

MEASURE HOMOLOGY AND SINGULAR HOMOLOGY ARE ISOMETRICALLY ISOMORPHIC

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ABSTRACT. Measure homology is a variation of singular homology designed by Thurston in his discussion of simplicial volume. Zastrow and Hansen showed independently that singular homology (with real coefficients) and measure homology coincide algebraically on the category of CW-complexes. It is the aim of this paper to prove that this isomorphism is *isometric* with respect to the ℓ^1 -seminorm on singular homology and the seminorm on measure homology induced by the total variation. This, in particular, implies that one can calculate the simplicial volume via measure homology – as already claimed by Thurston. For example, measure homology can be used to prove Gromov’s proportionality principle of simplicial volume.

1. INTRODUCTION

The simplicial volume is a topological invariant of oriented closed connected manifolds, measuring the complexity of the fundamental class. Despite its topological nature, the simplicial volume is linked to Riemannian geometry in various ways [4].

Originally, Gromov defined the simplicial volume to be the ℓ^1 -seminorm of the fundamental class (with real coefficients) [12]. In his famous lecture notes [15; Chapter 6], Thurston suggested an alternative description of the simplicial volume for smooth manifolds: he replaced singular homology and the ℓ^1 -seminorm by a new homology theory, called smooth measure homology, and a corresponding seminorm. However, except for the case of hyperbolic manifolds [16; Remark 0.1], there is no published proof that these two constructions result in the same simplicial volume. It is the purpose of the present article to close this gap.

More generally, we prove the following:

Theorem (1.1). *For all connected CW-complexes X , the inclusion $i_X: C_*(X) \rightarrow \mathcal{C}_*(X)$ of the singular chain complex into the measure chain complex induces a natural isomorphism*

$$H_*(X) \cong \mathcal{H}_*(X).$$

This isomorphism is isometric with respect to the ℓ^1 -seminorm on singular homology and the seminorm on measure homology induced by the total variation.

Theorem (1.2). *For all connected smooth manifolds M the canonical inclusions $i_M^s: C_*^s(M) \rightarrow C_*^s(M)$ (of the smooth singular chain complex into the smooth measure chain complex) and $j_M: C_*^s(M) \rightarrow C_*(M)$ induce a natural isomorphism*

$$H_*(i_M^s) \circ H_*(j_M)^{-1}: H_*(M) \rightarrow \mathcal{H}_*(M).$$

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This isomorphism is isometric with respect to the ℓ^1 -seminorm on singular homology and the seminorm on smooth measure homology induced by the total variation.

We will now explain the occurring terminology in more detail.

The ℓ^1 -seminorm on singular homology with real coefficients is the seminorm induced by the ℓ^1 -norm on the singular chain complex:

Definition (1.3). Let X be a topological space and let $k \in \mathbf{N}$. We define a norm $\|\cdot\|_1$ on the singular k -chains $C_k(X)$ with real coefficients by

$$\|c\|_1 := \sum_{\sigma \in \text{map}(\Delta^k, X)} |a_\sigma|$$

for all $c = \sum_{\sigma \in \text{map}(\Delta^k, X)} a_\sigma \cdot \sigma \in C_k(X)$. This norm induces a seminorm on the k -th singular homology $H_k(X)$ with real coefficients by

$$\|\alpha\|_1 := \inf \{ \|c\|_1 \mid c \in C_k(X), \partial(c) = 0, [c] = \alpha \}$$

for all $\alpha \in H_k(X)$. ◇

One of the most important properties of $\|\cdot\|_1$ is “functoriality,” i.e., for all continuous maps f the induced homomorphism $H_*(f)$ does not increase the seminorm. (Based on this property, Gromov introduced a more general framework of functorial seminorms [5; 5.34].)

Measure homology is a curious generalisation of singular homology: Let X be a topological space, let $k \in \mathbf{N}$ and let $S_k(X) \subset \text{map}(\Delta^k, X)$ be some set of singular simplices. The idea of measure homology is to think of a singular chain $\sum_{\sigma \in S_k(X)} a_\sigma \cdot \sigma$ with real coefficients as a signed measure on $S_k(X)$ having the mass a_σ on the set $\{\sigma\}$. The measure chain complex consists of all signed measures on $S_k(X)$ satisfying some finiteness condition.

Thus the measure chain complex is larger than the singular chain complex and hence provides more room for constructions such as “smearing” [15, 13; page 6.8ff, page 547ff]. The other side of the coin is that it is quite hard to gain a geometric intuition of more complicated measure chains.

Depending on the choice of the mapping spaces $S_k(X)$ and their topology, there are two main flavours of measure homology:

- One for general topological spaces using the compact-open topology on the set of all singular simplices – the corresponding chain complexes and homology groups are denoted by $\mathcal{C}_*(X)$ and $\mathcal{H}_*(X)$.
- One for smooth manifolds using the \mathcal{C}^1 -topology on the set of smooth singular simplices – the corresponding chain complexes and homology groups are denoted by $\mathcal{C}_*^s(X)$ and $\mathcal{H}_*^s(X)$.

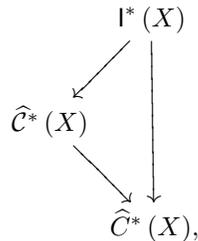
In both cases, measure homology is equipped with the seminorm induced by the total variation on the chain level.

Measure homology of the second kind (so-called smooth measure homology) was introduced by Thurston [15; page 6.6]. Some basic properties of smooth measure homology are also listed in Ratcliffe’s book [13; §11.5]. A thorough treatment of measure homology for general spaces is given in the papers of Zastrow and Hansen [16, 6]. In both papers it is shown that measure homology for CW-complexes coincides *algebraically* with singular homology with real coefficients.

The general idea of the proof of Theorem (1.1) and (1.2) is to take a dual point of view: singular homology and the ℓ^1 -seminorm admit a dual concept, called bounded

cohomology. The key property is that the canonical seminorm on bounded cohomology and the ℓ^1 -seminorm are intertwined by a duality principle (Theorem (3.3)).

Unlike the ℓ^1 -seminorm, bounded cohomology and its seminorm are rather well understood by Ivanov’s homological algebraic approach [7, 11]. In our proof of Theorem (1.1) and (1.2), we take advantage of a special cochain complex $I^*(X)$ computing bounded cohomology. More precisely, we construct a dual $\widehat{C}^*(X)$ of the measure chain complex (together with a corresponding duality principle) fitting into a commutative diagram



where $\widehat{C}^*(X)$ is the cochain complex defining bounded cohomology. The crux of this diagram is that the vertical arrow induces an isometric isomorphism on the level of cohomology and that the arrows on the left do not increase the seminorm. Then the duality principles allow us to deduce that the algebraic isomorphism $H_*(X) \cong \mathcal{H}_*(X)$ must be isometric.

For simplicity, we only prove Theorem (1.1) in detail. The smooth case, requiring a small detour to smooth singular homology, is considered briefly in Section 5 (and is also explained in the author’s diploma thesis [14]).

This paper is organised as follows: Measure homology (the non-smooth version) is defined in Section 2. Section 3 is concerned with the dual point of view, i.e., the construction of a dual for measure homology and the derivation of a corresponding duality principle. A proof of Theorem (1.1) is presented in Section 4. In Section 5, we have a glimpse at the smooth universe, that is at smooth measure homology and at a proof of Theorem (1.2). Finally, in Section 6, we list some applications to the simplicial volume, including Gromov’s proportionality principle and some of its consequences.

