A comment on the structure of graded modules over graded principal ideal domains in the context of persistent homology

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Abstract

The literature in persistent homology often refers to a "structure theorem for finitely generated graded modules over a graded principal ideal domain". We clarify the nature of this structure theorem in this context.

1 Introduction

The persistent homology with field coefficients of finite type filtrations can be described in terms of barcodes. Zomorodian and Carlsson promoted the elegant idea to view persistent homology with coefficients in a field K as a graded module over the graded polynomial ring $K[T]$ [ZC05]. They then suggest a general structure theorem for finitely generated graded modules over graded principal ideal domains [ZC05, Theorem 2.1]. Applying this structure theorem to the graded polynomial ring $K[T]$ gives a graded elementary divisor decomposition of persistent homology, which can be reinterpreted as barcodes [CZCG04] or, equivalently, as persistence diagrams [EH10].

However, there does not seem to be a proof of this general structure theorem in the literature in the form stated by Zomorodian and Carlsson. As this theorem is quoted multiple times in work on persistent homology and as it is a potential source of confusion, the goal of this expository note is to clarify the nature of this structure theorem (even though it might be clear to the experts).

We first give a precise formulation of the structure theorem; this formulation slightly differs from the statement of Zomorodian and Carlsson [ZC05, Theorem 2.1] (for a reason explained below):

Theorem 1.1 (structure theorem for graded modules over graded PIDs)**.** *Let R be a graded principal ideal domain with $R \neq R_0$ and let M be a finitely generated graded R-module. Then M admits a graded elementary divisor decomposition (Definition 2.8) and the signatures of all such graded decompositions of M coincide.*

The key observation of this note is that in fact every $\mathbb{N}$-graded principal ideal domain is

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- a principal ideal domain with the 0-grading or
- a polynomial ring over a field with a multiple of the canonical grading.

The proof is elementary [VO83, Remark 2.7] (Proposition 3.1).

For trivially graded principal ideal domains, in general, the graded elementary divisor version of the structure theorem does *not* hold (Example 4.1). This explains the additional hypothesis of $R \neq R_0$ in Theorem 1.1. In contrast, the graded prime power version of the structure theorem also holds if the grading is trivial (Proposition 4.2).

For polynomial rings, the graded uniqueness part can be deduced in a straightforward way from the ungraded uniqueness. However, for the graded existence part, there does not seem to be a "generic" derivation from the ungraded existence result – the difficulty being the graded direct sum splitting (as exhibited in the case of the trivially graded ring $\mathbb{Z}$). Finding such a splitting needs a careful inductive approach that establishes that the torsion submodule is graded and that avoids dividing out cyclic submodules in bad position/order. The graded existence part can be proved using specific properties of polynomial rings over fields.

In conclusion, the structure theorem for graded modules over graded principal ideal domains gives a helpful structural perspective on barcodes for persistent homology (and also for the computation of persistent homology [ZC05, SVJ13]), but its scope does not seem to go beyond the special case that is needed for persistent homology and it does not seem to provide a shortcut avoiding special properties of polynomial rings over fields.

Generalisations of $\mathbb{N}$-graded persistent homology such as zigzag persistence or $\mathbb{R}$-graded persistence (or more general indexing situations) are usually based on arguments from quiver representations [CdS10, BCB20]. Similarly to the $\mathbb{N}$-graded case, in these settings, it is also essential that the underlying coefficients are a field.

Organisation of this article

Basic notions on graded rings and modules are recalled in Section 2. In Section 3, we prove the observation on the classification of graded principal ideal domains (Proposition 3.1). The case of principal ideal domains with trivial gradings is considered in Section 4; the case of polynomial rings over fields is discussed in Section 5, where we give an elementary proof of the structure theorem.

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2 Graded rings and modules

We recall basic notions on graded rings and modules and decompositions of graded modules. As usual in (discrete) persistence, we consider only the case of discrete non-negative gradings, i.e., gradings over $\mathbb{N}$.

Definition 2.1 (graded ring)**.** A *graded ring* is a pair $(R, (R_n)_{n\in\mathbb{N}})$, where R is a ring and the R_n are additive subgroups of R with the following properties:

- The additive group $(R, +)$ is the internal direct sum of the $(R_n)_{n\in\mathbb{N}}$.
- For all $n, m \in \mathbb{N}$, we have $R_n \cdot R_m \subset R_{n+m}$.

