

ON THE SIMPLICIAL VOLUME AND THE EULER CHARACTERISTIC OF (ASPHERICAL) MANIFOLDS

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ABSTRACT. A well-known question by Gromov asks whether the vanishing of the simplicial volume of oriented closed aspherical manifolds implies the vanishing of the Euler characteristic.

We study various versions of Gromov's question and collect strategies towards affirmative answers and strategies towards negative answers to this problem. Moreover, we put Gromov's question into context with other open problems in low- and high-dimensional topology.

A special emphasis is put on a comparative analysis of the additivity properties of the simplicial volume and the Euler characteristic for manifolds with boundary. We explain that the simplicial volume defines a symmetric monoidal functor (TQFT) on the *amenable* cobordism category, but not on the whole cobordism category. In addition, using known computations of simplicial volumes, we conclude that the fundamental group of the 4-dimensional amenable cobordism category is not finitely generated.

We also consider new variations of Gromov's question. Specifically, we show that counterexamples exist among aspherical spaces that are only homology equivalent to oriented closed connected manifolds.

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1. INTRODUCTION

The simplicial volume $\|M\|$ is a homotopy invariant of oriented compact manifolds M , defined as the ℓ^1 -semi-norm of the singular \mathbb{R} -fundamental class. The simplicial volume is proportional to the Riemannian volume for

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hyperbolic manifolds and zero in the presence of amenability. By the Gauß–Bonnet theorem and Følner covering towers, a similar behaviour is also exhibited by the Euler characteristic of aspherical manifolds. But understanding the connection between the vanishing behaviour of the simplicial volume and the Euler characteristic of closed aspherical manifolds remains a mystery.

In particular, the following problem by Gromov is wide open:

Question 1.1 ([53, p. 232]). Let M be an oriented closed aspherical manifold. Does the following implication hold?

$$(SV_\chi) \quad \|M\| = 0 \implies \chi(M) = 0.$$

The main challenge in answering Question 1.1 is to find a common ground for the various conditions and invariants involved: asphericity, being a closed manifold, the vanishing of the simplicial volume, and the vanishing of the Euler characteristic.

In this article, we explore different approaches to Question 1.1, in search of both positive and negative examples, as well as study its connections with other open problems in low- and high-dimensional topology.

On the one hand, a key direction pursued in this paper is the comparative study of the additivity properties of the simplicial volume and of the Euler characteristic, which naturally leads us to look at the simplicial volume (and stable integral simplicial volume) of compact manifolds with boundary (see also Section 1.1 below).

On the other hand, a different direction considered in this paper is based on the observation that if the answer to Question 1.1 is affirmative, then Property (SV_χ) will hold for a general class of oriented closed manifolds which are only homology equivalent to an aspherical one (see Remark 5.1). Based on this, we consider generalized versions of Question 1.1 for aspherical spaces which are only homology equivalent to an oriented closed manifold and for oriented closed manifolds which are only homology equivalent to an aspherical space (see also Section 1.2 below).

A quick summary of various other strategies and examples is provided in Section 1.3.

1.1. Additivity. As the category of closed aspherical manifolds is difficult to handle structurally, we extend the setup to compact aspherical manifolds with π_1 -injective boundary (Section 2.3) and consider an (equivalent) version of Gromov’s question in this context.

Question (see Question 2.13). Let $(M, \partial M)$ be an oriented compact aspherical manifold with non-empty π_1 -injective aspherical boundary. Does the following implication hold?

$$(SV_\chi, \partial) \quad \|M, \partial M\| = 0 \implies \chi(M, \partial M) = 0.$$

We recall some examples and sufficient conditions for the vanishing of the simplicial volume in Section 3. Moreover, we show an extension of Gromov’s

vanishing theorem (Theorem 3.5) to the vanishing of the relative simplicial volume $\|M, \partial M\|$ for oriented compact connected manifolds $(M, \partial M)$ which admit open covers with certain amenability properties (Theorem 3.13). Also, in Section 6, we discuss the properties of the relative stable integral simplicial volume and the analogues of the above questions for this invariant.

The extension to manifolds with boundary generally allows us to compare additivity and filling properties of the simplicial volume and of the Euler characteristic more systematically. For example, we show that Question 1.1 is related to Edmonds' problem (Conjecture 3.30) as follows:

Proposition (see Proposition 3.31). *Suppose that the following hold:*

- (a) *Every oriented closed aspherical 3-manifold with amenable fundamental group is the π_1 -injective boundary of an oriented compact aspherical 4-manifold W with $\|W, \partial W\| = 0$.*
- (b) *All oriented closed aspherical 4-manifolds satisfy Property (SV_χ) .*

Then, there exists an oriented closed aspherical 4-manifold M with $\chi(M) = 1$.

The notion of a topological quantum field theory (TQFT), as a symmetric monoidal functor on the cobordism category, provides an efficient way of encoding additivity properties. The connections between the simplicial volume and (invertible) TQFTs are discussed in Section 4. Specifically, based on known additivity properties, we explain in Section 4.2 that the simplicial volume defines an invertible TQFT on a suitable *amenable cobordism category* with values in \mathbb{R} . In addition, we show that this functor cannot be extended to a functor on the whole cobordism category (Proposition 4.7). This contrasts the additivity behaviour of the relative Euler characteristic, which is unconditional and thus defines a TQFT on the whole cobordism category (Remark 4.9). We also obtain the following result about the fundamental group of the amenable cobordism category of 4-manifolds and related cobordism categories (see Section 4.2 for the precise definition of Cob_d^G):

Theorem (see Theorem 4.5). *Let G be a class of amenable groups that is closed under isomorphisms and let M be an object of Cob_4^G . Then the group $\pi_1(B\text{Cob}_4^G, [M])$ is not finitely generated.*

1.2. Aspherical spaces homology equivalent to closed manifolds.

Using the Kan–Thurston theorem, we show in Section 5 that Property (SV_χ) fails in general if closed aspherical manifolds are replaced by aspherical spaces homology equivalent to closed manifolds or closed manifolds homology equivalent to an aspherical space (see Section 5.2 for the definition of an acyclic map):

Theorem (see Theorem 5.7). *Let $n \in \mathbb{N}_{\geq 2}$ be even.*

- (1) *There exist aspherical spaces X that admit an acyclic map $X \rightarrow M$ to an oriented closed connected n -manifold M and satisfy $\|X\| = 0$*

and $\chi(X) \neq 0$. In particular, these aspherical spaces do not satisfy Property (SV χ).

- (2) There exist oriented closed connected n -manifolds M that admit an acyclic map $X \rightarrow M$ from an aspherical space X and satisfy $\|M\| = 0$ and $\chi(M) \neq 0$. In particular, these manifolds do not satisfy Property (SV χ).

1.3. Strategies and known examples. As the following examples show, the hypotheses in Question 1.1 cannot be reasonably weakened or modified in any straightforward way:

- In general, non-aspherical oriented closed connected manifolds do not satisfy Property (SV χ): For example, $\|S^2\| = 0$, but $\chi(S^2) = 2$.
- The converse implication of Property (SV χ) does not hold in general for aspherical manifolds: For example, oriented closed connected hyperbolic 3-manifolds have vanishing Euler characteristic, but their simplicial volume is non-zero.
- In general, Property (SV χ) does not hold for oriented compact connected manifolds with non-empty boundary without imposing additional conditions on the inclusion of the boundary (Remark 2.16).
- In general, Property (SV χ) does not hold for aspherical spaces that are only homology equivalent to oriented closed connected manifolds (Theorem 5.7).

We also refer to Section 3 for a survey of examples of oriented closed manifolds with vanishing or non-vanishing simplicial volume.

Various strategies have been developed to handle Question 1.1. In particular, this also led to a wide range of positive examples:

Direct computations of both sides. One of Gromov's original motivations to formulate Question 1.1 was the observation that the simplicial volume and the Euler characteristic share some common vanishing properties. Examples of this phenomenon include manifolds that admit non-trivial self-coverings, closed aspherical manifolds with amenable fundamental group, and closed aspherical manifolds that admit small amenable open covers (Section 3).

Boundedness properties of the Euler class. The simplicial volume of an oriented closed connected n -manifold M is closely related to the comparison map from bounded cohomology to singular cohomology in degree n (see Section 2.4 and Proposition 2.18). On the other hand, the Euler characteristic is related to the Euler class by duality. As we explain in Proposition 2.22, an immediate consequence is that Property (SV χ) can be reformulated in terms of the boundedness of the Euler class. The problem of the boundedness of the Euler class is well studied and understood in several cases [84, 103, 63, 17].

L^2 -Betti numbers. Gromov suggested to use the fact that the Euler characteristic can be computed as the alternating sum of the L^2 -Betti numbers and asked the following version of Question 1.1:

Question 1.2 ([53, p. 232]). Let M be an oriented closed aspherical manifold. Does the following implication hold?

$$(SVL^2) \quad \|M\| = 0 \implies (\forall_{k \in \mathbb{N}} b_k^{(2)}(M) = 0).$$

Assuming the Singer conjecture on the vanishing of the L^2 -Betti numbers of closed aspherical manifolds outside the middle dimension [35], Question 1.1 and Question 1.2 are equivalent. More concretely, Gromov [54, p. 306] proposed a definition of integral foliated simplicial volume, involving dynamical systems, and then

- to establish an upper bound of the L^2 -Betti numbers in terms of integral foliated simplicial volume (via Poincaré duality), and
- to investigate whether the vanishing of the simplicial volume of closed aspherical manifolds implies the vanishing of integral foliated simplicial volume.

The first step has been carried out by Schmidt [98]. The second step is an open problem, which is known to have a positive answer in many cases, e.g., for oriented closed aspherical manifolds

- that have amenable fundamental group [46],
- that carry a non-trivial smooth S^1 -action [39],
- that are generalised graph manifolds [40],
- that are smooth and have trivial minimal volume [11, (proof of Corollary 5.4)]. In particular, Question 1.1 with the simplicial volume replaced by the minimal volume has a positive answer [97].

Moreover, the integral foliated simplicial volume is related to the cost of the fundamental group [73] and to the stable integral simplicial volume [78, 46] (Section 6). In turn, the stable integral simplicial volume gives upper bounds for homology growth, torsion homology growth [46], and the rank gradient [72].

Functorial semi-norms. If the integral foliated simplicial volume is a functorial semi-norm on aspherical closed manifolds, then Question 1.1 has an affirmative answer [38, Theorem 2.2.2].

Geometric positivity results. Conversely, it is known that many examples of closed aspherical manifolds with potentially non-zero Euler characteristic have positive simplicial volume. Examples include oriented closed connected hyperbolic manifolds [101, 51], closed manifolds with negative sectional curvature [59], closed irreducible locally symmetric spaces of higher rank [69], closed manifolds with non-positive sectional curvature and sufficiently negative intermediate Ricci curvature [25], and closed manifolds with non-positive sectional curvature and strong enough conditions at a single point [26]. We refer to Section 3.1 for further examples of manifolds with (non-)vanishing simplicial volume.

Outlook. Supported by the wealth of positive examples, and in view of the existence of “exotic” aspherical manifolds, it seems plausible that Question 1.1 has a positive answer in the following special case:

Question 1.3. Let M be an oriented closed aspherical n -manifold whose universal covering is homeomorphic to \mathbb{R}^n . Does M satisfy Property (SV_χ) ?

Organisation of the article. In Section 2, we collect the definitions of simplicial volume (Section 2.1) and bounded cohomology (Section 2.4) as well as the duality principle which connects these (Section 2.4). Moreover, we discuss the behaviour of the simplicial volume with respect to glueings (Section 2.2) and introduce a relative version of Question 1.1 (Section 2.3). Finally, in Section 2.5, we discuss the boundedness properties of the Euler class in connection with Question 1.1.

Section 3 is mainly devoted to the vanishing of the simplicial volume. Some known examples are collected in Section 3.1. Vanishing results for the simplicial volume assuming the existence of amenable open covers are recalled in Section 3.2 and extended to manifolds with boundary in Section 3.3. Known results and open problems about the behaviour of the simplicial volume with respect to products are recalled in Section 3.4 and these are then discussed in connection with Question 1.1 (Proposition 3.24). Finally, Section 3.5 explains a connection between Question 1.1 and a conjecture of Edmonds in four-dimensional topology (Conjecture 3.30) via “fillings” of closed manifolds.

In Section 4, we define the amenable cobordism category and explain how to interpret the simplicial volume as an invertible TQFT on this cobordism category. Using the simplicial volume, we prove that the fundamental group of the 4-dimensional amenable cobordism category is not finitely generated (Theorem 4.5). Also, using known results about cobordism categories, we show that the simplicial volume does not extend to the whole cobordism category (Proposition 4.7).