2. MEASURE HOMOLOGY

In this section, our basic object of study, measure homology, is introduced. In Section 2.2, we describe the algebraic isomorphism between singular homology and measure homology. The smooth case is deferred to Section 5.

2.1. Definition of measure homology. Before stating the precise definition of measure homology, we recall some basics from measure theory:

Definition (2.1). Let (X, \mathcal{A}) be a measurable space.

- A map $\mu: \mathcal{A} \rightarrow \mathbf{R} \cup \{\infty, -\infty\}$ is called a **signed measure** if $\mu(\emptyset) = 0$, not both ∞ and $-\infty$ are contained in the image of μ , and μ is σ -additive.
- A **null set** of a signed measure μ on (X, \mathcal{A}) is a measurable set $A \in \mathcal{A}$ with $\mu(B) = 0$ for all $B \in \mathcal{A}$ with $B \subset A$.

- A **determination set** of a signed measure μ on (X, \mathcal{A}) is a subset D of X such that each measurable set contained in the complement of D is a μ -null set.
- The **total variation** of a signed measure μ on (X, \mathcal{A}) is given by

$$\|\mu\| := \sup_{A \in \mathcal{A}} \mu(A) - \inf_{A \in \mathcal{A}} \mu(A).$$

- For $x \in X$ the **atomic measure** concentrated in x is denoted by δ_x .
- If $f: (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ is a measurable map and μ is a signed measure on (X, \mathcal{A}) , then

$$\forall_{B \in \mathcal{B}} \mu^f(B) := \mu(f^{-1}(B))$$

defines a signed measure μ^f on (Y, \mathcal{B}) . \diamond

As indicated in the introduction, the measure homology chain complex consists of measures that respect some finiteness condition on the set of all singular simplices.

Definition (2.2). Let X be a topological space and let $k \in \mathbf{N}$.

- The **k -th measure chain group**, denoted by $\mathcal{C}_k(X)$, is the \mathbf{R} -vector space of signed measures on $\text{map}(\Delta^k, X)$ possessing a compact determination set and finite total variation. Here $\text{map}(\Delta^k, X)$ is equipped with the compact-open topology and the corresponding Borel σ -algebra. The elements of $\mathcal{C}_k(X)$ are called **measure k -chains**.
- For each $j \in \{0, \dots, k+1\}$ the inclusion $\partial_j: \Delta^k \rightarrow \Delta^{k+1}$ of the j -th face induces a continuous map $\text{map}(\Delta^{k+1}, X) \rightarrow \text{map}(\Delta^k, X)$ and hence a homomorphism (which we will also denote by ∂_j)

$$\begin{aligned} \partial_j: \mathcal{C}_{k+1}(X) &\longrightarrow \mathcal{C}_k(X) \\ \mu &\longmapsto \mu^{(\sigma \mapsto \sigma \circ \partial_j)}. \end{aligned}$$

The **boundary operator** of measure chains is then defined by

$$\partial := \sum_{j=0}^{k+1} (-1)^j \cdot \partial_j: \mathcal{C}_{k+1}(X) \longrightarrow \mathcal{C}_k(X).$$

- The \mathbf{R} -vector space $\mathcal{H}_k(X) := H_k(\mathcal{C}_*(X), \partial)$ is called the **k -th measure homology group** of X .
- The total variation $\|\cdot\|$ turns $\mathcal{C}_k(X)$ into a normed vector space and thus induces a seminorm on $\mathcal{H}_k(X)$ as follows: For all $\mu \in \mathcal{H}_k(X)$ we define

$$\|\mu\|_{\text{m}} := \inf \{ \|\nu\| \mid \nu \in \mathcal{C}_k(X), \partial(\nu) = 0, [\nu] = \mu \}. \quad \diamond$$

Zastrow showed that $(\mathcal{C}_*(X), \partial)$ indeed is a chain complex [16; Corollary 2.9]. Hence measure homology is well-defined. Each continuous map $f: X \rightarrow Y$ induces a chain map [16; Lemma-Definition 2.10(iv)]

$$\begin{aligned} \mathcal{C}_*(f): \mathcal{C}_*(X) &\longrightarrow \mathcal{C}_*(Y) \\ \mu &\longmapsto \mu^f \end{aligned}$$

which obviously does not increase the total variation. Therefore, we obtain a homomorphism $\mathcal{H}_*(f): \mathcal{H}_*(X) \rightarrow \mathcal{H}_*(Y)$ satisfying $\|\mathcal{H}_*(f)(\mu)\|_{\text{m}} \leq \|\mu\|_{\text{m}}$ for all $\mu \in \mathcal{H}_*(X)$. Clearly, this turns \mathcal{H}_* into a functor. Moreover, the functor \mathcal{H}_* is homotopy invariant [16; Lemma-Definition 2.10(vi)].

Analogously to singular homology, relative measure homology groups can be defined [16; Lemma-Definition 2.10(ii)].

2.2. The algebraic isomorphism. There is an obvious norm-preserving inclusion of the singular chain complex into the measure chain complex:

Definition (2.3). If X is a topological space and $k \in \mathbf{N}$, we write

$$i_X: C_k(X) \longrightarrow \mathcal{C}_k(X) \\ \sum_{\sigma \in \text{map}(\Delta^k, X)} a_\sigma \cdot \sigma \longmapsto \sum_{\sigma \in \text{map}(\Delta^k, X)} a_\sigma \cdot \delta_\sigma. \quad \diamond$$

Remark (2.4). This inclusion induces a natural chain map $C_*(X) \longrightarrow \mathcal{C}_*(X)$ (also denoted by i_X) which is norm-preserving. \square

Establishing the Eilenberg-Steenrod axioms for measure homology, Hansen [6] and Zastrow [16] independently proved the following theorem:

Theorem (2.5). *For all CW-complexes X , the inclusion $i_X: C_*(X) \longrightarrow \mathcal{C}_*(X)$ induces a natural isomorphism (of real vector spaces)*

$$H_*(X) \cong \mathcal{H}_*(X).$$

However, there are spaces for which singular homology and measure homology do not coincide [16; Section 6].

In Section 4, we prove the main result, namely that these isomorphisms are compatible with the induced seminorms on H_* and \mathcal{H}_* . Consequences of this theorem and of its smooth analogue (Theorem (1.2)) are discussed in Section 6.

3. A DUAL POINT OF VIEW

Rather than attempting to investigate the functorial seminorms $\|\cdot\|_1$ and $\|\cdot\|_m$ on singular homology and measure homology directly on the chain level, we take a dual point of view: we make use of bounded cohomology and the duality principle (Theorem (3.3)) to compute $\|\cdot\|_1$. Analogously, we also construct a dual for measure homology and derive a corresponding duality principle (Section 3.4). In Section 3.3, a special cochain complex for bounded cohomology is introduced, which turns out to be very convenient in our setting.

3.1. Bounded cohomology. Bounded cohomology is the functional analytic twin of singular cohomology. It is constructed via the topological dual of the singular chain complex instead of the algebraic one. The corresponding norm for singular cochains is therefore the supremum norm:

Definition (3.1). Let X be a topological space and $k \in \mathbf{N}$. For a singular cochain $f \in C^k(X)$ the (possibly infinite) **supremum norm** is defined by

$$\|f\|_\infty := \sup\{|f(\sigma)| \mid \sigma \in \text{map}(\Delta^k, X)\}.$$

This induces a seminorm on $H^k(X)$ by

$$\|\varphi\|_\infty := \inf\{\|f\|_\infty \mid f \in C^k(X), \delta(f) = 0, [f] = \varphi\}$$

for all $\varphi \in H^k(X)$. We write $\widehat{C}^k(X) := \{f \in C^k(X) \mid \|f\|_\infty < \infty\}$ for the vector space of **bounded k -cochains**. \diamond

It is easy to see that the coboundary operator δ on the singular cochain complex $C^k(X)$ satisfies $\delta(\widehat{C}^k(X)) \subset \widehat{C}^{k+1}(X)$. Thus $\widehat{C}^*(X)$ is a cochain complex.