For $n \in \mathbb{N}$, the elements in R_n are called *homogeneous of degree* n. An element of R is *homogenous* if there exists an $n \in \mathbb{N}$ such that the element is homogeneous of degree n.

A graded ring is a *graded principal ideal domain* if it is a domain and every homogeneous ideal (i.e., generated by homogeneous elements) is generated by a single element.

Example 2.2 (polynomial rings)**.** Let K be a ring. Then the usual degree on monomials in the polynomial ring $K[T]$ turns $K[T]$ into a graded ring via the canonical isomorphism $K[T] \cong_{\mathsf{Ab}} \bigoplus_{n\in\mathbb{N}} K \cdot T^n$. We will refer to this as the canonical grading on $K[T]$. If K is a field, then $K[T]$ is a principal ideal domain (graded and ungraded).

Definition 2.3 (graded module)**.** Let R be a graded ring. A *graded module over* R is a pair $(M, (M_n)_{n\in\mathbb{N}})$, consisting of an R-module M and additive subgroups M_n of M with the following properties:

- The additive group $(M, +)$ is the internal direct sum of the $(M_n)_{n\in\mathbb{N}}$.
- For all $n, m \in \mathbb{N}$, we have $R_n \cdot M_m \subset M_{n+m}$.

Elements of M_m are called *homogeneous of degree* m.

Remark 2.4 (the category of graded modules)**.** Let R be a graded ring. *Homomorphisms* between graded R-modules are R-linear maps that preserve the grading. Graded R-modules and graded homomorphisms of R-modules form the category ${}_R\mathsf{Mod}^*$ of graded R-modules.

Example 2.5 (shifted graded modules)**.** Let R be a graded ring, let M be a graded module over R, and let $n \in \mathbb{N}$. Then $\Sigma^n M$ denotes the graded R-module given by the n-shifted decomposition $0 \oplus \cdots \oplus 0 \oplus \bigoplus_{j\in\mathbb{N}_{\geq n}} M_{j-n}$.

Example 2.6 (direct sums and quotients of graded modules)**.** Let M and N be graded modules over a graded ring R. Then $M \oplus N$ is a graded R-module via the grading $(M_n \oplus N_n)_{n\in\mathbb{N}}$. If $M' \subset M$ is a graded submodule of M (i.e., it is generated by homogeneous elements), then $(M_n/(M' \cap M_n))_{n\in\mathbb{N}}$ turns M/M' into a graded R-module.

Persistent homology leads to persistence modules [ZC05]. Persistence modules in turn give rise to graded modules over graded polynomial rings [ZC05, Section 3.1]:

Example 2.7 (from persistence modules to graded modules)**.** Let K be a ring and let (M^*, f^*) be an $\mathbb{N}$-indexed persistence K-module. Then $M := \bigoplus_{n\in\mathbb{N}} M^n$ carries a $K[T]$-module structure, given by

$$\forall_{x\in M^n} \quad T \cdot x := f^n(x) \in M^{n+1}.$$

If we view $K[T]$ as a graded ring (Example 2.2), then this $K[T]$-module structure and this direct sum decomposition of M turn M into a graded $K[T]$-module. If (M^*, f^*) is of finite type, then M is finitely generated over $K[T]$.

Finally, we define the central types of decompositions arising in the structure theorems:

Definition 2.8 (graded elementary divisor decomposition)**.** Let R be a graded ring and let M be a graded module over R. A *graded elementary divisor decomposition* of M over R is an isomorphism

$$M \cong_{{}_R\mathsf{Mod}^*} \bigoplus_{j=1}^{N} \Sigma^{n_j} R/(f_j)$$

of graded R-modules with $N \in \mathbb{N}$, degrees $n_1, \dots, n_N \in \mathbb{N}$, and homogeneous elements $f_1, \dots, f_N \in R$ with $f_j | f_{j+1}$ for all $j \in \{1, \dots, N-1\}$. Here, the right-hand side carries the canonical grading. The elements $f_1, \dots, f_N$ are called *elementary divisors of* M.

