varepsilon \xrightarrow{f^\varepsilon} X^\varepsilon \xrightarrow{g} Y \to 0,$$

where, by abuse of notation, $g\colon X^\varepsilon \to Y$ is the $\mathbb{Z}$-chain map given by $g\colon X \to Y$ in degrees ≥ 0 and by zero in degree -1. Lemma 5.1 yields a $\mathbb{Z}$-chain map $h\colon Y \to X^\varepsilon$ such that there exists a chain homotopy $H\colon \mathrm{id}_{X^\varepsilon} \simeq h \circ g$. We define the chain maps $g^+\colon X \to Y^+$ and $h^+\colon Y^+ \to X$ by

$$g_j^+ := \begin{cases} g_j & \text{if } j \geq 1; \\ (g_0, \varepsilon) & \text{if } j = 0; \end{cases}$$

$$h_j^+ := \begin{cases} h_j & \text{if } j \geq 1; \\ h_0 \oplus H_{-1} & \text{if } j = 0. \end{cases}$$

We have constructed

$$\begin{array}{ccccc}
\cdots & & \cdots & & \cdots \\
\downarrow & & \downarrow & & \downarrow \\
X_2 & \xrightarrow{g_2} & Y_2 & \xrightarrow{h_2} & X_2 \\
\partial_2^X \downarrow & & \downarrow \partial_2^Y & & \downarrow \partial_2^X \\
X_1 & \xrightarrow{g_1} & Y_1 & \xrightarrow{h_1} & X_1 \\
\partial_1^X \downarrow & & \downarrow (\partial_1^Y, 0) & & \downarrow \partial_1^X \\
X_0 & \xrightarrow{(g_0,\varepsilon)} & Y_0 \oplus \mathbb{Z} & \xrightarrow{h_0 \oplus H_{-1}} & X_0
\end{array}$$

Then one easily checks that $H_{*\geq 0}$ provides a chain homotopy $\mathrm{id}_X \simeq h^+ \circ g^+$.

Now, suppose that X_j is finitely generated for all $j \leq n$. For all $j \leq n$, we have

$$\begin{aligned}
\mathrm{rk}_\mathbb{Z}(Y_j^+) &\leq T'^{-1} \,\mathrm{rk}_\mathbb{Z}(X_j) \leq T^{-1}\, \mathrm{rk}_\mathbb{Z}(X_j); \\
\|\partial_j^{Y^+}\| &\leq \|\partial_j^X\| \leq \exp(\kappa); \\
\|g_j^+\| &\leq 2.
\end{aligned}$$

Hence (X, Y^+) is a weak n-rebuilding of quality (T, κ). □

In the following, when we write $\lim_{i\to\infty} \frac{a_i}{b_i}$ for $a_i, b_i \in \mathbb{Z}$, we assume implicitly that $b_i \neq 0$ for all large enough $i \in \mathbb{N}$.

Proposition 5.3. *Let Γ be a residually finite infinite group and let $n \in \mathbb{N}$. Let X be a based free $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}$ (with the standard augmentation) such that the $\mathbb{Z}\Gamma$-module X_j is finitely generated for all $j \leq n$. Suppose that for all residual chains Λ_* in Γ and all large enough $i \in \mathbb{N}$, the based free $\mathbb{Z}$-chain complex $X^i := X_{\Lambda_i} = \mathbb{Z} \otimes_{\mathbb{Z}[\Lambda_i]} \mathrm{res}^\Gamma_{\Lambda_i} X$ admits a based subcomplex A^i that is augmentedly contractible in X^i such that for all $j \in \{0, \dots, n\}$, we have*

$$\lim_{i\to\infty} \frac{\mathrm{rk}_\mathbb{Z}(X_j^i) - \mathrm{rk}_\mathbb{Z}(A_j^i)}{\mathrm{rk}_\mathbb{Z}(X_j^i)} = 0. \tag{5.1}$$

Then $\Gamma \in \mathsf{CWR}_n$.