Definition (3.2). Let X be a topological space and $k \in \mathbf{N}$.

- The k -th bounded cohomology group of X is defined by

$$\widehat{H}^k(X) := H^k(\widehat{C}^*(X), \delta|_{\widehat{C}^*(X)}).$$

- The supremum norm on $\widehat{C}^k(X)$ induces a seminorm on $\widehat{H}^k(X)$ by

$$\|\varphi\|_\infty := \inf\{\|f\|_\infty \mid f \in \widehat{C}^k(X), \delta(f) = 0, [f] = \varphi\}.$$

for all $\varphi \in \widehat{H}^k(X)$. ◇

Overviews of bounded cohomology (of spaces) are given by Ivanov [7], Gromov [4], and Brooks [2], where also the more peculiar aspects of bounded cohomology are explained. For example, bounded cohomology depends only on the fundamental group [4, 7; Corollary (A) on page 40, Theorem (4.1)] and does not satisfy the excision axiom [2, 10; §3(a), §5].

3.2. Duality. The duality principle (Theorem (3.3)) shows an important aspect of bounded cohomology: bounded cohomology can be used to compute the ℓ^1 -seminorm (and hence the simplicial volume). Since bounded cohomology is much better understood (in view of the techniques presented in Ivanov's paper [7]) than the seminorm on homology, duality leads to interesting applications. For example, bounded cohomology can be used to give estimates for the simplicial volume of products and connected sums of manifolds [4; page 10].

Moreover, duality plays a central rôle in the proof that measure homology and singular homology are *isometrically* isomorphic (see Section 4).

Theorem (3.3) (Duality Principle). *Let X be a topological space, $k \in \mathbf{N}$ and $\alpha \in H_k(X)$.*

- (1) *Then $\|\alpha\|_1 = 0$ if and only if*

$$\forall_{\varphi \in \widehat{H}^k(X)} \langle \varphi, \alpha \rangle = 0.$$

- (2) *If $\|\alpha\|_1 > 0$, then*

$$\|\alpha\|_1 = \sup\left\{\frac{1}{\|\varphi\|_\infty} \mid \varphi \in \widehat{H}^k(X), \langle \varphi, \alpha \rangle = 1\right\}.$$

The angle brackets $\langle \cdot, \cdot \rangle$ refer to the Kronecker product on $\widehat{H}^*(X) \otimes H_*(X)$ defined by evaluation, just as the ordinary Kronecker product on $H^*(X) \otimes H_*(X)$.

This duality was discovered by Gromov [4]. A detailed proof – based on the Hahn-Banach theorem – is given in the book of Benedetti and Petronio [1; Proposition F.2.2].

3.3. A special cochain complex. Bounded cohomology, as defined in Definition (3.2), is quite hard to calculate. However, Ivanov found a homological algebraic route to bounded cohomology via strong relatively injective resolutions of Banach modules [7, 11]. In Section 4, the resolution discussed in the following paragraphs saves the day:

Definition (3.4). Let X be an arcwise connected space with universal covering \widetilde{X} , and let $k \in \mathbf{N}$.

- Then $\pi_1(X)$ acts from the left on the vector space $\text{map}(\tilde{X}^{k+1}, \mathbf{R})$ of continuous functions $\tilde{X}^{k+1} \rightarrow \mathbf{R}$ by

$$g \cdot f := ((x_0, \dots, x_k) \mapsto f(x_0 \cdot g, \dots, x_k \cdot g))$$

for all $f \in \text{map}(\tilde{X}^{k+1}, \mathbf{R})$ and all $g \in \pi_1(X)$.

- The subset of bounded functions in $\text{map}(\tilde{X}^{k+1}, \mathbf{R})$ is denoted by $\mathbf{B}^k(X)$ and we use the abbreviation $\mathbf{l}^k(X)$ for the functions in $\mathbf{B}^k(X)$ that are invariant under the above $\pi_1(X)$ -action. \diamond

How can we turn $\mathbf{B}^*(X)$ into a cochain complex? The vector space $\widehat{C}^k(\tilde{X})$ can be identified with the space of bounded functions $\text{map}(\Delta^k, \tilde{X}) \rightarrow \mathbf{R}$ (under this identification, the norm $\|\cdot\|_\infty$ just becomes the supremum norm). Now $\mathbf{B}^k(X)$ can be viewed as a subspace of $\widehat{C}^k(\tilde{X})$, namely as the space of those bounded functions $\text{map}(\Delta^k, \tilde{X}) \rightarrow \mathbf{R}$ that only depend on the vertices of the simplices (and are continuous in the vertices). In Gromov's terminology those functions would be called "straight bounded continuous cochains" [4; Section 2.3], inspiring the sans-serif notation.

It is clear that the coboundary operator on $\widehat{C}^*(\tilde{X})$ restricts to $\mathbf{B}^*(X)$ and that the operations of $\pi_1(X)$ on $\mathbf{B}^*(X)$ and $\widehat{C}^*(\tilde{X})$ are compatible with the above inclusion map. This makes $\mathbf{B}^*(X)$ a subcomplex of $\widehat{C}^*(\tilde{X})$. In other words, the homomorphism

$$\begin{aligned} u^k : \mathbf{B}^k(X) &\longrightarrow \widehat{C}^k(\tilde{X}) \\ f &\longmapsto (\sigma \mapsto f(\sigma(e_0), \dots, \sigma(e_k))) \end{aligned}$$

yields a $\pi_1(X)$ -equivariant cochain map. Here, e_0, \dots, e_k are the vertices of Δ^k .

Theorem (3.5). *Let X be a connected locally finite CW-complex and let $p: \tilde{X} \rightarrow X$ be its universal covering. For $\sigma \in \text{map}(\Delta^k, X)$ we denote by $\tilde{\sigma} \in \text{map}(\Delta^k, \tilde{X})$ some lift of σ with respect to p . The cochain map $v: \mathbf{l}^*(X) \rightarrow \widehat{C}^*(X)$ given by*

$$\begin{aligned} \mathbf{l}^k(X) &\longrightarrow \widehat{C}^k(X) \\ f &\longmapsto (\sigma \mapsto f(\tilde{\sigma}(e_0), \dots, \tilde{\sigma}(e_k))) \end{aligned}$$

induces an isometric isomorphism $H^*(\mathbf{l}^*(X)) \cong \widehat{H}^*(X)$.

Covering theory shows that v is a well-defined cochain map.