The *signature* of such a decomposition is the multiset of all pairs $(n_j, R^\times \cdot f_j)$ with $j \in \{1, \dots, N\}$.

Definition 2.9 (graded prime power decomposition)**.** Let R be a graded ring and let M be a graded module over R. A *graded prime power decomposition* of M over R is an isomorphism

$$M \cong_{{}_R\mathsf{Mod}} \bigoplus_{j=1}^{N} \Sigma^{n_j} R/(p_j^{k_j})$$

of graded R-modules with $N \in \mathbb{N}$, $n_1, \dots, n_N \in \mathbb{N}$, $k_1, \dots, k_N \in \mathbb{N}$, and homogeneous prime elements $p_1, \dots, p_N \in R$. Here, the right-hand side carries the canonical grading.

The *signature* of such a decomposition is the multiset of all pairs $(n_j, R^\times \cdot p_j^{k_j})$ with $j \in \{1, \dots, N\}$.

3 Graded principal ideal domains

For the sake of completeness, we provide a proof of the following observation [VO83, Remark 2.7].

Proposition 3.1 (graded PIDs)**.** *Let R be a graded principal ideal domain. Then R is of one of the following types:*

- *We have $R = R_0$, i.e., R is an ordinary principal ideal domain with the 0-grading.*
- *The subring R_0 is a field and R is isomorphic to the graded ring $R_0[T]$, where the grading on $R_0[T]$ is a multiple of the canonical grading.*

Proof. Let $R \neq R_0$ and let $n \in \mathbb{N}_{>0}$ be the minimal degree with $R_n \neq 0$. Then

$$R_{\geq n} := \bigoplus_{j \in \mathbb{N}_{\geq n}} R_j$$

is a homogeneous ideal in R; as R is a graded principal ideal domain, there exists a $t \in R$ with $R_{\geq n} = (t)$. We show that t is homogeneous of degree n: Let

$x \in R_n \setminus \{0\}$. Then t divides x and a straightforward computation shows that hence also t is homogeneous. The grading implies that t has degree n.

We show that the canonical R_0-algebra homomorphism $\varphi\colon R_0[T] \longrightarrow R$ given by $\varphi(T) := t$ is an isomorphism.

- We first show that φ is injective: Because R is graded and t is homogeneous, it suffices to show that $a \cdot t^k \neq 0$ for all $a \in R_0 \setminus \{0\}$ and all $k \in \mathbb{N}$. However, this is guaranteed by the hypothesis that R is a domain.
- Regarding surjectivity, let $y \in R$. It suffices to consider the case that y is homogeneous of degree $m \geq n$. Because $(t) = R_{\geq n}$, we know that t divides y, say $y = t \cdot y'$. Then y' is homogeneous and we can iterate the argument for y'. Proceeding inductively, we obtain that m is a multiple of n and that there exists an $a \in R_0$ with $y = a \cdot t^{m/n}$. Hence, φ is surjective.

This establishes that R is isomorphic as a graded ring to $R_0[T]$, where $R_0[T]$ carries the canonical grading on $R_0[T]$ scaled by n.

It remains to show that $R_0 \cong_{\mathsf{Ring}} R/(t)$ is a field. Thus, we are left to show that (t) is a maximal ideal in R. By construction, every ideal a that contains $(t) = R_{\geq n}$ is generated by (t) and a subset of R_0; in particular, a is homogeneous, whence principal. The grading shows that then $a = R$ or $a = (t)$. Thus, (t) is maximal and so R_0 is a field. □

In the setting of $\mathbb{Z}$-graded principal ideal domains, further examples appear, such as generalised Rees rings [PvG82].

4 Trivially graded principal ideal domains

Example 4.1 (elementary divisor decompositions over trivially graded PIDs)**.** Let R be a principal ideal domain with the 0-grading that contains two non-associated prime elements p and q (e.g., 2 and 3 in $\mathbb{Z}$). We consider the graded R-module

$$M := \Sigma^0 R/(p) \oplus \Sigma^1 R/(q).$$

This graded R module does *not* admit a *graded* elementary divisor decomposition: Indeed, if there were a graded elementary divisor decomposition of M, then the corresponding elementary divisors would have to coincide with the ungraded elementary divisors. The only ungraded elementary divisor of M is $p \cdot q$. However, M does *not* contain a homogenous element with annihilator ideal $(p \cdot q)$. Therefore, M does not admit a graded elementary divisor decomposition.