Section 5 is concerned with the study of Question 1.1 using known constructions that produce aspherical spaces. More precisely, in Sections 5.1–5.3, we recall the Kan–Thurston theorem and explain how to use this to prove the result stated in Section 1.2 (see Theorem 5.7). Then, in Section 5.4, we briefly review known constructions of closed aspherical manifolds, Davis’ reflection group trick and Gromov’s hyperbolization, in the context of Question 1.1.

Finally, Section 6 surveys the approach to Question 1.1 via the stable integral simplicial volume.

Notation. We use $\mathbb{N} = \{0, 1, 2, \dots\}$. We recall that aspherical spaces are assumed to be path-connected.

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2. SIMPLICIAL VOLUME

We recall the definition of the simplicial volume, basic glueing properties, and the role of the comparison map for bounded cohomology. Moreover, we consider and study a relative version of Question 1.1, and discuss the connection of Question 1.1 with (the boundedness of) the Euler class.

2.1. Simplicial volume. The simplicial volume originally appeared in Gromov's proof of Mostow rigidity as a homotopy invariant replacement of the hyperbolic volume [90, 51].

Definition 2.1 (Simplicial volume). Let M be an oriented closed connected n -manifold. The *simplicial volume* of M is defined as

$$\|M\| := \|[M]\|_1 \in \mathbb{R}_{\geq 0},$$

where $[M] \in H_n(M; \mathbb{R})$ is the \mathbb{R} -fundamental class of M and $\|\cdot\|_1$ denotes the semi-norm on $H_*(\cdot; \mathbb{R})$, induced by the ℓ^1 -norm on the singular chain complex $C_*(\cdot; \mathbb{R})$ with respect to the basis given by the singular simplices.

Definition 2.2 (Relative simplicial volume). Let $(M, \partial M)$ be an oriented compact connected n -manifold M with boundary ∂M . The *relative simplicial volume* of $(M, \partial M)$ is defined as

$$\|M, \partial M\| := \|[M, \partial M]\|_1 \in \mathbb{R}_{\geq 0},$$

where $[M, \partial M] \in H_n(M, \partial M; \mathbb{R})$ is the \mathbb{R} -fundamental class of $(M, \partial M)$ and $\|\cdot\|_1$ denotes the ℓ^1 -semi-norm on relative singular homology.

In the oriented, compact, non-connected case, we define the (relative) simplicial volume as the sum of the (relative) simplicial volumes of the components. In particular, $\|\emptyset\| = 0$.

Remark 2.3. The boundary of a relative fundamental cycle of $(M, \partial M)$ is a fundamental cycle of ∂M . This shows that for every oriented compact connected n -manifold M with non-empty boundary ∂M , we have

$$\|M, \partial M\| \geq \frac{\|\partial M\|}{n+1}.$$

In particular, $\|M, \partial M\| = 0$ implies $\|\partial M\| = 0$.

Note that in the case of compact 3-manifolds better estimates are available [15].

Remark 2.4. One can also define the (relative) simplicial volume with *integral coefficients* just by working with integral singular homology. More precisely, the *integral (relative) simplicial volume* of an oriented compact connected n -manifold M with (possibly empty) boundary ∂M is defined by

$$\|M, \partial M\|_{\mathbb{Z}} := \|[M, \partial M]_{\mathbb{Z}}\|_1 \in \mathbb{N},$$

where $[M, \partial M]_{\mathbb{Z}} \in H_n(M, \partial M; \mathbb{Z})$ is the \mathbb{Z} -fundamental class of $(M, \partial M)$. Notice that we still have $\|M, \partial M\|_{\mathbb{Z}} \geq \|\partial M\|_{\mathbb{Z}}/(n+1)$.

2.2. Simplicial volume and glueings of manifolds. In general, the simplicial volume is not additive with respect to the glueing of manifolds along submanifolds. However, (sub)additivity does hold in the case of amenable glueings:

Theorem 2.5 (Simplicial volume and glueings [51, 13][45, Theorem 7.6]). *Let I be a finite set and let $(M_i, \partial M_i)_{i \in I}$ be a family of oriented compact connected manifolds of the same dimension. Assume that all the boundary components have amenable fundamental group. Moreover, let $(M, \partial M)$ be obtained from $(M_i, \partial M_i)_{i \in I}$ by a pairwise glueing (along orientation reversing homeomorphisms) of a set of boundary components. Then, we have*

$$\|M, \partial M\| \leq \sum_{i \in I} \|M_i, \partial M_i\|.$$

If all glued boundary components are π_1 -injective in their original manifold, then

$$\|M, \partial M\| = \sum_{i \in I} \|M_i, \partial M_i\|.$$

Remark 2.6. Theorem 2.5 allows that boundary components of the given manifolds $(M_i, \partial M_i)$ are glued to boundary components of the same manifold $(M_i, \partial M_i)$; i.e., self-glueings are included. Furthermore, not all boundary components need to be glued (so that some of the components remain boundary components of M).

Remark 2.7. It is well-known that the Euler characteristic is *always* additive with respect to glueings: given oriented compact connected n -manifolds $(M, \partial M)$ and $(N, \partial N)$ with homeomorphic (or just homotopy equivalent) boundary components $M_1 \subseteq \partial M$ and $N_1 \subseteq \partial N$, we set $Z = M \cup_{M_1 \cong N_1} N$. Then we have:

$$\chi(Z) = \chi(M) + \chi(N) - \chi(M_1 \cong N_1)$$

and similarly:

$$\chi(Z, \partial Z) = \chi(M, \partial M) + \chi(N, \partial N) + \chi(M_1 \cong N_1).$$

Here $\chi(W, \partial W) := \chi(W) - \chi(\partial W)$ denotes the *relative* Euler characteristic of the compact manifold $(W, \partial W)$.

Assuming that M_1 is aspherical and has amenable fundamental group, then both the simplicial volume and the Euler characteristic of M_1 vanish (Example 3.2(1) and Theorem 3.6). In particular, the last formula simplifies in this case to a formula analogous to the one in Theorem 2.5:

$$\chi(Z, \partial Z) = \chi(M, \partial M) + \chi(N, \partial N).$$

Example 2.8 (Doubles). Given an oriented compact connected manifold M with non-empty boundary ∂M , we define the *double* of M to be

$$D(M) := M \cup_{\partial M \cong \partial(-M)} -M,$$

where $-M$ denotes a copy of M with the opposite orientation. It is easily seen that we always have subadditivity of the simplicial volume in this case:

$$\|D(M)\| \leq 2 \cdot \|M, \partial M\|.$$

Indeed, given a relative fundamental cycle c of M , we can set \bar{c} to be the relative fundamental cycle of $-M$ corresponding to $-c$. Then, $c' = c + \bar{c}$ is in fact a fundamental cycle of $D(M)$ with norm

$$|c'|_1 \leq |c|_1 + |\bar{c}|_1 = 2 \cdot |c|_1.$$

Then the subadditivity of the simplicial volume follows from taking the infimum over all such c . The same computation also works for integral coefficients.

Remark 2.9 (Doubles and asphericity). In general, the double of an oriented compact aspherical manifold with boundary is not necessarily aspherical; prototypical examples of this kind are the 3-ball or $S^1 \times D^2$.

Let $(M, \partial M)$ be a compact n -manifold, where M is aspherical and ∂M is connected, and let F denote the homotopy fibre of the inclusion $\partial M \subset M$. For simplicity, we write $H := \pi_1(\partial M, x)$ and $G := \pi_1(M, x)$ and denote by $\iota: H \rightarrow G$ the induced homomorphism. Moreover, let $G' := \text{im}(\iota)$ be the image of ι , and consider the corresponding diagram

$$\begin{array}{ccc} K(G, 1) & \longleftarrow \partial M & \longrightarrow K(G, 1) \\ \parallel & \downarrow q & \parallel \\ K(G, 1) & \longleftarrow K(G', 1) & \longrightarrow K(G, 1) \end{array}$$

induced by ι and the inclusion maps. Suppose that the (homotopy) pushout

$$D(M) = M \cup_{\partial M} M \simeq K(G, 1) \overset{h}{\cup}_{\partial M} K(G, 1)$$

is aspherical. Then the induced map between the homotopy pushouts

$$g: K(G, 1) \overset{h}{\cup}_{\partial M} K(G, 1) \rightarrow K(G, 1) \overset{h}{\cup}_{K(G', 1)} K(G, 1)$$

is a homotopy equivalence by the Seifert–van Kampen theorem. Passing to the homotopy fibres of the diagram above, regarded as a diagram over $K(G, 1)$, we obtain the following diagram (up to canonical homotopy equivalence)

$$\begin{array}{ccc} * & \longleftarrow F & \longrightarrow * \\ \parallel & \downarrow q' & \parallel \\ * & \longleftarrow D & \longrightarrow * \end{array}$$

where D is discrete with cardinality equal to the index $[G : G']$. We recall that the homotopy fibre of a map from a homotopy pushout is canonically identified up to homotopy equivalence with the homotopy pushout of the

respective homotopy fibres. Thus, since g is a homotopy equivalence, the induced map

$$g': \Sigma F \rightarrow \Sigma D,$$

between the homotopy fibres of the source and target of g as spaces over $K(G, 1)$, where Σ denotes here the (unreduced) suspension, is again a homotopy equivalence. Therefore, $\pi_0(F) \cong [G : G']$ and each path-component of F must have trivial integral homology. In addition, the homotopy fibre of

$$q: \partial M \rightarrow K(G', 1)$$

is identified with a path-component of F , so the map q is acyclic (see Section 5.2). In particular, q is an integral homology equivalence and arises as the plus construction associated to the kernel of ι . As a consequence, if ι is injective, then $q: \partial M \rightarrow K(G', 1)$ is a homotopy equivalence, so ∂M is again aspherical. Conversely, it is well known that the double is aspherical if M and ∂M are aspherical and ι is injective.

On the other hand, we do not know if the injectivity of ι is necessary for the asphericity of the double.

An interesting example of amenable glueings is given by connected sums:

Proposition 2.10 ([51][45, Corollary 7.7]). *Let $n \geq 1$ and let M and N be oriented closed connected n -manifolds. The following hold:*

- (1) $\chi(M\#N) = \chi(M) + \chi(N) - \chi(S^n)$;
- (2) If $n \geq 3$, then $\|M\#N\| = \|M\| + \|N\|$.

In particular, if $n \geq 3$, we have:

- (3) If n is even and $\chi(M) = 0 = \chi(N)$, then $\chi(M\#N) \neq 0$;
- (4) If $\|M\| = 0 = \|N\|$, then $\|M\#N\| = 0$.

Remark 2.11. Note that the connected sum of aspherical manifolds in dimension ≥ 3 is never aspherical [81, Lemma 3.2]. Thus, Proposition 2.10 cannot be used to produce counterexamples to Question 1.1.

Example 2.12. The formula in Proposition 2.10(2) fails in dimension 2. For example, hyperbolic surfaces have non-zero simplicial volume (Example 3.1(1)) but the two-dimensional torus has zero simplicial volume (Example 3.2.(1)).

2.3. A relative version of Gromov's question. We consider the following version of Question 1.1 for manifolds with boundary and show that it is a consequence of Property (SV χ) (Proposition 2.15).

Question 2.13. Let $(M, \partial M)$ be an oriented compact aspherical manifold with non-empty π_1 -injective aspherical boundary. Does the following implication hold?

$$(SV_\chi, \partial) \quad \|M, \partial M\| = 0 \implies \chi(M, \partial M) = 0.$$

Remark 2.14. For every oriented compact connected n -manifold M with boundary ∂M , we have $|\chi(M)| = |\chi(M, \partial M)|$. Indeed, when n is even, we know that $\chi(\partial M) = 0$, so $\chi(M, \partial M) = \chi(M) - \chi(\partial M) = \chi(M)$. On the other hand, when n is odd, we have $\chi(\partial M) = 2 \cdot \chi(M)$, so $\chi(M, \partial M) = \chi(M) - \chi(\partial M) = -\chi(M)$.

In order to disprove Property (SV χ) it suffices to find an example that does not satisfy Property (SV χ, ∂):

Proposition 2.15. *Let $n \geq 1$. If all oriented closed aspherical manifolds of dimension n or $n - 1$ satisfy Property (SV χ), then all oriented compact aspherical n -manifolds with non-empty π_1 -injective aspherical boundary satisfy Property (SV χ, ∂).*

Proof. Let $(M, \partial M)$ be an oriented compact aspherical n -manifold with non-empty π_1 -injective aspherical boundary. Then the double $D(M) := M \cup_{\partial M} M$ is an oriented closed aspherical n -manifold (Remark 2.9) and $\|D(M)\| \leq 2 \cdot \|M, \partial M\|$ (Example 2.8).