Remark (3.6). Since all connected smooth manifolds are triangulable (and hence locally finite CW-complexes), the theorem applies in particular to this case. \square

Proof. The universal covering map p induces an isometric isomorphism

$$\widehat{p}^* : \widehat{H}^*(X) \longrightarrow H^*(\widehat{C}^*(\tilde{X})^{\pi_1(X)})$$

where $\widehat{C}^*(\tilde{X})^{\pi_1(X)}$ denotes the subcomplex of $\pi_1(X)$ -fixed points under the canonical $\pi_1(X)$ -action on $\text{map}(\Delta^k, \tilde{X})$ induced by the $\pi_1(X)$ -action on \tilde{X} [7; proof of

Theorem (4.1)]. By construction, the triangle

$$\begin{array}{ccc} H^*(\mathbf{1}^*(X)) & \xrightarrow{H^*(v)} & \widehat{H}^*(X) \\ & \searrow^{H^*(u|)} & \swarrow_{\widehat{p}^*} \\ & & H^*(\widehat{\mathcal{C}}^*(\widetilde{X})^{\pi_1(X)}) \end{array}$$

is commutative (where $u|$ denotes the restriction of u to the $\pi_1(X)$ -fixed points). The fact that there exists a canonical isometric isomorphism

$$H^*(\mathbf{1}^*(X)) \cong H^*(\widehat{\mathcal{C}}^*(\widetilde{X})^{\pi_1(X)})$$

follows from work of Monod [11; Theorem 7.4.5]. The homomorphism u fits into the ladder

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{R} & \xrightarrow{\varepsilon} & \mathbf{B}^0(X) & \longrightarrow & \mathbf{B}^1(X) \longrightarrow \cdots \\ & & \text{id} \downarrow & & u^0 \downarrow & & u^1 \downarrow \\ 0 & \longrightarrow & \mathbf{R} & \xrightarrow{\varepsilon} & \widehat{\mathcal{C}}^0(\widetilde{X}) & \longrightarrow & \widehat{\mathcal{C}}^1(\widetilde{X}) \longrightarrow \cdots, \end{array}$$

whose rows are strong resolutions of the trivial Banach $\pi_1(X)$ -module \mathbf{R} by relatively injective $\pi_1(X)$ -modules [11, 7; Theorem 7.4.5, proof of Theorem (4.1)]. Hence we obtain that the induced map $H^*(u)$ must be this isometric isomorphism [11; Lemma 7.2.6]. Therefore, the composition $H^*(v) = (\widehat{p}^*)^{-1} \circ H^*(u|)$ is also an isometric isomorphism. \square

3.4. A dual for measure homology. In order to develop a duality principle in the setting of measure homology, we have to construct a “dual” $\widehat{\mathcal{H}}^*(X)$ playing the rôle of the bounded cohomology groups $\widehat{H}^*(X)$ in the singular theory:

If $c = \sum_{\sigma \in \text{map}(\Delta^k, X)} a_\sigma \cdot \sigma \in C_k(X)$ is a singular chain and $f \in \widehat{\mathcal{C}}^k(X)$ is a singular cochain, their Kronecker product is given by

$$\langle f, c \rangle = f(c) = \sum_{\sigma \in \text{map}(\Delta^k, X)} a_\sigma \cdot f(\sigma) \in \mathbf{R}.$$

If we think of c as a linear combination of atomic measures, this looks like an integration of f over c . Hence our “dual” in measure homology consists of (bounded) functions that can be integrated over measure chains:

Definition (3.7). Let X be a topological space and let $k \in \mathbf{N}$. We define

$$\widehat{\mathcal{C}}^k(X) := \{f: \text{map}(\Delta^k, X) \longrightarrow \mathbf{R} \mid f \text{ is Borel measurable and bounded}\}$$

and (where $f(\partial(\sigma))$ is an abbreviation for $\sum_{j=0}^{k+1} (-1)^j \cdot f(\sigma \circ \partial_j)$)

$$\begin{aligned} \delta: \widehat{\mathcal{C}}^k(X) &\longrightarrow \widehat{\mathcal{C}}^{k+1}(X) \\ f &\longmapsto (\sigma \mapsto (-1)^{k+1} \cdot f(\partial(\sigma))). \end{aligned} \quad \diamond$$

It is not hard to see that this map $\delta: \widehat{\mathcal{C}}^k(X) \longrightarrow \widehat{\mathcal{C}}^{k+1}(X)$ is indeed well-defined and that it turns $\widehat{\mathcal{C}}^*(X)$ into a cochain complex.

Definition (3.8). Let X be a topological space and let $k \in \mathbf{N}$. The k -th bounded measure cohomology group of X is given by $\widehat{\mathcal{H}}^k(X) := H^k(\widehat{\mathcal{C}}^*(X), \delta)$. We

write $\|\cdot\|_\infty$ for the seminorm on $\widehat{\mathcal{H}}^k(X)$ which is induced by the supremum norm on $\widehat{\mathcal{C}}^k(X)$. \diamond

As a second step, we have to generalise the Kronecker product to bounded measure cohomology. As indicated above, our Kronecker product relies on integration:

Definition (3.9). Let X be a topological space and let $k \in \mathbf{N}$. The **Kronecker product** of $\mu \in \mathcal{C}_k(X)$ and $f \in \widehat{\mathcal{C}}^k(X)$ is defined as

$$\langle f, \mu \rangle := \int f d\mu.$$

If μ is a measure cycle and f is a cocycle, we write $\langle [f], [\mu] \rangle := \langle f, \mu \rangle = \int f d\mu$. \diamond

The integral is defined and finite, since the elements of $\mathcal{C}_k(X)$ are (signed) measures of finite total variation and the elements of $\widehat{\mathcal{C}}^k(X)$ are bounded measurable functions. Moreover, the integral is obviously bilinear. The transformation formula implies that the definition of the Kronecker product on (co)homology does not depend on the chosen representatives. Hence the Kronecker product is well-defined and bilinear.

Lemma (3.10). *Let X be a topological space and let $k \in \mathbf{N}$. The Kronecker product defined above is compatible with the Kronecker product on bounded cohomology in the following sense: for all $f \in \widehat{\mathcal{C}}^k(X)$ and all $c \in \mathcal{C}_k(X)$,*

$$\langle v_2(f), c \rangle = \langle f, i_X(c) \rangle,$$

where $v_2(f)$ denotes the linear extension of $f: \text{map}(\Delta^k, X) \rightarrow \mathbf{R}$ to the vector space $\mathcal{C}_k(X)$. Passage to (co)homology yields for all $\varphi \in \widehat{\mathcal{H}}^k(X)$ and all $\alpha \in H_k(X)$

$$\langle H^k(v_2)(\varphi), \alpha \rangle = \langle \varphi, H_k(i_X)(\alpha) \rangle.$$

Proof. Since both Kronecker products are bilinear, it suffices to consider the case where c consists of a single singular simplex $\sigma \in \text{map}(\Delta^k, X)$. Then the left hand side – by definition – evaluates to $f(\sigma)$. Since $i_X(c) = i_X(\sigma) = \delta_\sigma$ is the atomic measure on $\text{map}(\Delta^k, X)$ concentrated in σ , we obtain

$$\langle f, i_X(c) \rangle = \int f d\delta_\sigma = 1 \cdot f(\sigma)$$

for the right hand side. The corresponding equality in (co)homology follows because $v_2: \widehat{\mathcal{C}}^*(X) \rightarrow \widehat{\mathcal{C}}^*(X)$ is easily recognised to be a chain map. \square

The above Kronecker product leads to the following (slightly weakened) duality principle:

Lemma (3.11) (Duality Principle of Measure Homology). *Let X be a topological space, let $k \in \mathbf{N}$, and $\alpha \in \mathcal{H}_k(X)$.*

- (1) *If $\|\alpha\|_{\mathbf{m}} = 0$, then $\langle \varphi, \alpha \rangle = 0$ for all $\varphi \in \widehat{\mathcal{H}}^k(X)$.*
- (2) *If $\|\alpha\|_{\mathbf{m}} > 0$, then*

$$\|\alpha\|_{\mathbf{m}} \geq \sup \left\{ \frac{1}{\|\varphi\|_\infty} \mid \varphi \in \widehat{\mathcal{H}}^k(X), \langle \varphi, \alpha \rangle = 1 \right\}.$$