Proposition 4.2 (prime power decompositions over trivially graded PIDs)**.** *Let R be a principal ideal domain with the* 0*-grading and let M be a finitely generated graded R-module. Then M admits a graded prime power decomposition and the signature of all such graded decompositions of M coincide.*

Proof. Because R is trivially graded, the grading on M decomposes M as a direct sum $\bigoplus_{n \in \mathbb{N}} M_n$ of R-submodules. In view of finite generation of M, only finitely many of these summands are non-trivial. We can now apply the ungraded structure theorem to each summand M_n to conclude. □

5 Polynomial rings over fields

In view of Proposition 3.1, Theorem 1.1 can equivalently be stated as follows (which is exactly the special case needed in persistent homology):

Theorem 5.1 (structure theorem for graded modules over polynomial rings)**.** *Let K be a field and let M be a finitely generated graded module over the graded ring $K[T]$. Then there exist $N \in \mathbb{N}$, $n_1, \dots, n_N \in \mathbb{N}$, and $k_1, \dots, k_N \in \mathbb{N}_{>0} \cup \{\infty\}$ with*

$$M \cong_{K[T]\,\mathsf{Mod}^*} \bigoplus_{j=1}^{N} \Sigma^{n_j} K[T]/(T^{k_j}).$$

Here, $T^\infty := 0$. The multiset of all (n_j, k_j) with $j \in \{1, \dots, N\}$ is uniquely determined by M.

The rest of this section contains an elementary and constructive proof of Theorem 5.1.

5.1 Uniqueness of graded decompositions

The uniqueness claim in Theorem 5.1 can be derived inductively from the ungraded uniqueness statement:

Let a decomposition as in Theorem 5.1 be given and let $\varphi \colon \bigoplus_{\dots} \cdots \longrightarrow M$ be a corresponding graded $K[T]$-isomorphism. Then

$$M' := \varphi(N') \text{ with } N' := \bigoplus_{j \in \{1,\dots,N\}, n_j = 0} \Sigma^{n_j} K[T]/(T^{k_j})$$

is a graded submodule of M and it is not difficult to see that $M' = \varphi(N') = \operatorname{Span}_{K[T]} M_0$. Moroever, M' is finitely generated over $K[T]$. Therefore, the ungraded structure theorem when applied to M' shows that the multiset of all pairs (n_j, k_j) with $n_j = 0$ is uniquely determined by M.

For the induction step, we pass to the quotient M/M', which is a finitely generated graded $K[T]$-module with $(M/M')_0 \cong 0$. We shift the degrees on M/M' by -1 and inductively apply the previous argument.

5.2 Homogeneous matrix reduction

The standard matrix reduction algorithm for the computation of persistent homology [EH10, ZC05] can be viewed as a proof of the existence part of Theorem 5.1.

We phrase the matrix reduction algorithm in the graded language to emphasise the connection with graded decompositions.

Definition 5.2 (graded matrix)**.** Let K be a field, let $r, s \in \mathbb{N}$, and let $n_1, \dots, n_r$, $m_1, \dots, m_s \in \mathbb{N}$ be two monotonically increasing sequences. A matrix $A \in M_{r \times s}(K[T])$ is (n_*, m_*)-*graded* if the following holds: For all $j \in \{1, \dots, r\}, k \in \{1, \dots, s\}$, we have that the entry $A_{jk} \in K[T]$ is a homogeneous polynomial and

- $A_{jk} = 0$ or
- $m_k = n_j + \deg A_{jk}$.

In a graded matrix, the degrees of matrix entries monotonically increase from the left to the right and from the bottom to the top.

Definition 5.3 (reduced matrix)**.** Let K be a field, let $r, s \in \mathbb{N}$, let $n_1, \dots, n_r$ and $m_1, \dots, m_s \in \mathbb{N}$ be two monotonically increasing sequences, and let $A \in M_{r\times s}(K[T])$ be an (n_*, m_*)-graded matrix.