Suppose $\|M, \partial M\| = 0$. Then $\|D(M)\| = 0$ and $\|\partial M\| = 0$ (Remark 2.3). From Property (SV χ) in dimension n and $n - 1$, respectively, we conclude

$$\chi(D(M)) = 0 \quad \text{and} \quad \chi(\partial M) = 0.$$

Therefore, we compute

$$\chi(M, \partial M) = \chi(M) - \chi(\partial M) = \frac{1}{2} \cdot (\chi(D(M)) + \chi(\partial M)) - \chi(\partial M) = 0.$$

Hence, $(M, \partial M)$ satisfies Property (SV χ, ∂). \square

Example 2.16. Note that Property (SV χ, ∂) does not hold for all oriented compact aspherical manifolds without the π_1 -injectivity condition on the boundary. For example, if we take an oriented closed connected hyperbolic even-dimensional manifold N and let $M := N \times D^2$, then M and $\partial M = N \times S^1$ are aspherical and $\|M, \partial M\| = 0$ (Proposition 3.19 or Example 3.2(4)). On the other hand, we have $\chi(M, \partial M) = \chi(N) \cdot \chi(D^2, S^1) = \chi(N) \neq 0$.

Remark 2.17. Note that by allowing the case of empty boundary, Proposition 2.15 can be formulated as an equivalence between Question 2.13 and Question 1.1. The extension of Gromov's question to manifolds with boundary allows us in particular to explore Question 1.1 by studying the properties of (vanishing of) the simplicial volume and the Euler characteristic along glueings of manifolds and compare their respective additivity properties. This viewpoint will also be explored in Section 4.

2.4. The comparison map. Dually, simplicial volume can be expressed in terms of bounded cohomology. *Bounded cohomology*

$$H_b^*(\cdot; \mathbb{R}) := H^*(C_b^*(\cdot; \mathbb{R}))$$

is the cohomology of the topological dual $C_b^*(\cdot; \mathbb{R})$ of the singular chain complex, where the dual is taken with respect to the ℓ^1 -norm. Bounded cohomology is then endowed with the ℓ^∞ -seminorm, denoted by $\|\cdot\|_\infty$.

The inclusion $C_b^*(\cdot; \mathbb{R}) \hookrightarrow C^*(\cdot; \mathbb{R})$ induces a natural transformation

$$\text{comp}^*: H_b^*(\cdot; \mathbb{R}) \implies H^*(\cdot; \mathbb{R}),$$

which is called the *comparison map*. A straightforward application of the Hahn–Banach Theorem shows:

Proposition 2.18 (Duality principle [51]). *Let (X, A) be a pair of spaces, let $k \in \mathbb{N}$, and let $\alpha \in H_k(X, A; \mathbb{R})$. Then*

$$\|\alpha\|_1 = \sup\{|\langle \text{comp}_{X,A}^k(\varphi), \alpha \rangle| \mid \varphi \in H_b^k(X, A; \mathbb{R}), \|\varphi\|_\infty \leq 1\}.$$

In particular: If $(M, \partial M)$ is an oriented compact connected n -manifold with (possibly empty) boundary, then $\|M, \partial M\|$ is the operator norm of the composition

$$H_b^n(M, \partial M; \mathbb{R}) \xrightarrow{\text{comp}_{M, \partial M}^n} H^n(M, \partial M; \mathbb{R}) \overset{\cap[M, \partial M]}{\cong} \mathbb{R}$$

and

$$\text{comp}_{(M, \partial M)}^n \text{ is surjective} \iff \|M, \partial M\| > 0.$$

Proposition 2.19. *Let M be an oriented closed connected n -manifold such that $\|M\| = 0$. Suppose that $x \in H^k(M; \mathbb{R})$ is bounded (i.e. x lies in the image of the comparison map comp_M^k) and let $x^* \in H^{n-k}(M; \mathbb{R})$ be such that $x \cup x^* \neq 0$. Then x^* is not bounded.*

Proof. Let $y \in H_b^k(M; \mathbb{R})$ be a class with $\text{comp}_M^k(y) = x$. Assume for a contradiction that $x^* \in H^{n-k}(M; \mathbb{R})$ lies in the image of comp_M^{n-k} , that is, there is $z \in H_b^{n-k}(M; \mathbb{R})$ with $\text{comp}_M^{n-k}(z) = x^*$. The usual explicit formula for the cup-product on singular cohomology shows that the cup-product lifts to a cup-product on bounded cohomology. Then

$$\text{comp}_M^n(y \cup z) = \text{comp}_M^k(y) \cup \text{comp}_M^{n-k}(z) = x \cup x^* \neq 0,$$

so comp_M^n maps surjectively onto $H^n(M; \mathbb{R}) \cong \mathbb{R}$. But this contradicts the assumption $\|M\| = 0$ according to Proposition 2.18. \square

The vanishing of the simplicial volume thus implies that not too many classes can be bounded; dually, the vanishing of the simplicial volume causes that there are many other classes with vanishing ℓ^1 -semi-norm.

Corollary 2.20. *Let M be an oriented closed connected n -manifold satisfying $\|M\| = 0$ and let $N_*(M; \mathbb{R}) := \{\alpha \in H_*(M; \mathbb{R}) \mid \|\alpha\|_1 = 0\}$. Then*

$$\sum_{k \in \mathbb{N}} \dim_{\mathbb{R}} N_k(M; \mathbb{R}) \geq \frac{1}{2} \cdot \sum_{k \in \mathbb{N}} \dim_{\mathbb{R}} H_k(M; \mathbb{R}).$$

Proof. On the one hand, by the duality principle (Proposition 2.18), we have

$$\dim_{\mathbb{R}} H_k(M; \mathbb{R}) - \dim_{\mathbb{R}} N_k(M; \mathbb{R}) = \dim_{\mathbb{R}} (\text{im } \text{comp}_M^k)$$

for all $k \in \mathbb{N}$. On the other hand, Poincaré duality and Proposition 2.19 imply that

$$\sum_{k \in \mathbb{N}} \dim_{\mathbb{R}}(\text{im comp}_M^k) \leq \frac{1}{2} \cdot \sum_{k \in \mathbb{N}} \dim_{\mathbb{R}} H^k(M; \mathbb{R}) = \frac{1}{2} \cdot \sum_{k \in \mathbb{N}} \dim_{\mathbb{R}} H_k(M; \mathbb{R}).$$

Combining both estimates gives the claim. \square

2.5. Boundedness of the Euler class. The Euler characteristic of an oriented closed connected smooth n -manifold M can be expressed in terms of the Euler class $e(M) \in H^n(M; \mathbb{R})$ [85] via

$$\chi(M) = \langle e(M), [M] \rangle.$$

The norm of the Euler class has been studied extensively in the literature, especially, in connection with the existence of *flat* structures (a detailed account of results in this direction is given in Frigerio's book [45]). The boundedness of the Euler class is also closely related to Question 1.1.

Question 2.21. Let M be an oriented closed aspherical smooth n -manifold. Does the following property hold?

(Eub) The Euler class $e(M) \in H^n(M; \mathbb{R})$ is bounded.

Proposition 2.22. *Let $n \in \mathbb{N}$ and let M be an oriented closed connected smooth n -manifold. Then the following are equivalent:*

- (1) *The manifold M satisfies Property (SV χ).*
- (2) *The manifold M satisfies Property (Eub).*

Proof. Let M satisfy Property (SV χ). If $\|M\| = 0$, then $\langle e(M), [M] \rangle = \chi(M) = 0$. By duality, this implies that $e(M) = 0$; in particular, $e(M)$ is bounded. On the other hand, if $\|M\| > 0$, then the comparison map is surjective (Proposition 2.18), hence $e(M)$ is also bounded. This shows that M satisfies Property (Eub).

Conversely, suppose that $e(M)$ is bounded. Then

$$|\chi(M)| = |\langle e(M), [M] \rangle| \leq \|e(M)\|_{\infty} \cdot \|M\|.$$

As a consequence, if $\|M\| = 0$, then $\chi(M) = 0$; i.e., M satisfies Property (SV χ). \square

Remark 2.23. Let $(M, \partial M)$ be an oriented compact connected manifold with boundary. We may define

$$e(M, \partial M) \in H^n(M, \partial M; \mathbb{R})$$

to be the Poincaré dual class to $\chi(M) \in \mathbb{Z} \cong H_0(M; \mathbb{Z})$. We recall that $|\chi(M)| = |\chi(M, \partial M)|$ (Remark 2.14). Then Property (SV χ , ∂) is equivalent to

(Eub, ∂) $e(M, \partial M) \in H^n(M, \partial M; \mathbb{R})$ is bounded.

The proof is the same as for Proposition 2.22.

Remark 2.24. It is well known that the Euler class of flat vector bundles is bounded [45, Section 13]. This shows that if $\|M\| = 0$ and the tangent bundle of M admits a flat connection, then $\chi(M) = 0$ [63][17][45, Theorem 13.11].

Moreover, it was conjectured [45, Conjecture 13.13] that the Euler class of topologically flat sphere bundles admits a bounded representative. Monod and Nariman [88, Theorem 1.8] have recently proved that the Euler class of the (discrete) group of orientation-preserving homeomorphisms of S^3 is unbounded.

Example 2.25. Assuming that Question 1.1 has an affirmative answer, then Proposition 2.22 has interesting implications for the existence of tangential maps between smooth manifolds. We recall that a map $f: M \rightarrow N$ between closed smooth manifolds is called *tangential* if the vector bundles TM and f^*TN are isomorphic. As a consequence, a tangential map $f: M \rightarrow N$ between oriented closed connected smooth manifolds preserves the Euler class up to sign. Assuming Question 1.1, it follows that there cannot exist tangential maps $f: M \rightarrow N$ if $\chi(M) \neq 0$ and N is aspherical with zero simplicial volume. Indeed, assuming that N satisfies Property (SV χ), it follows that $e(N) = 0$ (since $\chi(N) = 0$). Then, given a tangential map $f: M \rightarrow N$, the classes $f^*(e(N))$ and $e(M)$ agree up to sign, so $e(M) = 0$ (and therefore also $\chi(M) = 0$).

3. VANISHING OF THE SIMPLICIAL VOLUME

In this section, we collect some known results on the simplicial volume. We will be mainly interested in describing sufficient conditions for the vanishing of the simplicial volume. We also compare those situations with the respective behaviour of the Euler characteristic.

3.1. Computations of the simplicial volume. In general, computing exact values of the simplicial volume is difficult. For example, the problem of determining whether a given (triangulated) manifold has vanishing simplicial volume or not is undecidable [102, Chapter 2.6]. The two major sources for (non-)vanishing results are amenability (which leads to vanishing) and negative curvature (which leads to non-vanishing).

Example 3.1 (Non-vanishing). The following manifolds have positive simplicial volume:

- (1) Oriented closed connected hyperbolic manifolds [101, 51];
- (2) The compactification of oriented connected complete finite-volume hyperbolic manifolds [51, 49];
- (3) Oriented closed connected manifolds with negative sectional curvature [59];
- (4) Oriented closed connected locally symmetric spaces of non-compact type [18, 69];

- (5) Oriented closed connected manifolds with non-positive sectional curvature and sufficiently negative intermediate Ricci curvature [25];
- (6) Oriented closed connected manifolds with non-positive sectional curvature and strong enough conditions at a single point [26];
- (7) Oriented closed connected rationally essential (e.g., aspherical) manifolds of dimension ≥ 2 with non-elementary hyperbolic fundamental group (this follows via the duality principle from work of Mineyev on surjectivity of the comparison map [86]);
- (8) Oriented closed connected rationally essential manifolds of dimension ≥ 2 with non-elementary relatively hyperbolic fundamental group [7];
- (9) Non-vanishing of simplicial volume is inherited through a proportionality principle [51, 101, 70]: If M and N are oriented closed connected Riemannian manifolds with isometric universal coverings, then $\|M\| > 0$ if and only if $\|N\| > 0$.