Proof. Let $\varphi \in \widehat{\mathcal{C}}^k(X)$. Assume that $\mu \in \mathcal{C}_k(X)$ is a measure cycle representing α and $f \in \widehat{\mathcal{C}}^k(X)$ is a cocycle representing φ . If $\langle \varphi, \alpha \rangle = 1$, then

$$1 = |\langle \varphi, \alpha \rangle| = \left| \int f d\mu \right| \leq \|f\|_\infty \cdot \|\mu\|.$$

Taking the infimum over all representatives results in $1 \leq \|\varphi\|_\infty \cdot \|\alpha\|_m$. In particular, if there exists such a φ , then

$$\|\alpha\|_m \geq \frac{1}{\|\varphi\|_\infty} > 0.$$

Now the lemma is an easy consequence of this inequality. \square

Remark (3.12). A posteriori we will be able to conclude – in view of Theorem (1.1), Theorem (3.3) and Lemma (3.10) – that in the first part of the lemma “if and only if” is also true and that in the second part equality holds. \diamond

4. PROVING THE ISOMETRY

This section is devoted to the proof of Theorem (1.1). To show that the algebraic isomorphism between singular homology and measure homology is isometric, we proceed in two steps. First, we prove the theorem in the special case of connected locally finite CW-complexes. In the second step we generalise this result using a colimit argument.

4.1. First step – connected locally finite CW-complexes. We investigate the dual $\widehat{\mathcal{C}}^*(X)$ by means of the complex $l^*(X)$ introduced in Definition (3.4): recall that the vector space $l^k(X)$ is the set of all bounded functions in $\text{map}(\widetilde{X}^{k+1}, \mathbf{R})$ that are $\pi_1(X)$ -invariant. Then the key to the proof of Theorem (1.1) is a careful analysis of the diagram

$$\begin{array}{ccc} & l^*(X) & \\ & \swarrow v_1 & \downarrow v \\ \widehat{\mathcal{C}}^*(X) & & \widehat{\mathcal{C}}^*(X), \\ & \searrow v_2 & \end{array}$$

the maps being defined as follows:

- For $k \in \mathbf{N}$ let $s_k: \text{map}(\Delta^k, X) \longrightarrow \text{map}(\Delta^k, \widetilde{X})$ be a Borel section of the map induced by the universal covering map. The existence of such a section is guaranteed by the following theorem – whose (elementary, but rather technical) proof is exiled to the Appendix:

Theorem (4.1). *Let X be a connected locally finite CW-complex or a manifold. Then the map*

$$\begin{aligned} P: \text{map}(\Delta^k, \widetilde{X}) &\longrightarrow \text{map}(\Delta^k, X) \\ \sigma &\longmapsto p \circ \sigma \end{aligned}$$

induced by the universal covering map $p: \widetilde{X} \longrightarrow X$ admits a Borel section.

If $f \in \mathcal{I}^k(X)$, we write

$$\begin{aligned} v_1(f) : \text{map}(\Delta^k, X) &\longrightarrow \mathbf{R} \\ \sigma &\longmapsto f((s_k(\sigma))(e_0), \dots, (s_k(\sigma))(e_k)). \end{aligned}$$

Since s_k is Borel, $v_1(f)$ is also Borel.

- The map v_2 is given by linear extension (cf. Lemma (3.10)).
- The map v is defined in Theorem (3.5): For $f \in \mathcal{I}^k(X)$, the homomorphism $v(f)$ is the linear extension of

$$\begin{aligned} \text{map}(\Delta^k, X) &\longrightarrow \mathbf{R} \\ \sigma &\longmapsto f((s_k(\sigma))(e_0), \dots, (s_k(\sigma))(e_k)). \end{aligned}$$

Remark (4.2). Since the functions living in $\mathcal{I}^*(X)$ are both $\pi_1(X)$ -invariant and bounded, v_1 is a well-defined cochain map, and $H^*(v_1)$ does not increase the norm. By construction, the diagram is commutative. In particular, $H^*(v_2)$ is surjective by Theorem (3.5). \square

The crux of the above diagram is that the vertical arrow induces an isometric isomorphism on the level of cohomology and that v_1 does not increase the seminorm. Hence the duality principles allow us to deduce that the algebraic isomorphism must be isometric for all connected locally finite CW-complexes:

Proof (of Theorem (1.1) for connected locally finite CW-complexes). According to Theorem (2.5), the induced homomorphism $H_*(i_X) : H_*(X) \longrightarrow \mathcal{H}_*(X)$ is an isomorphism. Therefore, it remains to show that $H_*(i_X)$ is compatible with the seminorms.

Let $k \in \mathbf{N}$ and $\alpha \in H_k(X)$. Since $i_X : C_*(X) \longrightarrow \mathcal{C}_*(X)$ is norm preserving, it is immediate that $\|H_k(i_X)(\alpha)\|_m \leq \|\alpha\|_1$.

The proof of the reverse inequality is split into two cases:

- (1) Suppose $\|H_k(i_X)(\alpha)\|_m = 0$. From Lemma (3.10) and Lemma (3.11) we obtain

$$\langle H^k(v_2)(\varphi), \alpha \rangle = \langle \varphi, H_k(i_X)(\alpha) \rangle = 0$$

for all $\varphi \in \widehat{\mathcal{H}}^k(X)$. By the previous remark, $H^k(v_2)$ is surjective. Hence

$$\forall_{\psi \in \widehat{H}^k(X)} \langle \psi, \alpha \rangle = 0,$$

implying $\|\alpha\|_1 = 0$ by duality (Theorem (3.3)).

- (2) Let $\|H_k(i_X)(\alpha)\|_m > 0$. In this case, the duality principle for measure homology (Lemma (3.11)) and Lemma (3.10) yield

$$\begin{aligned} \|H_k(i_X)(\alpha)\|_m &\geq \sup \left\{ \frac{1}{\|\varphi\|_\infty} \mid \varphi \in \widehat{\mathcal{H}}^k(X), \langle \varphi, H_k(i_X)(\alpha) \rangle = 1 \right\} \\ &= \sup \left\{ \frac{1}{\|\varphi\|_\infty} \mid \varphi \in \widehat{\mathcal{H}}^k(X), \langle H^k(v_2)(\varphi), \alpha \rangle = 1 \right\}. \end{aligned}$$

We compare the last set with the corresponding set of Theorem (3.3): Let $\psi \in \widehat{H}^k(X)$ such that $\langle \psi, \alpha \rangle = 1$. Since $H^k(v)$ is an isometric isomorphism (Theorem (3.5)), there exists a $\xi \in H^k(\mathcal{I}^*(X))$ satisfying

$$H^k(v)(\xi) = \psi \quad \text{and} \quad \|\xi\|_\infty = \|\psi\|_\infty.$$

Then $\varphi := H^k(A)(\xi) \in \widehat{\mathcal{H}}^k(X)$ possesses the following properties:

- By construction, $H^k(v_2)(\varphi) = (H^k(v_2) \circ H^k(v_1))(\xi) = H^k(v)(\xi) = \psi$, and hence

$$\langle H^k(v_2)(\varphi), \alpha \rangle = \langle \psi, \alpha \rangle = 1.$$

- Furthermore, we get from Remark (4.2) that

$$\|\varphi\|_\infty = \|H^k(v_1)(\xi)\|_\infty \leq \|\xi\|_\infty = \|\psi\|_\infty.$$