Proof. Set $\kappa := \max\{1, \max\{\log_+ \|\partial_j^X\| \mid j \leq n\}\}$. Let $T \in \mathbb{R}_{\geq 1}$ and let Λ_* be a residual chain in Γ. For i large enough, we denote by $Y^i := X^i/A^i$ the quotient chain complex. For all $j \in \{0, \dots, n\}$, we have

$$\lim_{i\to\infty} \frac{\mathrm{rk}_\mathbb{Z}(Y_j^i)}{\mathrm{rk}_\mathbb{Z}(X_j^i)} = 0$$

by assumption. Moreover, in degree 0 we have

$$\lim_{i\to\infty} \frac{\operatorname{rk}_{\mathbb{Z}}(Y_0^i)+1}{\operatorname{rk}_{\mathbb{Z}}(X_0^i)} = 0$$

using that $\lim_{i\to\infty} \operatorname{rk}_{\mathbb{Z}}(X_0^i) = \infty$ because Γ is infinite. Together, we have

$$\lim_{i\to\infty} \frac{\operatorname{rk}_{\mathbb{Z}}(X_j^i)}{\operatorname{rk}_{\mathbb{Z}}\big((Y^i)_j^+\big)} = \infty.$$

Hence there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$

$$\min\left\{ \frac{\operatorname{rk}_{\mathbb{Z}}(X_j^i)}{\operatorname{rk}_{\mathbb{Z}}\big((Y^i)_j^+\big)} \;\middle|\; 0 \leq j \leq n \right\} \geq T.$$

By Lemma 4.12 (iii), we have

$$\|\partial_j^{X^i}\| \leq \|\partial_j^X\| \leq \exp(\kappa).$$

The based free $\mathbb{Z}\Gamma$-resolution X of $\mathbb{Z}$ is in particular augmented over $\mathbb{Z}$ and thus, so is the based free $\mathbb{Z}$-chain complex X^i. Hence Lemma 5.2 applies and yields a weak n-rebuilding $(X^i, (Y^i)^+)$ of quality (T, κ). □

Condition (5.1) means that asymptotically the subcomplex A^i is "large" in X^i. We will apply Proposition 5.3 in the context of amenable groups to appropriate subcomplexes coming from Følner sequences.

Let Γ be a finitely generated group and let S be a finite generating set of Γ. For a subset F of Γ, we write $\partial_S F := \{\gamma \in F \mid \gamma \cdot s \in \Gamma \setminus F \text{ for some } s \in S\}$. We also write $\operatorname{int}_S(F) := F \setminus \partial_S F$. Recall that a *Følner sequence* in Γ with respect to S is a sequence $F_* = (F_i)_{i\in\mathbb{N}}$ of non-empty finite subsets of Γ satisfying

$$\lim_{i\to\infty} \frac{\#\partial_S F_i}{\#F_i} = 0.$$

A finitely generated group is *amenable* if and only if it admits a Følner sequence. Residually finite amenable groups admit Følner sequences satisfying additional properties:

Theorem 5.4 (Weiss [Wei01][KKN17, Theorem 7])**.** *Let Γ be a finitely generated amenable group that is residually finite and infinite. Let S be a finite generating set of Γ and let Λ_* be a residual chain in Γ. Then there exists a Følner sequence F_* in Γ with respect to S such that for all $i \in \mathbb{N}$, the set F_i is a set of right-coset representatives for $\Lambda_i\backslash\Gamma$.*

We say that a Følner sequence F_* as in Theorem 5.4 is *compatible* with the residual chain Λ_*.

Theorem 5.5. *Let $n \in \mathbb{N}$ and let Γ be a residually finite infinite amenable group of type FP_n. Then $\Gamma \in \mathsf{CWR}_n$.*