In this paper, we focus our attention on vanishing results for the simplicial volume. The following example contains some known vanishing results:

Example 3.2 (Vanishing). The following manifolds have zero simplicial volume:

- (1) Oriented closed connected n -manifolds with amenable fundamental group and $n > 0$ [51] or, more generally, with n -boundedly acyclic fundamental group [89, 51]; finitely presented non-amenable boundedly acyclic groups have been recently constructed [42, 87];
- (2) Oriented compact connected n -manifolds M with non-empty boundary such that both the fundamental groups of M and of ∂M are amenable [51];
- (3) More generally, oriented compact connected n -manifolds M with non-empty boundary such that $\|\partial M\| = 0$ and the connecting homomorphism $H_b^{n-1}(\partial M) \rightarrow H_b^n(M, \partial M)$ is surjective. Manifolds satisfying the latter condition can be constructed by taking manifolds whose boundary inclusion is π_1 -surjective and such that their fundamental group lies in Lex [10]. Recall that Lex groups are those groups Γ such that every epimorphism $\Lambda \twoheadrightarrow \Gamma$ induces an injective map in bounded cohomology in every degree. Examples of Lex groups contain free groups, amenable groups [10], boundedly acyclic groups and certain extensions of these [42, Remark 3.8].
- (4) Oriented compact connected manifolds with (possibly empty) boundary that admit a self-map f of degree $\deg(f) \notin \{0, 1, -1\}$ [51];
- (5) Oriented closed connected manifolds that are the boundary of an oriented compact connected manifolds with zero simplicial volume (Remark 2.3);
- (6) Oriented closed connected n -manifolds that admit a smooth non-trivial S^1 -action [104]. More generally, manifolds admitting an F -structure also have zero simplicial volume [24, 91];

- (7) Oriented closed aspherical manifolds supporting an affine structure whose holonomy map is injective and contains a pure translation [14];
- (8) Oriented closed connected smooth manifolds with zero minimal volume [51, 8] or zero minimal volume entropy [51, p. 37][4];
- (9) Oriented closed connected graph 3-manifolds [99, 51];
- (10) All mapping tori of oriented closed connected 3-manifolds [16]; however, the general behaviour of simplicial volume of general mapping tori is very diverse [65].

Remark 3.3. In view of Question 1.1, it would be interesting to understand whether all oriented closed aspherical manifolds that admit a self-map f of degree $\deg(f) \notin \{0, 1, -1\}$ must have zero Euler characteristic. For surfaces this is clearly the case (the only candidate being the torus). More generally, this is known to be true whenever the fundamental group of the aspherical manifold is Hopfian [2]. The statement was claimed in full generality by Sullivan [100, Footnote 23, p. 318] without a proof.

3.2. Amenable covers: The closed case. A useful approach to investigate the vanishing of simplicial volume is to consider *amenable covers*. This idea dates back to Gromov [51] and it was then developed further by many authors [60, 50, 47, 44, 79, 62, 21, 3, 76, 93]. We use the terminology of amenable category [50, 21, 76]:

Definition 3.4 (Amenable covers and category).

- (1) Let X be a topological space and let U be a subset of X . We say that U is *amenable* in X if for every $x \in U$ the image

$$\text{im}(\pi_1(U \hookrightarrow X, x)) \leq \pi_1(X, x)$$

is amenable. We say that an open cover of X is *amenable*, if it consists of amenable sets.

- (2) The *amenable category* of X , denoted by $\text{cat}_{\text{Am}}(X)$, is the minimal integer n such that X admits an open amenable cover with cardinality n . If no such integer exists, we simply set $\text{cat}_{\text{Am}}(X) = +\infty$.

The vanishing results for open amenable covers are usually stated in terms of assumptions on the multiplicity of the cover instead of the cardinality. These assumptions essentially are the same when working with paracompact Hausdorff spaces [21, Remark 3.13].

The importance of amenable covers in our setting is demonstrated by the following two results:

Theorem 3.5 (Gromov’s vanishing theorem [51, p. 40]). *Let M be an oriented closed connected n -manifold. Then,*

$$\text{cat}_{\text{Am}}(M) \leq n \implies \|M\| = 0.$$

A similar result for the Euler characteristic has been proved by Sauer [97]:

Theorem 3.6 (Euler characteristic and amenable covers). *Let M be an oriented closed aspherical n -manifold. Then,*

$$\text{cat}_{\text{Am}}(M) \leq n \implies \chi(M) = 0.$$

Remark 3.7. In Theorem 3.6, the asphericity assumption is crucial: every even-dimensional sphere provides a counterexample in the non-aspherical setting.

In particular, Theorems 3.5 and 3.6 show that all oriented closed aspherical manifolds M with $\text{cat}_{\text{Am}}(M) \leq \dim(M)$ satisfy Property (SV χ).

Example 3.8. Certain fibre bundles yield examples of the situation arising in Theorems 3.5 and 3.6. Let $N \hookrightarrow M \rightarrow B$ be a fibre bundle of oriented closed connected manifolds and suppose that

$$\text{cat}_{\text{Am}}(N) \leq \frac{\dim(M)}{\dim(B) + 1}.$$

Then, we have $\text{cat}_{\text{Am}}(M) \leq \dim(M)$ [76, Corollary 1.2] (and $\dim(M) \geq \dim(B) + 1 \geq 1$). Using Theorems 3.5 and 3.6, we conclude:

- (a) $\|M\| = 0$ (Theorem 3.5).
- (b) If M is aspherical, then also $\chi(M) = 0$ (Theorem 3.6).

Concerning (b), it should be noted that in this case the asphericity of N is also sufficient in order to conclude that $\chi(M) = 0$. Indeed the hypothesis on $\text{cat}_{\text{Am}}(N)$ shows that

$$\text{cat}_{\text{Am}}(N) \leq \frac{\dim(M)}{\dim(B) + 1} = \frac{\dim(B) + \dim(N)}{\dim(B) + 1} = \frac{\dim(B)}{\dim(B) + 1} + \frac{\dim(N)}{\dim(B) + 1}.$$

Since $\text{cat}_{\text{Am}}(N)$ is an integer, it follows that $\text{cat}_{\text{Am}}(N) \leq \dim(N)$. Thus, $\chi(N) = 0$ (Theorem 3.6) and so $\chi(M) = \chi(N) \cdot \chi(B) = 0$.

Recently, Gromov's vanishing theorem was extended to weaker situations, such as weakly boundedly acyclic open covers and other more general homotopy colimit decompositions [62, 93]. This new context suggests in particular the following question:

Question 3.9. Let M be an oriented closed aspherical n -manifold. Assuming there exists a weakly boundedly acyclic open cover of M in the sense of Ivanov [62] with cardinality at most n , does it follow that $\chi(M) = 0$?

A negative answer to Question 3.9 would also produce closed aspherical examples that do not satisfy Property (SV χ). In fact, no example of an oriented closed aspherical manifold M with vanishing simplicial volume and $\text{cat}_{\text{Am}}(M) = \dim(M) + 1$ seems to be known.

3.3. Amenable covers: The case with boundary. The vanishing theorem can also be extended to the relative setting via Gromov's vanishing-finiteness theorem [51, 47, 61]. We quickly recall the formulation of the vanishing-finiteness theorem and mention two convenient special cases for

compact manifolds with boundary: the relative vanishing theorem and the case of locally co-amenable subcomplexes.

For an oriented connected (possibly non-compact) manifold M without boundary, the *locally finite simplicial volume* is defined by

$$\|M\|^{\text{lf}} := \inf\{|c|_1 \mid c \in C_n^{\text{lf}}(M; \mathbb{R}) \text{ is a fundamental cycle of } M\},$$

where $C_*^{\text{lf}}(\cdot; \mathbb{R})$ denotes the (singular) locally finite chain complex of M . Because locally finite chains are not necessarily ℓ^1 , the locally finite simplicial volume is not always finite.

Theorem 3.10 (Vanishing-finiteness theorem [51, p. 58][47, Section 7.2][61, Theorem 4.6]). *Let M be an oriented connected n -manifold without boundary and let $(U_i)_{i \in \mathbb{N}}$ be an open cover of M with the following properties:*

- (i) *For each $i \in \mathbb{N}$, the subset $U_i \subset M$ is amenable and relatively compact.*
- (ii) *The sequence $(U_i)_{i \in \mathbb{N}}$ is amenable at infinity, i.e., there exists an increasing sequence $(K_i)_{i \in \mathbb{N}}$ of compact subsets of M such that:*
 - (a) *The family $(M \setminus K_i)_{i \in \mathbb{N}}$ is locally finite;*
 - (b) *For all $i \in \mathbb{N}$, we have $U_i \subset M \setminus K_i$;*
 - (c) *For all sufficiently large $i \in \mathbb{N}$, the set U_i is amenable in $M \setminus K_i$.*
- (iii) *The multiplicity of $(U_i)_{i \in \mathbb{N}}$ is at most n .*

Then $\|M\|^{\text{lf}} = 0$.

Remark 3.11. Properties (a) and (b) in (ii) show that every sequence $(U_i)_{i \in \mathbb{N}}$ which is amenable at infinity is necessarily locally finite.

Vanishing of the locally finite simplicial volume leads to vanishing results for the relative simplicial volume:

Remark 3.12. If $(M, \partial M)$ is an oriented compact connected manifold, then $\|M, \partial M\| \leq \|\text{int}(M)\|^{\text{lf}}$ [51][71, Proposition 5.12]. In general, this inequality is strict [51, p. 10][71, Example 6.17]: E.g., $\|[0, 1], \{0, 1\}\| < \infty = \|(0, 1)\|^{\text{lf}}$.

On the other hand, there is no known example for which $\|\text{int}(M)\|^{\text{lf}}$ is finite and distinct from $\|M, \partial M\|$.

Theorem 3.13 (Relative vanishing theorem). *Let $(M, \partial M)$ be an oriented compact connected n -manifold that admits an amenable open cover $(U_i)_{i \in I}$ with the following properties:*

- (i) *The multiplicity of $(U_i)_{i \in I}$ is at most n .*
- (ii) *The multiplicity of $(U_i \cap \partial M)_{i \in I}$ is at most $n - 1$.*
- (iii) *For each $i \in I$, the set $U_i \cap \partial M$ is amenable in ∂M .*

Then $\|M, \partial M\| = 0$.

Proof. In view of Remark 3.12, it suffices to show that the vanishing-finiteness theorem (Theorem 3.10) can be applied to $\text{int}(M)$. To this end, we modify the given open cover $(U_i)_{i \in I}$ as follows. Up to homeomorphism, we can write

$$\text{int}(M) \cong M \cup_{\partial M} (\partial M \times [0, +\infty)).$$

We extend $(U_i)_{i \in I}$ to the right-hand side by replacing the sets V of $(U_i)_{i \in I}$ intersecting ∂M with $V \cup ((V \cap \partial M) \times [0, \infty))$. Let $(V_i)_{i \in I}$ be the resulting open cover of $\text{int}(M)$. We now upgrade this cover $(V_i)_{i \in I}$ to a locally finite cover made of relatively compact sets without increasing the multiplicity; because the intersection of $(U_i)_{i \in I}$ with ∂M has multiplicity at most $n - 1$, this is indeed possible by a standard procedure [47, Proof of Theorem 11.2.3, p. 144]. The resulting open cover satisfies all the conditions required by the vanishing-finiteness theorem (Theorem 3.10). \square

Based on Theorem 3.13, the following question is a special case of Question 2.13:

Question 3.14. Let $(M, \partial M)$ be an oriented compact aspherical n -manifold with non-empty π_1 -injective aspherical boundary that admits an open cover as in Theorem 3.13. Then, do we have $\chi(M, \partial M) = 0$?

Amenable covers as in the vanishing-finiteness theorem (Theorem 3.10) appear naturally in the presence of locally co-amenable subcomplexes [51, 47]:

Definition 3.15 (Locally co-amenable subcomplex [51, p. 59][47, Definition 11.2.1]). Let M be an oriented compact connected PL-manifold with non-empty boundary and let P be a simplicial complex such that $M \cong |P|$. Assume that there exists a simplicial complex $K \subset P$ such that $|K| \subset \text{int}(M)$ and M is homeomorphic to a closed regular neighbourhood of K inside P [47, Definition 11.1.4]. Suppose also that K has codimension at least 2 in M . Then, K is called *locally co-amenable in P (or in M)* if for each vertex $v \in (K'')^0$ of the second barycentric subdivision K'' of K we have that

$$\pi_1(|S_v \setminus (S_v \cap K)|)$$

is amenable. Here, S_v denotes the simplicial sphere in P'' centered at v .

Remark 3.16. If K is locally co-amenable in M , then M is homotopy equivalent to K .

Remark 3.17. If both M and ∂M are aspherical and M admits a locally co-amenable subcomplex K , then the boundary inclusion is *not* π_1 -injective: Indeed, if $\partial M \hookrightarrow M$ were π_1 -injective, then $\pi_1(\partial M)$ would be isomorphic to a subgroup of $\pi_1(M)$. So, asphericity and the Shapiro lemma show that

$$\text{cd } \pi_1(M) \geq \text{cd } \pi_1(\partial M) = \dim(M) - 1.$$

However, as M is homotopy equivalent to K (Remark 3.16), also K is aspherical and thus

$$\text{cd } \pi_1(M) \leq \dim(K) \leq \dim(M) - 2,$$

which is a contradiction.