- For $k \in \{1, \dots, s\}$, we define

  $$\mathrm{low}_A(k) := \max\bigl\{j \in \{1, \dots, r\} \bigm| A_{jk} \neq 0\bigr\} \in \mathbb{N}$$

  (with $\max \emptyset := 0$). I.e., $\mathrm{low}_A(k)$ is the index of the "lowest" matrix entry in column k that is non-zero.

- The matrix A is *reduced* if all columns have different low-indices: For all $k, k' \in \mathbb{N}$ with $\mathrm{low}_A(k) \neq 0$ and $\mathrm{low}_A(k') \neq 0$, we have $\mathrm{low}_A(k) \neq \mathrm{low}_A(k')$.

Graded matrices can be transformed into reduced matrices via elementary column operations; these reduced matrices then lead to module decompositions:

Algorithm 5.4 (homogeneous matrix reduction)**.** Given a field K, $r, s \in \mathbb{N}$, monotonically increasing sequences $n_1, \dots, n_r$ and $m_1, \dots, m_s \in \mathbb{N}$, and an (n_*, m_*)-graded matrix $A \in M_{r\times s}$, do the following:

- For each k from 1 up to s (in ascending order):

  Let $\ell := \mathrm{low}_A(k)$.

  If $\ell \neq 0$, then:

  - For each j from ℓ down to 1 (in descending order):
    If $A_{jk} \neq 0$ and there exists $k' \in \{1, \dots, k-1\}$ with $\mathrm{low}_A(k') = j$, then:
    - Update the matrix A by subtracting $A_{jk}/A_{jk'}$-times the column k' from column k.
      [Loop invariant observation: Because A is graded, $A_{jk}/A_{jk'}$ indeed is a homogeneous polynomial over K and the resulting matrix is (n_*, m_*)-graded. This eliminates the entry $A_{jk'}$.]

- Return the resulting matrix A.

Proposition 5.5. *Let K be a field, let $r, s \in \mathbb{N}$, let $n_1, \dots, n_s$ and $m_1, \dots, m_r \in \mathbb{N}$ be monotonically increasing, and let $A \in M_{r\times s}(K[T])$ be an (n_*, m_*)-graded matrix. Then:*

1. *The homogeneous matrix reduction algorithm (Algorithm 5.4) terminates on this input after finitely many steps (relative to the arithmetic on K).*

2. *The resulting matrix A' is reduced and there is a graded $s \times s$-matrix B over $K[T]$ that admits a graded inverse and satisfies $A' = A \cdot B$.*

3. *The low-entries of the resulting matrix A' are the elementary divisors of A over $K[T]$.*

4. *We have*

$$F/\operatorname{im} A \cong_{K[T]\mathsf{Mod}^*} \bigoplus_{j\in I} \Sigma^{n_j} K[T]/(T^{m_{k(j)}-n_j}) \oplus \bigoplus_{j\in I'} \Sigma^{n_j} K[T],$$

where $F := \bigoplus_{j=1}^{r} \Sigma^{n_j} K[T]$ *and* $I := \{\mathrm{low}_{A'}(k) \mid k \in \{1,\dots,s\}\} \setminus \{0\}$ *as well as* $I' := \{1,\dots,r\} \setminus I$*. For* $j \in I$*, let* $k(j) \in \{1,\dots,s\}$ *be the unique (!) index with* $\mathrm{low}_{A'}(k(j)) = j$*.*

Proof. Ad 1. Well-definedness follows from the observation mentioned in the algorithm: As every homogeneous polynomial in $K[T]$ is of the form $\lambda \cdot T^d$ with $\lambda \in K$ and $d \in \mathbb{N}$ and as the matrix is graded, the corresponding division can be performed in $K[T]$ and the gradedness of the matrix is preserved by the elimination operation. Termination is then clear from the algorithm.

Ad 2. As we traverse the columns from left to right, a straightforward induction shows that no two columns can remain that have the same non-zero value of "low_A". The product decomposition comes from the fact that we only applied elementary homogeneous column operations without swaps.

Ad 3. Because the resulting matrix A' is obtained through elementary column operations from A, the elementary divisors of A' and A coincide. Applying Lemma 5.6 to A' proves the claim.