Combining these properties with the above estimate results in

$$\|H_k(i_X)(\alpha)\|_m \geq \sup \left\{ \frac{1}{\|\psi\|_\infty} \mid \psi \in \widehat{H}^k(X), \langle \psi, \alpha \rangle = 1 \right\}.$$

Since $\|\alpha\|_1 \geq \|H_k(i_X)(\alpha)\|_m > 0$, we can use the duality principle (Theorem (3.3)) to conclude that

$$\|H_k(i_X)(\alpha)\|_m \geq \|\alpha\|_1. \quad \square$$

4.2. Second step – the general case. We can now reduce the general case of connected CW-complexes to the case of connected finite CW-complexes:

Proof (of Theorem (1.1) – the general case). Let X be a connected CW-complex. Again, Theorem (2.5) states that $H_*(i_X): H_*(X) \rightarrow \mathcal{H}_*(X)$ is an isomorphism and it remains to prove that $H_*(i_X)$ is isometric:

Let $\alpha \in H_k(X)$. Then clearly $\|\alpha\|_1 \geq \|H_*(i_X)(\alpha)\|_m$. For the converse inequality, let $\mu \in \mathcal{C}_k(X)$ be a measure chain representing $H_*(i_X)(\alpha)$. Hansen showed in his proof that measure homology respects certain colimits [6; proof of Proposition 5.1] that we can find a compact subspace $A \subset X$ and a measure chain $\nu \in \mathcal{C}_k(A)$ such that $\mathcal{C}_k(i)(\nu) = \mu$, where $i: A \hookrightarrow X$ is the inclusion. Then, as one can check easily, $\|\nu\|_m = \|\mu\|_m$.

Since A is compact and X is a connected CW-complex, we can assume that A is a connected finite subcomplex of X .

The first step of our proof shows that the isomorphism $H_*(i_A): H_*(A) \rightarrow \mathcal{H}_*(A)$ is isometric. In particular, the preimage $\beta := H_*(i_A)^{-1}([\nu])$ satisfies

$$\|\beta\|_1 = \|[\nu]\|_m \leq \|\nu\|_m = \|\mu\|_m.$$

By construction, $H_*(i)(\beta) = H_*(i_X)^{-1} \circ \mathcal{H}_*(i) \circ H_*(i_A)(\beta) = \alpha$, and therefore functoriality implies

$$\|\alpha\|_1 = \|H_*(i)(\beta)\|_1 \leq \|\beta\|_1 \leq \|\mu\|_m.$$

Taking the infimum over all representatives μ of $H_*(i_X)(\alpha)$ gives the desired inequality $\|\alpha\|_1 \leq \|H_*(i_X)(\alpha)\|_m$. \square

5. A GLIMPSE AT THE SMOOTH UNIVERSE

In this section, a short exposition of the smooth version of the isometric isomorphism (Theorem (1.2)) is given. We first state a precise definition of smooth measure homology. Section 5.2 introduces smooth singular homology which is the building bridge between singular homology and smooth measure homology. In Section 5.3 and 5.4, the corresponding algebraic and isometric isomorphisms are explained.

5.1. Definition of smooth measure homology. In order to define smooth measure homology, we have to make precise what smooth simplices are and what the topology on the corresponding mapping spaces looks like. Then the definition is completely analogous to the definition of measure homology:

Definition (5.1). Let M be a smooth manifold and let $k \in \mathbf{N}$.

- A singular simplex $\sigma: \Delta^k \rightarrow M$ is called **smooth** if it can be extended to a smooth map on an open neighbourhood of Δ^k . We write $\text{map}_\infty(\Delta^k, M)$ for the set of all smooth singular simplices.
- The **\mathcal{C}^1 -topology** on $\text{map}_\infty(\Delta^k, M)$ is the unique topology that turns the differential $\text{map}_\infty(\Delta^k, M) \rightarrow \text{map}(T\Delta^k, TM)$ into a homeomorphism onto the image, where $\text{map}(T\Delta^k, TM)$ is endowed with the compact-open topology.
- The **k -th smooth measure chain group** $\mathcal{C}_k^s(M)$ is defined like the k -th measure chain group but using $\text{map}_\infty(\Delta^k, M)$ with the \mathcal{C}^1 -topology instead of $\text{map}(\Delta^k, M)$ with the compact-open topology.
- The **k -th smooth measure homology group** $\mathcal{H}_k^s(M)$ is the k -th homology group of $\mathcal{C}_*^s(M)$, where the boundary operator is defined as in the non-smooth case.
- The total variation on $\mathcal{C}_*^s(M)$ induces a seminorm on smooth measure homology, which is denoted by $\|\cdot\|_m^s$. \diamond

5.2. Smooth singular (co)homology. There is no reasonable chain map between $C_*(M)$ and $\mathcal{C}_*^s(M)$, so we take a small detour to smooth singular homology. On (co)homology, it turns out – as one would suspect – that smooth singular homology and singular homology are isometrically isomorphic.

Definition (5.2). Let M be a smooth manifold. Then $C_*^s(M)$ stands for the subcomplex of $C_*(M)$ generated by all smooth simplices. We write

$$H_*^s(M) := H_*(C_*^s(M), \partial|_{C_*^s(M)})$$

for **smooth singular homology** (with real coefficients). Furthermore, we obtain a seminorm on $H_*^s(M)$ induced by the ℓ^1 -norm on $C_*^s(M)$. \diamond

Proposition (5.3). *Let M be a smooth manifold. The inclusion $j_M: C_*^s(M) \hookrightarrow C_*(M)$ induces a natural (isometric) isomorphism*

$$H_*^s(M) \cong H_*(M).$$

Proof. Via the Whitney approximation theorem, a smoothing operator

$$s: C_*(M) \rightarrow C_*^s(M)$$

can be constructed [8; page 417], satisfying the following conditions: The map s is a chain map with $s \circ j_M = \text{id}$ and $j_M \circ s \simeq \text{id}$, and for each singular simplex $\sigma \in \text{map}(\Delta^k, M)$ the image $s(\sigma) \in C_k^s(M)$ consists of just one smooth simplex.

The first part implies that $H_*(j_M): H_*^s(M) \rightarrow H_*(M)$ is an isomorphism with inverse $H_*(s)$. Moreover, the second property ensures that $H_*(s)$ does not increase the seminorm. Hence $H_*(j_M)$ is isometric. \square

Moreover, we need a smooth version of bounded cohomology:

Definition (5.4). If M is a smooth manifold, we define $\widehat{C}_s^*(M)$ as the set of all homomorphisms $f: C_k^s(M) \rightarrow \mathbf{R}$ satisfying

$$\sup\{|f(\sigma)| \mid \sigma \in \text{map}_\infty(\Delta^k, M)\} < \infty.$$

As in the non-smooth case, $\widehat{C}_s^*(M)$ can be equipped with a coboundary operator δ and we write $\widehat{H}_s^*(M) := H^*(\widehat{C}_s^*(M), \delta)$ for **smooth bounded cohomology**. \diamond

Proposition (5.5). *Let M be a smooth manifold. Then the restriction homomorphism $\widehat{C}^*(M) \rightarrow \widehat{C}_s^*(M)$ induces a natural isometric isomorphism*

$$\widehat{H}^*(M) \cong \widehat{H}_s^*(M).$$

Moreover, there is a duality principle (in the sense of Theorem (3.3)) for smooth bounded cohomology.