Proposition 3.18. *If M is an oriented compact connected PL-manifold with non-empty boundary that admits a locally co-amenable subcomplex, then $\|M, \partial M\| = 0$.*

Proof. Under the given assumptions, the vanishing-finiteness theorem (Theorem 3.10) applies to $\text{int}(M)$ [47, Theorem 11.2.3]. Therefore, $\|M, \partial M\| = 0$ (Remark 3.12). \square

3.4. Products of manifolds. While the Euler characteristic is multiplicative with respect to products, the product behaviour for the simplicial volume is more delicate. If one of the factors is closed, the vanishing behaviour of simplicial volume is controlled by the factors:

Proposition 3.19 (Simplicial volume and products [51, 12][71, Proposition C.7]). *Let M be an oriented closed connected m -manifold and let N be an oriented compact connected n -manifold with (possibly empty) boundary. Then, we have*

$$\|M\| \cdot \|N, \partial N\| \leq \|M \times N, \partial(M \times N)\| \leq \binom{n+m}{m} \cdot \|M\| \cdot \|N, \partial N\|.$$

The exact values in general are unknown; the only known non-zero computation is the product of two closed surfaces [19].

On the other hand, the product of at least three compact manifolds with non-empty boundary always has vanishing simplicial volume:

Proposition 3.20. *Let M_1, M_2, M_3 be oriented compact connected PL-manifolds with non-empty boundary. Then, we have*

$$\|M_1 \times M_2 \times M_3, \partial(M_1 \times M_2 \times M_3)\| = 0.$$

Proof. In this situation, $M_1 \times M_2 \times M_3$ admits a locally co-amenable subcomplex [51, Example (a), p. 59][47, Theorem 14]. Therefore, we can apply Proposition 3.18.

An alternative proof is as follows: Let $n = \dim(M_1 \times M_2 \times M_3)$. The homotopy fibre of the boundary inclusion in this case has trivial fundamental group and thus has trivial bounded cohomology. Hence the induced map in bounded cohomology $H_b^k(M_1 \times M_2 \times M_3) \rightarrow H_b^k(\partial(M_1 \times M_2 \times M_3))$ is an isomorphism in every degree [51, 89]. The long exact sequence of the pair then implies that $H_b^n(M_1 \times M_2 \times M_3, \partial(M_1 \times M_2 \times M_3))$ is trivial, whence $\|M_1 \times M_2 \times M_3, \partial(M_1 \times M_2 \times M_3)\| = 0$ (Proposition 2.18). \square

A nice application of the previous result is the following:

Example 3.21. Let $(T^2)^\times$ denote a 2-dimensional torus with an open disk removed. Then the simplicial volume of the product of (at least) three copies of $(T^2)^\times$ vanishes. On the other hand, $\chi((T^2)^\times \times (T^2)^\times \times (T^2)^\times) = -1$.

While triple products have zero relative simplicial volume, the situation remains undecided in the following case of products of two compact manifolds [47, Question 8]:

Question 3.22. Let M and N be oriented compact connected manifolds with non-empty connected boundaries. Does it follow that

$$(SV \times) \quad \|M \times N, \partial(M \times N)\| = 0 ?$$

Remark 3.23. Without the connectedness assumption on the boundary, there are products that do not satisfy Property (SV \times). For example, it is known that the product of a compact hyperbolic surface with boundary and a closed interval has non-zero relative simplicial volume [51, p. 17] [71, Corollary 6.2].

The following proposition shows an interesting connection between Property (SV \times) and Question 1.1.

Proposition 3.24. *Let M and N be oriented compact aspherical manifolds with non-empty π_1 -injective aspherical boundary that satisfy $\chi(M) \cdot \chi(N) \neq 0$. Furthermore, suppose that M and N have dimensions of different parity and satisfy Property (SV \times). Then $\partial(M \times N)$ does not satisfy Property (SV χ).*

Proof. As M and N satisfy Property (SV \times), we have $\|M \times N, \partial(M \times N)\| = 0$. In particular, $\|\partial(M \times N)\| = 0$ (Remark 2.3). Moreover, the boundary $\partial(M \times N) = (\partial M \times N) \cup_{\partial M \times \partial N} (M \times \partial N)$ is aspherical and has even dimension. So it suffices to show that $\partial(M \times N)$ has non-zero Euler characteristic. Since $\chi(M \times N) = \chi(M) \cdot \chi(N) \neq 0$ and $M \times N$ is odd-dimensional, we have $\chi(\partial(M \times N)) = 2 \cdot \chi(M \times N) \neq 0$. \square

3.5. Small aspherical fillings. We now come to a higher order version of vanishing, which asks for “small” aspherical fillings of the given manifold with vanishing Euler characteristic/simplicial volume. We will be mainly interested in filling aspherical 3-manifolds with amenable fundamental group.

Definition 3.25 ([37, p. 3]). Let M be an oriented closed aspherical 3-manifold. We define

$$\text{Fill}_\chi(M) := \min_{W \in F(M)} |\chi(W)|,$$

where $F(M)$ denotes the class of all oriented compact aspherical 4-manifolds W with π_1 -injective boundary M .

Question 3.26 (Edmonds [37, p. 3]). Does there exist an oriented closed connected 3-manifold M with $\text{Fill}_\chi(M) \neq 0$?

In the same spirit, we could ask the corresponding question for the simplicial volume:

Definition 3.27. Let M be an oriented closed connected 3-manifold. We say that M admits a small aspherical filling if there exists $W \in F(M)$ such that $\|W, M\| = 0$.

The previous definition suggests the following question:

Question 3.28. Does every oriented closed aspherical 3-manifold satisfy the following implication?

(Fill) $\pi_1(M)$ is amenable $\implies M$ admits a small aspherical filling.

Question 3.28 can be interpreted as a manifold variant of the *uniform boundary condition* (UBC) [82]. Recall that a space X satisfies UBC in dimension n if there exists a constant $K > 0$ such that every boundary $c \in \text{im } \partial_{n+1} \subset C_n(X; \mathbb{R})$ can be filled K -efficiently, i.e., there exists a chain $b \in C_{n+1}(X; \mathbb{R})$ such that $\partial_{n+1}b = c$ and $|b|_1 \leq K \cdot |c|_1$. Spaces with amenable fundamental groups satisfy UBC in all dimensions [82]. Therefore, in a similar way, Question 3.28 asks whether the small fundamental cycles of oriented closed connected 3-manifolds M with amenable fundamental group can be filled efficiently using relative fundamental cycles of 4-manifolds with M as π_1 -injective boundary.

Similar quantified bordism problems have been successfully studied in more geometric contexts [22].

Remark 3.29. Property (Fill) does *not* hold in dimension 1. Indeed, the only surfaces that have the circle as π_1 -injective boundary are hyperbolic surfaces with totally geodesic boundary. All these have uniformly positive simplicial volume (Example 3.1(2)).

Property (Fill) holds in dimension 2. The only candidate to check is the 2-torus, which is the π_1 -injective boundary of $S^1 \times (T^2)^\times$.

Our interest in Question 3.28 is motivated by the following open problem in 4-dimensional topology:

Conjecture 3.30 ([37, Conjecture 1]). *There exists an oriented closed aspherical 4-manifold M with $\chi(M) = 1$.*

The last conjecture and Question 1.1 are connected as follows:

Proposition 3.31. *Suppose that the following hold:*

- (a) *Every oriented closed aspherical 3-manifold satisfies Property (Fill);*
- (b) *All oriented closed aspherical 4-manifolds satisfy Property (SV χ).*

Then Conjecture 3.30 holds.

Proof. Edmonds [37] constructed an oriented compact aspherical 4-manifold W with non-empty π_1 -injective aspherical boundary and $\chi(W) = 1$ [37, Proposition 4.1]. Moreover, ∂W is a torus bundle over the circle [37, Proposition 4.1]. This shows that ∂W has amenable fundamental group.

Using Property (Fill), there exists an oriented compact aspherical 4-manifold W' with π_1 -injective boundary $\partial W' \cong \partial W$ (orientation-reversing) and $\|W', \partial W'\| = 0$. Moreover, by hypothesis, Property (SV χ) is satisfied both for all 3- and 4-dimensional oriented closed aspherical manifolds (Property (SV χ) is automatically satisfied in odd dimensions). Hence, Proposition 2.15 shows that $(W', \partial W')$ satisfies Property (SV χ, ∂), and so $\chi(W', \partial W') = 0$. Therefore,

$$M := W \cup_{\partial W \cong \partial W'} W'$$

is an oriented closed aspherical 4-manifold with

$$\chi(M) = \chi(W) + \chi(W', \partial W') = 1 + 0 = 1 .$$

Therefore M provides the required example for Conjecture 3.30. \square

4. SIMPLICIAL VOLUME AND COBORDISM CATEGORIES

In this section, we will introduce the amenable cobordism category and explain how the simplicial volume extends to a symmetric monoidal functor on this cobordism category. In other words, the simplicial volume defines an invertible TQFT in this restricted sense. Interestingly, it will be shown that the simplicial volume does not extend to a functor on the whole cobordism category of smooth oriented manifolds. This fact reflects the (non-)additivity properties of the simplicial volume.

Viewing the simplicial volume as an invertible TQFT will allow us to obtain some interesting information about the fundamental group of the amenable cobordism category and its variations. Specifically, we will show that this fundamental group is *not* finitely generated (Theorem 4.5). This result is based on the following computations of simplicial volume in dimension 4:

Remark 4.1. For $n \in \mathbb{N}$, let $SV(n) \subset \mathbb{R}_{\geq 0}$ denote the set of simplicial volumes of all oriented closed connected n -manifolds. Then $SV(n)$ is a countable submonoid of $(\mathbb{R}_{\geq 0}, +)$ [58, Remark 2.3].

If $n \geq 4$, then $SV(n)$ has no gap at zero [58] and thus is non-discrete. Moreover, $SV(3)$ contains the set of all volumes of oriented closed connected hyperbolic 3-manifolds (scaled by $1/v_3$) and thus is non-discrete [101]. Therefore, if $n \geq 3$, then the additive monoid $SV(n)$ is *not* finitely generated.

Moreover, $SV(4)$ contains an infinite family of values that are linearly independent over \mathbb{Q} [57].

We first consider the simpler case of the connected sum monoid and prove that it is not finitely generated (Section 4.1). In Section 4.2, we explain how to view the simplicial volume as a symmetric monoidal functor on the amenable cobordism category and use this description to deduce the non-finite generation of the fundamental group of the 4-dimensional amenable cobordism category (Theorem 4.5). Finally, we prove that the functor of simplicial volume cannot be extended to an invertible TQFT on the whole cobordism category (Proposition 4.7).

4.1. The connected sum monoid. For $n \in \mathbb{N}$, let $\text{Mfd}_n^\#$ denote the monoid, whose elements are diffeomorphism classes of oriented closed connected smooth n -manifolds and whose operation is given by connected sum. By the classification of surfaces, the monoid $\text{Mfd}_2^\#$ is generated by the 2-torus. This finite generation fails in higher dimensions:

Proposition 4.2. *Let $n \in \mathbb{N}_{\geq 3}$. Then the monoid $\text{Mfd}_n^\#$ is not finitely generated.*

Proof. As simplicial volume is additive in dimension ≥ 3 with respect to connected sums (Proposition 2.10), we can view the simplicial volume as a

monoid homomorphism

$$S: \text{Mfd}_n^\# \rightarrow \mathbb{R}_{\geq 0}$$

from $\text{Mfd}_n^\#$ to the additive monoid $(\mathbb{R}_{\geq 0}, +)$. The submonoid $S(\text{Mfd}_n^\#)$ is not finitely generated (Remark 4.1). Because finite generation is preserved by monoid homomorphisms, we conclude that $\text{Mfd}_n^\#$ is *not* finitely generated. \square

Remark 4.3. As suggested by the referee, the previous result also admits a more geometric proof: In every dimension $n \geq 3$ there exist *infinitely many* hyperbolic n -manifolds (and none of them is a non-trivial connected sum).

4.2. Simplicial volume as a TQFT. The simplicial volume can be viewed as an (invertible) TQFT defined on an appropriate cobordism category of oriented smooth manifolds. This is essentially a basic consequence of known additivity properties of the simplicial volume. For background material about cobordism categories and TQFTs, we refer the interested reader to the work of Abrams [1] and the book by Kock [68], both of which focus especially on the 2-dimensional case, and to the lecture notes of Debray, Galatius, and Palmer [34], which contain an excellent exposition of the classification of invertible TQFTs following major recent developments in the field.