Proof. Since Γ is of type FP_n, there exists a based free $\mathbb{Z}\Gamma$-resolution X of $\mathbb{Z}$ such that the free $\mathbb{Z}\Gamma$-module X_j is finitely generated for all $j \leq n$. The differentials $\partial_1, \dots, \partial_n$ are given by (right-)multiplication with matrices $M_1, \dots, M_n$, respectively, whose entries are in $\mathbb{Z}\Gamma$. Let S be a finite generating set of Γ containing all group elements appearing in the entries of the matrices $M_1, \dots, M_n$. For $j \geq 0$, denote by S^j the finite set of all words with letters in S of length $\leq j$. Let Λ_* be a residual chain in Γ. By Theorem 5.4, there exists a Følner sequence F_* in Γ with respect to the finite generating set S^{n+1} that is compatible with the residual chain Λ_*. For all $i \in \mathbb{N}$, we consider the based free $\mathbb{Z}$-chain complex $X^i := X_{\Lambda_i} = \mathbb{Z} \otimes_{\mathbb{Z}[\Lambda_i]} \operatorname{res}^{\Gamma}_{\Lambda_i} X$. For i large enough, we will construct a based subcomplex A^i of X^i that is augmentedly contractible in X^i such that for all $j \in \{0, \ldots, n\}$, we have

$$\lim_{i \to \infty} \frac{\mathrm{rk}_{\mathbb{Z}}(X^i_j) - \mathrm{rk}_{\mathbb{Z}}(A^i_j)}{\mathrm{rk}_{\mathbb{Z}}(X^i_j)} = 0.$$

Then Proposition 5.3 yields that Γ lies in CWR_n.

Now, since the $\mathbb{Z}\Gamma$-resolution X is based free, for all j we have an isomorphism of $\mathbb{Z}\Gamma$-modules $X_j \cong \bigoplus_{I_j} \mathbb{Z}\Gamma$, where I_j is the set of basis elements. The differential $\partial_j \colon X_j \to X_{j-1}$ is given by (right-)multiplication with the $(I_j \times I_{j-1})$-matrix $M_j = (m_j^{kl})_{k \in I_j, l \in I_{j-1}}$, where $m_j^{kl} = \sum_{s \in S} \lambda_j^{kl}(s)s \in \mathbb{Z}\Gamma$ with $\lambda_j^{kl}(s) \in \mathbb{Z}$. Consider the isomorphism of $\mathbb{Z}[\Lambda_i]$-modules

$$\begin{aligned} \mathrm{res}^{\Gamma}_{\Lambda_i} \mathbb{Z}\Gamma &\cong \bigoplus_{F_i} \mathbb{Z}[\Lambda_i] \\ f &\hookleftarrow e_f \\ \gamma &\mapsto \gamma t(\gamma)^{-1} e_{t(\gamma)} \end{aligned}$$

where e_f is the $\mathbb{Z}[\Lambda_i]$-basis element corresponding to $f \in F_i$, and $t(\gamma) \in S$ is such that $\Lambda_i \gamma = \Lambda_i t(\gamma)$. We can identify the $\mathbb{Z}$-chain modules of X^i as

$$X^i_j = \mathbb{Z} \otimes_{\mathbb{Z}[\Lambda_i]} \mathrm{res}^{\Gamma}_{\Lambda_i} X_j \cong \mathbb{Z} \otimes_{\mathbb{Z}[\Lambda_i]} \bigoplus_{I_j} \bigoplus_{F_i} \mathbb{Z}[\Lambda_i] \cong \bigoplus_{I_j \times F_i} \mathbb{Z}.$$

Under this identification, the differential $\partial^i_j \colon X^i_j \to X^i_{j-1}$ maps the $\mathbb{Z}$-basis element $e_{(k,f)} \in X^i_j$ for $(k, f) \in I_j \times F_i$ to

$$\sum_{l \in I_{j-1}} \sum_{s \in S} \lambda_j^{kl}(s) e_{(l, t(fs))} \in X^i_{j-1}.$$

Define the based subcomplex A^i of X^i by

$$A^i_j := \begin{cases} \bigoplus_{I_j \times \mathrm{int}_{S^{j+1}}(F_i)} \mathbb{Z} & \text{if } j \in \{0, \ldots, n\}; \\ 0 & \text{if } j \geq n+1. \end{cases}$$