Proof. The dual

$$\begin{aligned} \widehat{C}^*(M) &\longrightarrow \widehat{C}^{*-1}(M) \\ f &\longmapsto (-1)^* \cdot f \circ h_{*-1} \end{aligned}$$

of a (bounded) chain homotopy $h: j_M \circ s \simeq \text{id}$ shows that $\widehat{H}_s^*(M) \cong \widehat{H}^*(M)$. Note that h can be chosen to be bounded in each degree [8; page 417ff], so that the above cochain homotopy is indeed well-defined. Furthermore, the duals $\widehat{C}^*(j_M)$ and $\widehat{C}^*(s)$ do not increase the seminorm. Therefore, $\widehat{H}^*(j_M)$ is an isometric isomorphism.

The duality principle follows easily from Theorem (3.3) and the fact that both $H_*(j_M)$ and $\widehat{H}^*(j_M)$ are isometric isomorphisms. \square

5.3. The algebraic isomorphism. Analogously to the non-smooth case, we can compare smooth singular homology and smooth measure homology:

Definition (5.6). If M is a smooth manifold and $k \in \mathbf{N}$, we write

$$\begin{aligned} i_M^s: C_k^s(M) &\longrightarrow C_k^s(M) \\ \sum_{\sigma \in \text{map}_\infty(\Delta^k, M)} a_\sigma \cdot \sigma &\longmapsto \sum_{\sigma \in \text{map}_\infty(\Delta^k, M)} a_\sigma \cdot \delta_\sigma. \end{aligned} \quad \diamond$$

Clearly, this inclusion induces a natural cochain map $C_*^s(M) \rightarrow C_*^s(M)$ (also denoted by i_M^s) which is norm-preserving.

Theorem (5.7). *Let M be a smooth manifold. Then i_M^s induces a natural isomorphism (of real vector spaces)*

$$H_*^s(M) \cong \mathcal{H}_*^s(M).$$

Proof. Zastrow explains how one can translate his proofs of the Eilenberg-Steenrod axioms for measure homology to the smooth case [16; Theorem 3.4]. Then an “induction” similar to Milnor’s proof of Poincaré duality [9, 14; page 351, Theorem (4.10)] shows that i_M^s induces an isomorphism between smooth singular homology and smooth measure homology. \square

Corollary (5.8). *For all smooth manifolds, singular homology and smooth measure homology are naturally isomorphic.* \square

5.4. The isometric isomorphism. As in Section 4.1, we can apply duality to see that smooth singular homology and smooth measure homology are isometrically isomorphic.

The duals $\widehat{\mathcal{C}}_s^*(M)$ and $\widehat{\mathcal{H}}_s^*(M)$ are defined like $\widehat{\mathcal{C}}^*(M)$ and $\widehat{\mathcal{H}}^*(M)$ where the mapping space $\text{map}(\Delta^k, M)$ (equipped with the compact-open topology) is replaced by $\text{map}_\infty(\Delta^k, M)$ (with the \mathcal{C}^1 -topology). Literally the same proof as for Theorem (3.11) yields that there is a duality principle for $\widehat{\mathcal{H}}_s^*(M)$ and $\mathcal{H}_s^*(M)$.

We now have collected all the tools necessary to prove Theorem (1.2):

Theorem (5.9). *Let M be a connected smooth manifold. Then the natural isomorphism $H_*(i_M^s) \circ H_*(j_M)^{-1} : H_*(M) \rightarrow \mathcal{H}_s^*(M)$ induced by the canonical inclusions $i_M^s : C_*^s(M) \rightarrow \mathcal{C}_*^s(M)$ and $j_M : C_*^s(M) \rightarrow C_*(M)$ is isometric.*

Proof. Analogously to Section 4.1, we can consider the commutative diamond

$$\begin{array}{ccc}
 & \mathbb{I}^*(M) & \\
 v_1 \swarrow & & \searrow v \\
 \widehat{\mathcal{C}}_s^*(M) & & \widehat{\mathcal{C}}^*(M), \\
 v_2 \searrow & & \swarrow v_3 \\
 & \widehat{\mathcal{C}}_s^*(M) &
 \end{array}$$

where v_1 , v_2 and v are defined as in Section 4.1. The cochain map v_3 is given by restriction, which is an isometric isomorphism by Proposition (5.5). Since we have established duality principles for smooth singular homology and smooth measure homology, the same proof as in Section 4.1 shows that $H_*(i_M^s)$ is an isometric isomorphism. Now the theorem follows with help of Proposition (5.3). \square

6. APPLICATIONS

We can apply the isometric isomorphisms of Theorem (1.1) and (1.2) to compute the simplicial volume in terms of (smooth) measure homology. In fact, this was the motivation for Thurston to invent measure homology.

Definition (6.1). If M is an oriented closed connected manifold, its **simplicial volume** is given by

$$\|M\| := \|[M]_{\mathbf{R}}\|_1,$$

where $[M]_{\mathbf{R}} \in H_{\dim M}(M)$ denotes the (real) fundamental class of M . \diamond

The simplicial volume of spheres and tori is zero since these manifolds admit selfmaps of degree larger than 1. On the other hand, the simplicial volume of hyperbolic manifolds is non-zero [1; Theorem C.4.2], e.g., $\|F_g\| = 4g - 4$ for all oriented closed surfaces F_g of genus $g \geq 2$. An amazing facet of the simplicial volume is that it is a *topological* invariant bounding the minimal volume from below (modulo a constant factor depending only on the dimension) [4; page 12].

Corollary (6.2). *Let M be an oriented closed connected manifold. Then*

$$\|M\| = \|H_*(i_M)([M]_{\mathbf{R}})\|_{\mathfrak{m}}.$$

If M is smooth, then

$$\|M\| = \|H_*(i_M^s) \circ H_*(j_M)^{-1}([M]_{\mathbf{R}})\|_{\mathfrak{m}}^s.$$

Proof. The first part follows because all closed manifolds are CW-complexes (up to homotopy) and hence Theorem (1.1) is applicable. The second part is a direct consequence of Theorem (1.2). \square

For hyperbolic manifolds, this result is well-known [16; Remark 0.1].

The above corollary makes it possible to apply Thurston's smearing technique [15, 13, 14; page 6.8ff, page 547ff, Chapter 5] to prove the proportionality principle of simplicial volume [15; page 6.9] (more details are given in [14; Chapter 5]):

Theorem (6.3) (Proportionality Principle of Simplicial Volume). *Let M and N be Riemannian manifolds with isometric universal Riemannian coverings. Then*

$$\frac{\|M\|}{\text{vol } M} = \frac{\|N\|}{\text{vol } N}.$$

Gromov's original proof of the proportionality principle is sketched in *Volume and bounded cohomology* [4; page 11].

Corollary (6.4). *The simplicial volume of oriented closed connected smooth flat manifolds vanishes.*

Proof. If we scale a flat manifold, the simplicial volume and the universal Riemannian covering space remain the same but the volume changes. Hence the proportionality principle implies that the simplicial volume must be zero.

Note that this result can also be obtained by Gromov's estimate of the minimal volume [4; page 12] or by the boundedness results for the Euler class of Milnor and Sullivan [4; page 23]. \square

Another consequence of the proportionality principle is the following mapping theorem, due to Gromov [5; 5.36].