4.2.1. Cobordism categories. For $d \in \mathbb{N}$, let Cob_d denote the d -dimensional (discrete) cobordism category of oriented manifolds [68, 34]. The objects of Cob_d are oriented closed smooth $(d - 1)$ -manifolds M , one from each diffeomorphism class. A morphism from M to N in Cob_d is an equivalence class of d -dimensional oriented smooth cobordisms $(W; \partial_{\text{in}}W, \partial_{\text{out}}W)$ equipped with orientation-preserving diffeomorphisms $-M \xrightarrow{\cong} \partial_{\text{in}}W$ (incoming boundary) and $N \xrightarrow{\cong} \partial_{\text{out}}W$ (outgoing boundary). The equivalence relation is given by orientation-preserving diffeomorphisms that preserve the boundary pointwise. Composition of morphisms in Cob_d is given by glueing of cobordisms, using the given identifications of the boundary components. The category Cob_d is a symmetric monoidal category under the operation of disjoint union.

4.2.2. Amenability conditions. Let G be a class of groups that is closed under isomorphisms. We consider the subcategory $\text{Cob}_d^G \subset \text{Cob}_d$ defined as follows: The objects are those manifolds with fundamental group in G (for each component). The morphisms are the cobordisms $(W; M, N)$ such that $M \hookrightarrow W$ and $N \hookrightarrow W$ are π_1 -injective (for all components). It should be noted that Cob_d^G is indeed a subcategory of Cob_d , i.e., that Cob_d^G is closed under composition. To see this, we only need to check that the π_1 -injectivity of the boundary components is preserved under composition of cobordisms. This can be shown inductively by glueing one pair of components at a time and applying the Seifert–van Kampen theorem as well as

the normal form theorems for amalgamated free products and HNN extensions [96, Chapter 11]. These guarantee at each stage that the remaining boundary components are π_1 -injective in the resulting manifold.

The symmetric monoidal pairing of Cob_d clearly restricts to a symmetric monoidal pairing on Cob_d^G . When $G = \text{Am}$ is the class of all amenable groups, we will refer to Cob_d^{Am} as the *amenable cobordism category*.

4.2.3. Simplicial volume as a TQFT on the amenable cobordism category. Let $\mathbb{R} = (\mathbb{R}, +)$ denote the additive (abelian) group of real numbers, regarded as a symmetric monoidal groupoid with one object. Moreover, let G be a class of amenable groups that is closed under isomorphisms. The additivity of the simplicial volume with respect to amenable glueings (Theorem 2.5) and disjoint union shows that we obtain a symmetric monoidal functor with values in the abelian group \mathbb{R} (regarded as a symmetric monoidal category):

$$\|-\|: \text{Cob}_d^G \rightarrow \mathbb{R}, (W; M, N) \mapsto \|W, \partial W\|.$$

In other words, the simplicial volume defines a TQFT on Cob_d^G . Because this TQFT takes values in an abelian group (hence Picard groupoid), it is invertible.

4.2.4. The fundamental group of $B\text{Cob}_d^G$. Let $B\text{Cob}_d^G$ denote the classifying space of the cobordism category Cob_d^G . An object M of Cob_d^G determines a point (0-simplex) $[M] \in B\text{Cob}_d^G$ and we denote by $\Omega_M B\text{Cob}_d^G$ the loop space of the classifying space $B\text{Cob}_d^G$ based at $[M]$. Note that the monoid of path-components of $B\text{Cob}_d^G$ is a group (similarly to $B\text{Cob}_d$). Thus, $B\text{Cob}_d^G$ is an infinite loop space, therefore, all of its path components have the same homotopy type. After passing to the classifying spaces, the functor $\|-\|$ induces a group homomorphism:

$$\varphi_M: \pi_1(B\text{Cob}_d^G, [M]) \cong \pi_0(\Omega_M B\text{Cob}_d^G) \rightarrow \pi_0(\Omega B\mathbb{R}) \cong \mathbb{R}.$$

Here, we have used the homotopy equivalence $\Omega B\Gamma \simeq \Gamma$ for groups Γ . The group homomorphisms φ_M (for all basepoints M) uniquely determine the functor $\|-\|$; similar facts hold more generally for functors whose target is a groupoid (see, for example, [34]).

Remark 4.4. Every endomorphism $(W; M, M)$ in Cob_d^G defines an element $[W] \in \pi_0(\Omega_M B\text{Cob}_d^G)$ whose image under the group homomorphism φ_M is the relative simplicial volume $\|W, \partial W\|$. In particular, if $[W] = [W']$, then $\|W, \partial W\| = \|W', \partial W'\|$.

Theorem 4.5. *Let $G \subset \text{Am}$ be a class of groups that is closed under isomorphisms and let M be an object of Cob_4^G . Then the group $\pi_1(B\text{Cob}_4^G, [M])$ is not finitely generated.*

Proof. The relative simplicial volume of 4-manifolds induces a group homomorphism

$$S: \pi_1(B\text{Cob}_4^G, [\emptyset]) \rightarrow \mathbb{R}.$$

The image of this group homomorphism contains the subset $SV(4)$ (Remark 4.4), which contains an infinite family of elements that are linearly independent over \mathbb{Q} (Remark 4.1). Therefore, the abelian group in S is not finitely generated and so $\pi_1(B \text{Cob}_4^G, [\emptyset])$ is not finitely generated.

As explained above, $\pi_1(B \text{Cob}_4^G, [M])$ is independent of the choice of basepoint $[M]$, so the result follows. \square

Remark 4.6. We also expect corresponding results in higher dimensions. However, currently, not enough is known about the structure of $SV(d)$ for $d \geq 5$.

4.2.5. *Non-extendability to Cob_d .* Since the simplicial volume does not satisfy additivity in general [45, Remark 7.9], it does not define a functor on Cob_d . However, it is still interesting to ask whether there might be a different extension of the simplicial volume to general oriented compact manifolds with boundary which is always additive. This question is closely related to the problem of extending the functor $\|-\|: \text{Cob}_d^G \rightarrow \mathbb{R}$ to the whole cobordism category Cob_d . Based on the classification of functors with values in a groupoid (see, e.g., [34]), this problem is essentially equivalent to the question of extending the homomorphism φ_\emptyset to $\pi_1(B \text{Cob}_d, \emptyset)$.

In contrast to $\pi_1(B \text{Cob}_4^G, [M])$ (Theorem 4.5), the fundamental group of the d -dimensional cobordism category $\pi_1(B \text{Cob}_d, \emptyset)$ is well known and simpler to describe. We first note that it agrees with the fundamental group of the standard *topologised* cobordism category \mathcal{C}_d [34, Section 2.4]. This is again independent of the choice of basepoint and can be identified with the Reinhart bordism group \mathfrak{R}_d [95][36, Appendix A]. We recall that \mathfrak{R}_d can be described as the group of equivalence classes of oriented closed d -manifolds where the equivalence relation is defined by cobordisms whose tangent bundle is equipped with a nowhere-vanishing vector field that extends the normal fields on the boundary components. This refined bordism group is known to be a split extension of the usual oriented bordism group Ω_d^{SO} by a cyclic group whose generator is represented by the d -sphere. More precisely, there is a split exact sequence:

$$(*) \quad 0 \rightarrow \mathbb{Z}/\text{Eul}_{d+1} \xrightarrow{[1] \mapsto [S^d]} \mathfrak{R}_d \rightarrow \Omega_d^{\text{SO}} \rightarrow 0$$

where $\text{Eul}_n = \{0\}$ if n is odd, $\text{Eul}_n = 2\mathbb{Z}$ if $n \equiv 2 \pmod{4}$, and $\text{Eul}_n = \mathbb{Z}$ if n is a multiple of 4. We refer to the literature [95][36, Appendix A][9] for the properties of the bordism group \mathfrak{R}_d and the description of the homotopy groups of $B\mathcal{C}_d$ in terms of bordism classes.

Using this description of $\mathfrak{R}_d \cong \pi_1(B \text{Cob}_d, [\emptyset])$, we conclude below that the simplicial volume of oriented closed d -manifolds cannot be extended to a functor on the cobordism category Cob_d , i.e., there is no additive extension of the simplicial volume $\|-\|$ (analogous to Theorem 2.5) to all oriented compact d -manifolds.

Let \mathcal{M}_d denote the monoid of endomorphisms of \emptyset in Cob_d , that is, the monoid of diffeomorphism classes of oriented closed d -manifolds under the operation of disjoint union.

Proposition 4.7. *Let $d \geq 2$.*

- (1) *There is no functor $\text{Cob}_d \rightarrow \mathbb{R}$ that extends the restriction of the simplicial volume $\|-\|_{|\mathcal{M}_d}: \mathcal{M}_d \rightarrow \mathbb{R}$ to oriented closed d -manifolds.*
- (2) *Let $G \subset \text{Am}$ be a class of groups that is closed under isomorphisms. The functor $\|-\|: \text{Cob}_d^G \rightarrow \mathbb{R}$ does not admit an extension to a functor on Cob_d .*

Proof. For (1), note that such an extension of $\text{Cob}_d \rightarrow \mathbb{R}$ would imply a factorization of $\|-\|_{|\mathcal{M}_d}: \mathcal{M}_d \rightarrow \mathbb{R}$ through $\mathfrak{R}_d \cong \pi_1(B\text{Cob}_d, [\emptyset])$ (see Remark 4.4). In particular, this would imply that $\|-\|$ is invariant under the Reinhart bordism relation. Moreover, since $\|S^d\| = 0$, it would further follow from the exact sequence (*) that $\|-\|$ is invariant under oriented bordism. This is obviously false in general, e.g., note that $M \sqcup (-M)$ is null-bordant as oriented closed d -manifold, but its simplicial volume is non-trivial in general. Claim (2) follows directly from (1). \square

Remark 4.8. The fact that $\|-\|$ is not invariant under oriented bordism can also be shown as follows. Note that for $d \in \{2, 3\}$, this fails because $\|-\|$ is non-trivial but $\Omega_d^{\text{SO}} \cong 0$. Then the result follows in all dimensions by taking suitable products. We note also that for $d = 4$, this property can be shown to fail also because the oriented bordism group $\Omega_4^{\text{SO}} \cong \mathbb{Z}$ is finitely generated, whereas $\text{SV}(4)$ is not finitely generated by Remark 4.1.

Remark 4.9. In contrast to simplicial volume, the (relative) Euler characteristic defines a (symmetric monoidal) functor $\chi: \text{Cob}_d \rightarrow \mathbb{Z}$ (invertible TQFT), which sends $(W; M, N)$ to $\chi(W, M)$. Indeed, the Euler characteristic is invariant under the Reinhart bordism relation [95].

5. ASPHERICALISATIONS

The construction of aspherical closed manifolds with vanishing simplicial volume is a key problem for Question 1.1. There are several known constructions of aspherical closed manifolds from non-aspherical or non-closed manifolds. Important examples of such constructions are Davis' reflection group trick [27, 28, 30] and Gromov's hyperbolization [52, 32, 33]. The general difficulty with using these constructions to obtain (counter)examples to Question 1.1 has to do with the difficulty of computing the simplicial volume of the resulting aspherical closed manifolds.

In this section, we consider extensions of the class of aspherical closed manifolds and look for interesting (counter)examples in these contexts. In particular, we will prove that the class of aspherical spaces that are homology equivalent to closed manifolds, as well as the class of closed manifolds that are homology equivalent to an aspherical space, do not satisfy Property (SV χ) in general (Theorem 5.7). We introduce the simplicial volume

of such spaces in Section 5.1. The proof of Theorem 5.7 will be given in Section 5.3; the proof is based on the Kan–Thurston theorem [64], which we recall in Section 5.2. Finally, we end with some brief comments on known constructions of aspherical closed manifolds and their possible connections with Question 1.1 (Section 5.4). Besides their independent interest, we hope that the results of this section, especially, combined with the aforementioned constructions of aspherical closed manifolds, might provide useful tools for promoting non-aspherical or non-closed examples to closed aspherical (counter)examples to Question 1.1.

5.1. Simplicial volume of spaces homology equivalent to manifolds.

Our goal in this section is to extend the definition of simplicial volume to spaces that are only homology equivalent to an oriented closed manifold and discuss the main properties of this invariant. This is motivated by the following basic observation:

Remark 5.1. Suppose that M is an oriented closed connected manifold with $\|M\| = 0$. Let $f: M \rightarrow N$ be a homology equivalence to an oriented closed connected manifold N ; in particular, this map has degree ± 1 and so $\|N\| = 0$ (this conclusion holds more generally if the degree of f is non-zero). Moreover, because f is a homology equivalence, it follows that $\chi(M) = \chi(N)$. In this sense, Property (SV χ) is inherited under homology equivalences between oriented closed connected manifolds.