Ad 4. In view of the second part, we have that $F/\operatorname{im} A \cong_{K[T]\mathsf{Mod}^*} F/\operatorname{im} A'$. Therefore, the claim is a direct consequence of Lemma 5.6. □

Lemma 5.6. *Let K be a field, let $r, s \in \mathbb{N}$, let $n_1,\dots,n_r$ and $m_1,\dots,m_s \in \mathbb{N}$ be monotonically increasing, and let $A \in M_{r\times s}(K[T])$ be an (n_*, m_*)-graded matrix that is reduced. Then:*

1. *The* low*-entries of A are the elementary divisors of A over $K[T]$.*
2. *Let* $F := \bigoplus_{j=1}^{r} \Sigma^{n_j} K[T]$ *and* $I := \{\mathrm{low}_A(k) \mid k \in \{1,\dots,s\}\} \setminus \{0\}$ *as well as* $I' := \{1,\dots,r\} \setminus I$*. Then*

$$F/\operatorname{im} A \cong_{K[T]\mathsf{Mod}^*} \bigoplus_{j\in I} \Sigma^{n_j} K[T]/(T^{m_{k(j)}-n_j}) \oplus \bigoplus_{j\in I'} \Sigma^{n_j} K[T]$$

Proof. Ad 1. Let $k \in \{1,\dots,s\}$ with $\ell := \mathrm{low}_A(k) \neq 0$. Then we can clear out all the entries of A in column k above ℓ by elementary row operations (again, the gradedness of A ensures that this is possible). Swapping zero rows and columns appropriately thus results in a matrix in rectangle "diagonal" form; moreover, as all the "diagonal" entries are monomials, we can swap rows and columns to obtain a matrix A' in Smith normal form that both

- has the same elementary divisors as A and
- whose elementary divisors are precisely the low-entries of A.

In particular, these elementary divisors must coincide.

Ad 2. The claim is clear if A is already in Smith normal form. By construction, there are square matrices B and C that are invertible over $K[T]$ and represent graded $K[T]$-isomorphisms with

$$A' = C \cdot A \cdot B.$$

In particular, $F/\operatorname{im} A \cong_{K[T]\mathsf{Mod}^*} (C \cdot F)/\operatorname{im} A'$. By construction, the values of $\mathrm{low}_{A'}$ and the degrees of A' differ from the ones of A only by compatible index permutations. Therefore, the claim follows. □

5.3 Existence of a graded decomposition

To prove existence in Theorem 5.1 we can follow the standard proof pattern of first finding a (graded) finite presentation and then applying (homogeneous) matrix reduction.

Let M be a finitely generated graded $K[T]$-module. Then M also has a finite generating set consisting of homogeneous elements. This defines a surjective graded $K[T]$-homomorphism

$$\varphi\colon F := \bigoplus_{j=1}^{r} \Sigma^{n_j} K[T] \longrightarrow M$$

for suitable $r \in \mathbb{N}$ and monotonically increasing $n_1, \dots, n_r \in \mathbb{N}$. As φ is a graded homomorphism, $\ker \varphi \subset F$ is a graded $K[T]$-submodule and we obtain an isomorphism

$$M \cong_{K[T]\mathsf{Mod}^*} F/\operatorname{im} \ker \varphi$$

of graded $K[T]$-modules.

Because $K[T]$ is a principal ideal domain, the graded submodule $\ker \varphi \subset F$ is finitely generated over $K[T]$. Because $\ker \varphi$ is a graded submodule, $\ker \varphi$ has a finite homogeneous generating set. (In fact, there also exists a homogeneous free $K[T]$-basis for $\ker \varphi$, as can be seen from a straightforward inductive splitting argument [Web85, Lemma 1].) In particular, there exist $s \in \mathbb{N}$, monotonically increasing $m_1, \dots, m_s \in \mathbb{N}$, and a graded $K[T]$-homomorphism

$$\psi\colon E := \bigoplus_{k=1}^{s} \Sigma^{m_k} K[T] \longrightarrow F$$

with $\operatorname{im} \psi = \ker \varphi$. Because ψ is graded and n_*, m_* are monotonically increasing, the $r \times s$-matrix A over $K[T]$ that represents ψ with respect to the canonical homogeneous bases of E and F is graded in the sense of Definition 5.2.