The chain complex A^i is well-defined, as the differential $\partial^i_j \colon X^i_j \to X^i_{j-1}$ restricts to a differential $A^i_j \to A^i_{j-1}$. For $j \leq n$, the index set I_j is finite by assumption. Hence, for all $j \leq n$, we have

$$\lim_{i \to \infty} \frac{\mathrm{rk}_{\mathbb{Z}}(X^i_j) - \mathrm{rk}_{\mathbb{Z}}(A^i_j)}{\mathrm{rk}_{\mathbb{Z}}(X^i_j)} = \lim_{i \to \infty} \frac{\#I_j \cdot \#\partial_{S^{j+1}}(F_i)}{\#I_j \cdot \#F_i} \leq \lim_{i \to \infty} \frac{\#\partial_{S^{n+1}}(F_i)}{\#F_i} = 0.$$

It remains to show that the inclusion $A^i \to X^i$ is augmentedly nullhomotopic. The inclusion $A^i \to X^i$ factors through the projection $X \to X^i$ via the $\mathbb{Z}$-chain map $A^i \to X$ that maps the $\mathbb{Z}$-basis element $e_{(k,f)} \in A^i_j \cong \bigoplus_{I_j \times \mathrm{int}_{S^{j+1}(F_i)}} \mathbb{Z}$ to $f \cdot e_k \in X_j \cong \bigoplus_{I_j} \mathbb{Z}\Gamma$. In particular, we obtain a factorisation of the augmented inclusion $(A^i)^\varepsilon \to (X^i)^\varepsilon$ through X^ε. Since X is a free $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}$, the augmented $\mathbb{Z}$-chain complex X^ε is contractible. Hence the augmented inclusion $(A^i)^\varepsilon \to (X^i)^\varepsilon$ is nullhomotopic. □

By Lemma 4.22, we recover the following known result [CG86, LLS11, KKN17].

Corollary 5.6. *Let $n \in \mathbb{N}$ and let Γ be a residually finite infinite amenable group of type FP_n. Then Γ lies in T_{n-1} and $\mathsf{H}_n(\mathbb{F})$ for every field $\mathbb{F}$.*

As a consequence of the modified Bootstrapping Theorem 4.24 for CWR_* we obtain:

Corollary 5.7. *Let Γ be a residually finite fundamental group of a finite graph of groups and let $n \in \mathbb{Z}$. If all vertex groups are infinite amenable of type FP_n and all edge groups are infinite elementary amenable of type FP_∞, then $\Gamma \in \mathsf{T}_{n-1}$.*

Remark 5.8 (ℓ^2-Torsion)**.** A group Γ is *of type* FL if the trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$ admits a finite free resolution. Li–Thom [LT14, Theorem 1.3] have shown that non-trivial amenable groups of type FL have vanishing ℓ^2-torsion. Hence the fundamental group of a finite graph of non-trivial amenable groups that are of type FL has vanishing ℓ^2-torsion [Lüc02, Theorem 3.93].

Let Γ be a residually finite fundamental group of a finite graph of groups with non-trivial amenable vertex groups of type FL and non-trivial elementary amenable edge groups of type FL. Then the ℓ^2-torsion $\rho^{(2)}(\Gamma)$ vanishes by the above, and Γ lies in T_∞ by Corollary 5.7. In particular, we have

$$\rho^{(2)}(\Gamma) = 0 = \sum_{j \geq 0} (-1)^j \cdot \widehat{t}_j(\Gamma, \Lambda_*)$$

for all residual chains Λ_* in Γ. This confirms a conjecture of Lück [Lüc13, Conjecture 1.11 (3)] for the group Γ.

Acknowledgements. We thank Mikołaj Frączyk for answering our questions about the article [ABFG24] and Uli Bunke for suggesting to consider Novikov–Shubin invariants.

K.L., C.L., and M.U. were supported by the CRC 1085 "Higher Invariants" (Universität Regensburg, funded by the DFG).

M.M. was supported by the ERC "Definable Algebraic Topology" DAT – Grant Agreement number 101077154.

M.M. was funded by the European Union – NextGenerationEU under the National Recovery and Resilience Plan (PNRR) – Mission 4 Education and research – Component 2 From research to business – Investment 1.1 Notice Prin 2022 – DD N. 104 del 2/2/2022, from title "Geometry and topology of manifolds", proposal code 2022NMPLT8 – CUP J53D23003820001.

R.S. was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – project number 338540207.

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