Thus, in connection with Question 1.1, it would be interesting to understand the class of manifolds which are homology equivalent to an oriented closed aspherical manifold with vanishing simplicial volume.

Definition 5.2. Let X be a topological space, let M be an oriented closed connected n -manifold and let $f: X \rightarrow M$ be an integral homology equivalence. We define the (\mathbb{R} -)fundamental class of (X, f) by

$$[X]_f := H_n(f; \mathbb{R})^{-1}([M]) \in H_n(X; \mathbb{R})$$

and the *simplicial volume of X* by

$$\|X\| := \|[X]_f\|_1 \in \mathbb{R}_{\geq 0}.$$

Remark 5.3. The simplicial volume of such spaces is well-defined in the following sense. Let (X, M^n, f) be as above. In particular, $H_k(X; \mathbb{Z})$ vanishes for $k > n$ and $H_n(X; \mathbb{Z}) \cong H_n(M; \mathbb{Z}) \cong \mathbb{Z}$. Therefore, $H_n(f; \mathbb{Z})^{-1}([M]_{\mathbb{Z}})$ is one of the two generators of $H_n(X; \mathbb{Z})$, which only differ by a sign. In particular, the \mathbb{R} -fundamental class of (X, f) is independent of M and f up to sign. Therefore, the simplicial volume of X is independent of the choice of M and f . Clearly the definition of $\|X\|$ applies more generally whenever the map $f: X \rightarrow M$ induces an isomorphism on $H_n(-; \mathbb{Z})$.

Remark 5.4 (Degree estimate). Let (X, M, f) and (Y, N, g) be as in Definition 5.2, where M and N are oriented closed connected manifolds of the same dimension n . If $h: X \rightarrow Y$ is a continuous map, then the *unsigned*

homological degree $|\deg h|$ is defined to be the unique natural number $d \in \mathbb{N}$ with

$$H_n(h; \mathbb{R})[X]_f = \pm d \cdot [Y]_g \in H_n(Y; \mathbb{R}).$$

As in the manifold case, we clearly have

$$|\deg h| \cdot \|Y\| \leq \|X\|$$

and it follows that the simplicial volume of X is homotopy invariant. Moreover, if X admits a self-map $h: X \rightarrow X$ with $|\deg h| \geq 2$, then $\|X\| = 0$. Furthermore, if $h: X \rightarrow Y$ is a (finite) covering map, then $|\deg h| \cdot \|Y\| = \|X\|$, as can be seen from the same argument as in the manifold case.

This extension of the simplicial volume to a homotopy invariant of spaces that are only homology equivalent to an oriented closed manifold should not be confused with the fact that the simplicial volume is *not* invariant under homology equivalences:

Example 5.5. There exist oriented closed connected non-positively curved (and hence aspherical) homology 4-spheres M [94]; in particular, M is homology equivalent to S^4 and a result by Fujiwara and Manning [48, Corollary 2.5] shows that $\|M\| > 0 = \|S^4\|$.

5.2. Acyclic maps and plus constructions. We review briefly the definition and basic properties of acyclic maps and refer to the literature [56, 92] for more details. A map $f: X \rightarrow Y$ is *acyclic* if the induced homomorphism

$$H_*(f; \mathcal{A}): H_*(X; f^*\mathcal{A}) \rightarrow H_*(Y; \mathcal{A})$$

is an isomorphism for every local coefficient system \mathcal{A} of abelian groups on Y ; in particular, f induces isomorphisms on singular homology and cohomology with both integral and real coefficients. Equivalently, a map $f: X \rightarrow Y$ is acyclic if its homotopy fibers have trivial integral homology. Every acyclic map $f: X \rightarrow Y$ between path-connected based spaces arises up to weak homotopy equivalence as the plus construction $\iota_P: X \rightarrow X_P^+$ with respect to a normal perfect subgroup $P \trianglelefteq \pi_1(X)$. In this case, we have

$$\pi_1(X_P^+) \cong \pi_1(X)/P.$$

Theorem 5.6 (Kan–Thurston [64, 5, 83]). *For every path-connected based topological space X , there is a group G_X together with an acyclic (based) map $f_X: K(G_X, 1) \rightarrow X$. Moreover, G_X and f_X can be chosen to be natural in X .*

Proof. The original functorial construction of $(G_X, f_X: K(G_X, 1) \rightarrow X)$ is due to Kan and Thurston [64]. Alternative constructions and refinements were obtained by Baumslag, Dyer, Heller [5], and Maunder [83]. (These constructions are also shown to preserve properties of homotopy finiteness, but they satisfy weaker functoriality properties in general.) \square

5.3. Using the Kan–Thurston theorem. The Kan–Thurston theorem (Theorem 5.6) has the following consequence in connection with Question 1.1.

Theorem 5.7. *Let $n \in \mathbb{N}_{\geq 2}$ be even.*

- (1) *There exist aspherical CW-complexes X that admit an acyclic map $X \rightarrow M$ to an oriented closed connected n -manifold M , and satisfy both $\|X\| = 0$ and $\chi(X) \neq 0$. In particular, these aspherical spaces do not satisfy Property (SV χ).*
- (2) *There exist oriented closed connected n -manifolds M that admit an acyclic map $X \rightarrow M$ from an aspherical CW-complex X , and satisfy both $\|M\| = 0$ and $\chi(M) \neq 0$. In particular, these manifolds do not satisfy Property (SV χ).*

Proof. Let M be an oriented closed connected n -manifold that has a (based) self-map $h: M \rightarrow M$ with $|\deg h| \geq 2$ and satisfies $\chi(M) \neq 0$. For example, as n is even, we may choose $M = S^n$.

Ad 1. By Theorem 5.6 (and the functoriality of the construction), there exists an aspherical CW-complex X with an acyclic map $f: X \rightarrow M$ and a map $H: X \rightarrow X$ that makes the following square commutative:

$$\begin{array}{ccc} X & \xrightarrow{H} & X \\ f \downarrow & & \downarrow f \\ M & \xrightarrow{h} & M. \end{array}$$

It follows that $|\deg H| = |\deg h| \geq 2$. Thus, $\|X\| = 0$ (Remark 5.4). Moreover, as f is a homology equivalence, it follows that $\chi(X) = \chi(M) \neq 0$.

Ad 2. It is sufficient to apply Theorem 5.6 to M . \square

We give some further context on possible improvements of Theorem 5.7:

Remark 5.8 (Poincaré duality). Hausmann [55] proved that the group G_X in the Kan–Thurston theorem (Theorem 5.6) can be chosen to be a duality group when X is homotopy finite. (A related interesting refinement of the Kan–Thurston theorem has also been obtained more recently by R. Kim [66].) We do not know whether we can obtain homotopy finite examples in Theorem 5.7 and whether Hausmann’s construction can also be made sufficiently functorial for the purpose of the proof above. Thus, it remains open whether Theorem 5.7 can be strengthened to produce examples where the fundamental group is a duality group.

We note that it remains an open problem whether every finitely presented Poincaré duality group is the fundamental group of a closed aspherical manifold [31]. In this connection, we also recall the following question [64, Question (ii) p. 254]: Is every oriented closed connected n -manifold, $n \geq 4$, homology equivalent to an oriented closed aspherical manifold? If this question has an affirmative answer, then there exist aspherical homology spheres

in all high dimensions. This would contradict a version of the Hopf conjecture, which claims that the Euler characteristic of every oriented closed aspherical $2k$ -manifold is either zero or its sign is $(-1)^k$ [29].

5.4. Further comments. Davis' reflection group trick [27, 28, 30] takes an oriented compact aspherical n -manifold $(W, \partial W)$ and constructs an oriented *closed* aspherical n -manifold M by reflecting W along so-called *pieces* of the boundary of W . The construction also yields a retraction

$$r: M \rightarrow W$$

i.e., $r \circ i = \text{id}_W$, where $i: W \rightarrow M$ is the inclusion. In particular, $\pi_1(r)$ and $H_*(r)$ are epimorphisms; $H_b^*(r)$ and $H^*(r)$ are monomorphisms. Starting from “exotic” W , this method can be used to construct “exotic” closed aspherical manifolds. Note that we may choose W to be an oriented compact manifold that is homotopy equivalent to X as in Theorem 5.7(1), assuming that the space X can also be chosen to be homotopy finite [5].

There exists an explicit formula for computing the Euler characteristic of the manifold M in terms of the Euler characteristic of the input manifold W and the combinatorics of the pieces of ∂W [28, p. 218]. However, in the case of the simplicial volume, the situation is more delicate. The Davis reflection group trick can be viewed as a refined version of doubling manifolds with boundary, where the refinement is given by the combinatorics of the pieces of ∂W . In order to generalise Example 2.8 to this setting, it would be desirable to find input manifolds W with $\|W, \partial W\| = 0$ and where additionally the simplicial volume of W can be realised by small relative fundamental cycles, whose behaviour on ∂W is adapted to the combinatorics on the pieces of ∂W . Particularly interesting input candidates would be the examples from Remark 2.16 or Proposition 3.20.

On the other hand, Gromov's hyperbolization [52, 32, 33] is a construction that takes an oriented closed connected triangulated manifold N and produces an oriented closed aspherical manifold $h(N)$ together with a degree 1 map

$$c: h(N) \rightarrow N.$$

In particular,

$$\|h(N)\| \geq \|N\| .$$

In addition, $h(N)$ is a smooth manifold if N is smooth, and h preserves the stable tangent bundle, i.e., the vector bundles $T(h(N))$ and c^*TN are stably isomorphic. This implies that hyperbolization preserves the characteristic classes and numbers of closed smooth manifolds. Also, $h(N)$ and N are (oriented) cobordant. Since the mod 2 Euler characteristic of N is determined by the bordism class of N , it is natural to consider the hyperbolization in connection with the following weak version of Question 1.1:

Question 5.9. Let M be an oriented closed aspherical manifold. Does the following implication hold?

$$(\text{SV}_\chi(\text{mod}2)) \quad \|M\| = 0 \implies \chi(M) \text{ is even?}$$

Assuming that M is smooth, the property “ $\chi(M)$ is even” is equivalent to the vanishing of the top Stiefel-Whitney class of M .

It would be interesting to find N as above with the property $\|h(N)\| = 0$. We note here that the simplicial volume is always positive in the case of *strict* hyperbolization (in the sense of Charney and Davis [23]). A relative version of this construction, which might still be relevant in connection with Question 2.13, has also been studied by Belegradek [6].

6. STABLE INTEGRAL SIMPLICIAL VOLUME

Stable integral simplicial volume and integral foliated simplicial volume are versions of the simplicial volume that admit Poincaré duality estimates for Betti numbers and for the Euler characteristic.

In this section, we recall definitions, basic properties, and known examples of stable integral simplicial volume, with a focus on the relative case and the connection with Property $(\text{SV}_\chi, \partial)$. Moreover, we quickly outline the relation with the integral foliated simplicial volume.

Definition 6.1 (Stable integral simplicial volume). Let $(M, \partial M)$ be an oriented compact connected manifold M with (possibly empty) boundary ∂M . The *stable integral simplicial volume* of $(M, \partial M)$ is defined as

$$\|M, \partial M\|_{\mathbb{Z}}^{\infty} := \inf \left\{ \frac{\|N, \partial N\|_{\mathbb{Z}}}{|\deg f|} \mid (N, f) \in C(M) \right\},$$

where $C(M)$ denotes the class of all finite (connected) coverings of M .

6.1. Estimates for the Betti numbers and the Euler characteristic.

The key observation is that Poincaré duality leads to Betti number estimates for simplicial volumes with respect to sufficiently integral coefficient rings:

Proposition 6.2 ([80, Example 14.28][54, p. 307][67, Proposition 3.2]). *Let R be a normed principal ideal domain with $|x| \geq 1$ for all $x \in R \setminus \{0\}$. Let $(M, \partial M)$ be an oriented compact connected n -manifold with (possibly empty) boundary ∂M . Then, for all $k \in \mathbb{N}$,*

$$b_k(M; R) \leq \|M, \partial M\|_R$$

In particular, $|\chi(M, \partial M)| = |\chi(M)| \leq (n+1) \cdot \|M, \partial M\|_R$.