Applying the homogeneous matrix reduction algorithm to A shows that

$$M \cong_{K[T]\mathsf{Mod}^*} F/\operatorname{im} A,$$

has the desired decomposition (Proposition 5.5; after discarding the irrelevant terms of the form $\Sigma^n K[T]/(T^0)$).

This completes the proof of the structure theorem (Theorem 5.1).

Remark 5.7. There is a general matrix reduction for a slighlty different notion of "graded" matrices over ($\mathbb{Z}$-)graded principal ideal domains [PvG82]. However, one should be aware that such "graded" matrices in general only lead to graded homomorphisms once one is allowed to change the grading on the underlying free modules. This explains why this general matrix reduction does not contradict the counterexample in the case of 0-graded principal ideal rings in Example 4.1.

5.4 Barcodes

For the sake of completeness, we recall the relation between graded decompositions and barcodes:

Remark 5.8 (barcodes of persistence modules)**.** Let K be a field and let (M^*, f^*) be an $\mathbb{N}$-indexed persistence K-module of finite type. We equip $M := \bigoplus_{n\in\mathbb{N}} M^n$ with the canonical graded $K[T]$-module structure (Example 2.7). By the graded structure theorem (Theorem 5.1), there exist $N \in \mathbb{N}$, $n_1, \dots, n_N \in \mathbb{N}$, and $k_1, \dots, k_N \in \mathbb{N}_{>0} \cup \{\infty\}$ with

$$M \cong_{K[T]\mathsf{Mod}^*} \bigoplus_{j=1}^{N} \Sigma^{n_j} K[T]/(T^{k_j}).$$

Let B be the multiset of all $(n_j, k_j - 1)$ with $j \in \{1, \dots, N\}$; then B is uniquely determined by M and this multiset B is the *barcode of* (M^*, f^*).

The barcode contains the full information on the isomorphism type of the graded $K[T]$-module M (and the underlying persistence module) and describes the birth, death, and persistence of elements as specified by the "elder rule": If (n, p) is an element of the barcode, this means that a new independent class is born at stage n, it persists for p stages, and it dies (if $p \neq \infty$) at stage $n+p+1$.

In particular, this leads to the notion of barcodes of persistent homology (in a given degree) of finite type persistence chain complexes and finite type filtrations in topology.

References

[BCB20] Magnus Bakke Botnan and William Crawley-Boevey. Decomposition of persistence modules. *Proc. Amer. Math. Soc.*, 148(11):4581–4596, 2020. Cited on page: 2

[CdS10] Gunnar Carlsson and Vin de Silva. Zigzag persistence. *Found. Comput. Math.*, 10(4):367–405, 2010. Cited on page: 2

[CZCG04] Gunnar Carlsson, Afra Zomorodian, Anne Collins, and Leonidas Guibas. Persistence Barcodes for Shapes. In Roberto Scopigno and Denis Zorin, editors, *Symposium on Geometry Processing*. The Eurographics Association, 2004. Cited on page: 1

[EH10] Herbert Edelsbrunner and John L. Harer. *Computational topology*. American Mathematical Society, Providence, RI, 2010. An introduction. Cited on page: 1, 6

[PvG82] R. Puystjens and J. van Geel. Diagonalization of matrices over graded principal ideal domains. *Linear Algebra Appl.*, 48:265–281, 1982. Cited on page: 5, 9

[SVJ13] Primoz Skraba and Mikael Vejdemo-Johansson. Persistence modules: Algebra and algorithms. 2013. arXiv:1302.2015 [cs.CG]. Cited on page: 2

[VO83] F. Van Oystaeyen. Generalized Rees rings and arithmetical graded rings. *J. Algebra*, 82(1):185–193, 1983. Cited on page: 2, 4

[Web85] Cary Webb. Decomposition of graded modules. *Proc. Amer. Math. Soc.*, 94(4):565–571, 1985. Cited on page: 9

[ZC05] Afra Zomorodian and Gunnar Carlsson. Computing persistent homology. *Discrete Comput. Geom.*, 33(2):249–274, 2005. Cited on page: 1, 2, 3, 6

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