Since the Euler characteristic is multiplicative with respect to finite coverings, this estimate also implies corresponding estimates for the stable integral simplicial volume [43, Proposition 6.1]:

Corollary 6.3. *Let $(M, \partial M)$ be an oriented compact connected n -manifold with (possibly empty) boundary ∂M . Then*

$$|\chi(M, \partial M)| \leq (n+1) \cdot \|M, \partial M\|_{\mathbb{Z}}^{\infty}.$$

6.2. Integral approximation problems. This estimate for the Euler characteristic (Corollary 6.3) suggests the following question:

Question 6.4. Let M be an oriented closed aspherical manifold with residually finite fundamental group. Does the following implication hold?

$$(SV_{\mathbb{Z}}^{\infty}) \quad \|M\| = 0 \implies \|M\|_{\mathbb{Z}}^{\infty} = 0.$$

The corresponding approximation question for non-zero values in general has a negative answer. For example, oriented closed connected hyperbolic manifolds M of dimension at least 4 satisfy $\|M\| < \|M\|_{\mathbb{Z}}^{\infty}$ [43]. Moreover, Proposition 2.10 and Corollary 6.3 show that Property $(SV_{\mathbb{Z}}^{\infty})$ does not hold for the connected sum of oriented closed aspherical manifolds with zero simplicial volume and zero Euler characteristic (of even dimension at least 4). However, these manifolds are never aspherical (Remark 2.11).

In Question 6.4, one usually adds the hypothesis of *residual finiteness* to ensure the existence of “enough” finite coverings. However, there are also no known examples of oriented closed aspherical manifolds with non-residually-finite fundamental group such that the vanishing behaviour of the ordinary simplicial volume is different from that of the stable integral simplicial volume. One possible strategy to produce such examples is to use Davis’ reflection group trick (Section 5.4) to construct oriented closed aspherical manifolds whose fundamental group is not residually finite. However, as explained in Section 5.4, it seems to be difficult to gain enough control on the (stable integral) simplicial volume when performing this construction.

What about Question 6.4 for manifolds with boundary? Similarly to Property (SV_{χ}, ∂) , also in the case of the stable integral simplicial volume, we need to impose additional boundary conditions:

Example 6.5. Let M be the product of three punctured tori as in Example 3.21. Then M is aspherical and

$$\|M, \partial M\| = 0 \quad \text{and} \quad \chi(M, \partial M) = -1.$$

Hence, Corollary 6.3 implies that $\|M, \partial M\|_{\mathbb{Z}}^{\infty} \neq 0$, even though $\pi_1(M)$ is residually finite.

Question 6.6. Let M be an oriented compact aspherical manifold with residually finite fundamental group and non-empty π_1 -injective aspherical boundary ∂M . Does the following implication hold?

$$(SV_{\mathbb{Z}}^{\infty}, \partial) \quad \|M, \partial M\| = 0 \implies \|M, \partial M\|_{\mathbb{Z}}^{\infty} = 0.$$

We observe the following diagram of implications as a consequence of Corollary 6.3 and Proposition 2.15:

$$\begin{array}{ccc} \text{Property } (SV_{\chi}) & \implies & \text{Property } (SV_{\chi}, \partial) \\ \uparrow & & \uparrow \\ \text{Property } (SV_{\mathbb{Z}}^{\infty}) & & \text{Property } (SV_{\mathbb{Z}}^{\infty}, \partial) \end{array}$$

We do not know whether the diagram can be completed with a lower horizontal implication. The main issue is the lack of suitably general additivity results concerning the integral simplicial volume. For example, given an oriented compact aspherical manifold M with residually finite fundamental group and non-empty π_1 -injective aspherical boundary, it is not clear what the vanishing of $\|D(M)\|_{\mathbb{Z}}^{\infty}$ has to say about the vanishing of $\|M, \partial M\|_{\mathbb{Z}}^{\infty}$.

Conversely, however, the vanishing of $\|M, \partial M\|_{\mathbb{Z}}^{\infty}$ implies the vanishing of $\|D(M)\|_{\mathbb{Z}}^{\infty}$:

Proposition 6.7. *Let $(M, \partial M)$ be an oriented compact connected n -manifold with non-empty boundary ∂M . Then:*

- (1) $\|M, \partial M\|_{\mathbb{Z}}^{\infty} \geq \|\partial M\|_{\mathbb{Z}}^{\infty} / (n + 1)$;
- (2) $\|D(M)\|_{\mathbb{Z}}^{\infty} \leq 2 \cdot \|M, \partial M\|_{\mathbb{Z}}^{\infty}$.

Proof. *Ad 1.* First recall that $\|N, \partial N\|_{\mathbb{Z}} \geq \|\partial N\|_{\mathbb{Z}} / (n + 1)$ for every oriented compact connected n -manifold N with boundary ∂N (Remark 2.4). Suppose that $\|M, \partial M\|_{\mathbb{Z}}^{\infty} = T \in \mathbb{R}_{>0}$, then for every $\varepsilon > 0$ there exists an oriented compact connected finite covering N_{ε} of degree d_{ε} such that

$$T \leq \frac{\|N_{\varepsilon}, \partial N_{\varepsilon}\|_{\mathbb{Z}}}{d_{\varepsilon}} < T + \varepsilon .$$

Hence, we also have

$$\frac{1}{n + 1} \cdot \frac{\|\partial N_{\varepsilon}\|_{\mathbb{Z}}}{d_{\varepsilon}} < T + \varepsilon .$$

Notice that the boundary ∂N_{ε} might consist of several different connected components S_1, \dots, S_k that cover ∂M with degrees d_1, \dots, d_k , respectively, such that $d_{\varepsilon} = \sum_{i=1}^k d_i$. By the pigeonhole principle, there exists a $j \in \{1, \dots, k\}$ with

$$\frac{1}{n + 1} \cdot \frac{\|S_j\|_{\mathbb{Z}}}{d_j} < T + \varepsilon ,$$

because $\sum_{i=1}^k d_i / d_{\varepsilon} = 1$ and

$$T + \varepsilon > \frac{1}{n + 1} \cdot \frac{\|\partial N_{\varepsilon}\|_{\mathbb{Z}}}{d_{\varepsilon}} = \sum_{i=1}^k \frac{1}{n + 1} \cdot \frac{\|S_i\|_{\mathbb{Z}}}{d_{\varepsilon}} = \sum_{i=1}^k \frac{d_i}{d_{\varepsilon}} \cdot \frac{1}{n + 1} \cdot \frac{\|S_i\|_{\mathbb{Z}}}{d_i} .$$

This shows that $\|\partial M\|_{\mathbb{Z}}^{\infty} / (n + 1) \leq T + \varepsilon$. Letting $\varepsilon \rightarrow 0$ proves the claim.

Ad 2. If $(N, \partial N) \rightarrow (M, \partial M)$ is a finite covering of degree d , then the induced map $D(N) \rightarrow D(M)$ between the doubles is also a finite covering of degree d . Moreover, by reflecting fundamental cycles (Example 2.8), we have $\|D(N)\|_{\mathbb{Z}} \leq 2 \cdot \|N, \partial N\|_{\mathbb{Z}}$. \square

Remark 6.8. Let $(M, \partial M)$ be an oriented compact connected manifold with (possibly empty) boundary. Then Proposition 6.7 gives an alternative way to derive an Euler characteristic estimate for $\|M, \partial M\|_{\mathbb{Z}}^{\infty}$ from the closed case. First, suppose that $n := \dim M$ is even. In this case, $\chi(D(M)) =$

$2 \cdot \chi(M, \partial M)$. Hence, the closed case of Corollary 6.3 and the second part of Proposition 6.7 show that

$$2 \cdot |\chi(M, \partial M)| = |\chi(D(M))| \leq (n+1) \cdot \|D(M)\|_{\mathbb{Z}}^{\infty} \leq 2 \cdot (n+1) \cdot \|M, \partial M\|_{\mathbb{Z}}^{\infty}.$$

Suppose now that M has odd dimension n . Then, we know that $\chi(\partial M) = 2 \cdot \chi(M)$. Hence, by the closed case of Corollary 6.3 and the first part of Proposition 6.7, we have

$$\begin{aligned} 2 \cdot |\chi(M, \partial M)| &= 2 \cdot |\chi(M) - \chi(\partial M)| = |\chi(\partial M)| \\ &\leq n \cdot \|\partial M\|_{\mathbb{Z}}^{\infty} \leq n \cdot (n+1) \cdot \|M, \partial M\|_{\mathbb{Z}}^{\infty}. \end{aligned}$$

Example 6.9. For the following manifolds, the stable integral simplicial volume equals the classical simplicial volume; in particular, these examples satisfy Property $(SV_{\mathbb{Z}}^{\infty})$ or Property $(SV_{\mathbb{Z}}^{\infty}, \partial)$, respectively:

- (1) All oriented compact aspherical surfaces [51, 67];
- (2) All oriented compact aspherical 3-manifolds with toroidal (or empty) boundary [40, 41];
- (3) All oriented closed connected generalised aspherical graph manifolds with residually finite fundamental group [40];
- (4) All oriented closed aspherical manifolds with residually finite amenable fundamental group [46];
- (5) All oriented compact aspherical smooth manifolds with residually finite fundamental group admitting a non-trivial smooth S^1 -action [39];
- (6) All oriented compact aspherical smooth manifolds with residually finite fundamental group admitting an F -structure [77];
- (7) All oriented closed aspherical smooth manifolds with residually finite fundamental group admitting a regular circle foliation with finite holonomy groups [20];
- (8) Every oriented closed aspherical manifold M with residually finite fundamental group and $\text{cat}_{\text{Am}}(M) \leq \dim(M)$ [77]. This applies e.g. to manifolds that are the total space of a fibre bundle $M \rightarrow B$ with oriented closed connected fiber N such that $\text{cat}_{\text{Am}}(N) \leq \dim(M)/(\dim(B)+1)$ and to manifolds of dimension $n \geq 4$ whose fundamental group Γ contains an amenable normal subgroup A with $\text{cd}_{\mathbb{Z}}(\Gamma/A) < n$.

Remark 6.10. The arguments discussed in this section can be extended to principal ideal domains with norm bounded from below by 1. Interesting examples of such rings include, for example, the finite fields with the trivial norm [75]. In this setting, the proof of Proposition 6.7 also applies verbatim.

6.3. The dynamical version. The Poincaré duality arguments also apply to the dynamical version of the (integral) simplicial volume, i.e., to the integral foliated simplicial volume:

Definition 6.11 (Integral foliated simplicial volume [54, 98, 40]). Let M be an oriented compact connected n -manifold with (possibly empty) boundary ∂M .

- If $\alpha = \pi_1(M) \curvearrowright (X, \mu)$ is a probability measure-preserving action on a standard Borel probability space, then we set

$$|M, \partial M|^\alpha := \|[M, \partial M]^\alpha\|_1^\alpha,$$

where $[M, \partial M]^\alpha \in H_n(M, \partial M; L^\infty(X; \mathbb{Z}))$ denotes the image of the usual fundamental class $[M, \partial M]_{\mathbb{Z}}$ under the inclusion of \mathbb{Z} into the *twisted* coefficient module $L^\infty(X; \mathbb{Z})$. The norm $|\cdot|_1^\alpha$ on the twisted chain complex is taken with respect to the L^1 -norm on $L^\infty(X; \mathbb{Z})$.

- The *(relative) integral foliated simplicial volume* of $(M, \partial M)$ is defined as

$$|M, \partial M| := \inf_{\alpha \in A(\pi_1(M))} |M, \partial M|^\alpha,$$

where $A(\pi_1(M))$ denotes the class of all probability measure-preserving $\pi_1(M)$ -actions on standard Borel probability spaces.

Remark 6.12. For technical reasons, in the setting of manifolds with boundary, it is recommended to work with actions of the fundamental groupoid instead of the fundamental group [40, 41].

Proposition 6.13. *Let $(M, \partial M)$ be an oriented compact connected n -manifold with (possibly empty) boundary ∂M . Then, for all $k \in \mathbb{N}$,*

$$b_k^{(2)}(M) \leq |M, \partial M|.$$

In particular, $|\chi(M, \partial M)| = |\chi(M)| \leq (n+1) \cdot |M, \partial M|$.

Proof. This is a relative version of the L^2 -Betti number estimate which was shown in the closed case by Schmidt [98] based on ideas of Gromov [54, p. 306]. When phrasing the proof in terms of L^2 -Betti numbers of standard equivalence relations, literally the same proof as in the closed case [74, Theorem 6.4.5] can be applied to the twisted Poincaré–Lefschetz duality isomorphism. \square

Many positive examples for Question 1.1 have been established using the integral foliated simplicial volume (Section 1.3) and most of the known computations of the stable integral simplicial volume are based on ergodic theoretic methods and the fact that

$$\|M, \partial M\|_{\mathbb{Z}}^\infty = |M, \partial M|^{\widehat{\pi_1(M)}}$$

holds [41, Proposition 2.12][78, 46], where $\widehat{\pi_1(M)}$ denotes the dynamical system given by the profinite completion of $\pi_1(M)$.

A summary of computations of integral foliated simplicial volume and of these ergodic theoretic methods can be found in the literature [74, Chapter 6].

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