Corollary (6.5) (Gromov's mapping theorem). *Let $n \in \mathbf{N}$ and let S_1, \dots, S_n and S'_1, \dots, S'_n be hyperbolic surfaces. If $f: S_1 \times \dots \times S_n \rightarrow S'_1 \times \dots \times S'_n$ is a continuous map of degree d , then*

$$|\chi(S_1 \times \dots \times S_n)| \geq d \cdot |\chi(S'_1 \times \dots \times S'_n)|.$$

Proof. For brevity, we write $S := S_1 \times \dots \times S_n$ and $S' := S'_1 \times \dots \times S'_n$. Since the surfaces involved are all hyperbolic, the universal Riemannian coverings of S and S' coincide. Therefore, we obtain from the proportionality principle and by functoriality of $\|\cdot\|_1$ that

$$\frac{\|S'\|}{\text{vol } S'} = \frac{\|S\|}{\text{vol } S} \geq \frac{d \cdot \|S'\|}{\text{vol } S}.$$

Moreover, $\|S'\| \neq 0$ [4; example on page 9 and (1) on page 10], and both the volume and the Euler characteristic are multiplicative with respect to the product. Since the hyperbolic volume and the absolute value of the Euler characteristic of oriented closed connected surfaces are proportional (by Gauß-Bonnet), the result follows. \square

APPENDIX – EXISTENCE OF A BOREL SECTION

To complete the proof of Theorem (1.1), we still have to provide a proof of Theorem (4.1). As a first step we prove the following (stronger) statement:

Lemma (A.1). *Let X be a locally path-connected, semi-locally simply connected space such that the universal covering \tilde{X} is metrisable (e.g., let X be a locally finite CW-complex). Then the map*

$$\begin{aligned} P: \text{map}(\Delta^k, \tilde{X}) &\longrightarrow \text{map}(\Delta^k, X) \\ \sigma &\longmapsto p \circ \sigma \end{aligned}$$

induced by the universal covering map $p: \tilde{X} \longrightarrow X$ is a local homeomorphism.

In the proof of this lemma we use the following notation for the sub-basic sets of the compact-open topology:

Definition (A.2). In the above situation, if $K \subset \Delta^k$ is compact and $U \subset X$ (or $U \subset \tilde{X}$) is open, we write U^K for the set of all $f \in \text{map}(\Delta^k, X)$ (or all $f \in \text{map}(\Delta^k, \tilde{X})$ respectively) satisfying $f(K) \subset U$. \diamond

Proof. Let $\sigma \in \text{map}(\Delta^k, \tilde{X})$. Then there is a small neighbourhood U of $\sigma(e_0)$ in \tilde{X} :

Definition (A.3). An open subset $U \subset \tilde{X}$ is called **small** if p is trivial over $p(U)$ and the restriction $p|_U: U \longrightarrow p(U)$ is a homeomorphism. \diamond

In particular, the image $p(U) \subset X$ is open because covering maps are open.

Remark (A.4). Since $p: \tilde{X} \longrightarrow X$ is a covering map and since X is locally path-connected, each point in \tilde{X} possesses a basic family of small neighbourhoods. \diamond

We show that $P(U^{\{e_0\}})$ is open and that the restriction $P|_{U^{\{e_0\}}}: U^{\{e_0\}} \longrightarrow P(U^{\{e_0\}})$ is a homeomorphism.

The set $P(U^{\{e_0\}})$ is open in $\text{map}(\Delta^k, X)$. By definition, $P(U^{\{e_0\}}) \subset (p(U))^{\{e_0\}}$. On the other hand, for each $\tau \in (p(U))^{\{e_0\}}$ there exists a lift $\tilde{\tau}: \Delta^k \longrightarrow \tilde{X}$ such that $\tilde{\tau}(e_0) \in U$ because Δ^k is simply connected. Thus

$$P(U^{\{e_0\}}) = (p(U))^{\{e_0\}}.$$

Since $p(U)$ is open in X , this is an open subset of $\text{map}(\Delta^k, X)$.

The restriction $P|_{U^{\{e_0\}}}: U^{\{e_0\}} \longrightarrow (p(U))^{\{e_0\}} = P(U^{\{e_0\}})$ is bijective. Since U is small, $p|_U$ is injective. Hence the uniqueness of lifts (prescribed on e_0 by the property to map into U) shows injectivity of $P|_{U^{\{e_0\}}}$ (Δ^k is connected).

The restriction $P|_{U^{\{e_0\}}}: U^{\{e_0\}} \longrightarrow (p(U))^{\{e_0\}} = P(U^{\{e_0\}})$ is a homeomorphism. By definition of the compact-open topology, P is continuous. It therefore remains to prove that the restriction $P|_{U^{\{e_0\}}}$ is open (which is the lion share of the proof):

Let $\tilde{\tau} \in U^{\{e_0\}}$ and let A be an open neighbourhood of $\tilde{\tau}$ in $U^{\{e_0\}}$. We have to show that $P|_{U^{\{e_0\}}}(A) \subset \text{map}(\Delta^k, X)$ is open: Since $P|_{U^{\{e_0\}}}$ is injective, this restriction is compatible with unions and intersections. By definition of the compact-open topology, it is therefore sufficient to consider the case $A = V^K \cap U^{\{e_0\}}$, where $K \subset \Delta^k$ is compact and $V \subset \tilde{X}$ is open.

In the following, we make use of especially small subsets of \tilde{X} :

Definition (A.5). A family $(U_i)_{i \in I}$ of subsets of \tilde{X} is **tiny** if the following conditions hold: all U_i are small and whenever $U_i \cap U_j \neq \emptyset$, then the union $U_i \cup U_j$ is also small. \diamond

Proposition (A.6). *Let $Y \subset \tilde{X}$ a compact subset and let $(U_i)_{i \in I}$ be a family of open subsets of \tilde{X} covering Y . Then there is a number $\varepsilon \in \mathbf{R}_{>0}$ with the following property: If $x \in Y$ and $B_\varepsilon(x)$ is the open ball in \tilde{X} around x with radius ε , then $B_\varepsilon(x) \subset U_i$ for some $i \in I$. Such an ε is called a **Lebesgue number** of the family $(U_i)_{i \in I}$.*

Proof. One can use literally the same proof as in Dugundji's book for the existence of a Lebesgue number (in a slightly weaker context) [3; Theorem XI.4.5]. \square

Corollary (A.7). *Let $Y \subset \tilde{X}$ be a compact subset covered by a family $(V_j)_{j \in J}$ of open sets. Then there is a (finite) tiny family $(U_i)_{i \in I}$ covering Y subordinate to $(V_j)_{j \in J}$.*

Proof. It is possible to cover Y by a family $(V'_j)_{j \in J'}$ of small sets subordinate to $(V_j)_{j \in J}$ (Remark (A.4)). Let $\varepsilon \in \mathbf{R}_{>0}$ be a Lebesgue-number, in the above sense, of this covering. Since \tilde{X} is locally path-connected, there is a covering $(U_i)_{i \in I}$ of Y by small subsets satisfying

$$\text{diam}(U_i) < \frac{\varepsilon}{2}$$

for all $i \in I$. Then $(U_i)_{i \in I}$ is tiny: Let $i, j \in I$ with $U_i \cap U_j \neq \emptyset$. Thus

$$\text{diam}(U_i \cup U_j) < \varepsilon.$$

By construction of ε , there is an $\ell \in J$ such that $U_i \cup U_j \subset V'_\ell$. Since U_i and U_j are open, so is their union $U_i \cup U_j$. Now $U_i \cup U_j \subset V'_\ell$ implies that $U_i \cup U_j$ is small. Furthermore, we can choose I to be finite because Y is compact. \square

Using the above corollary, we obtain a tiny covering $(V_j)_{j \in J}$ of $\tilde{\tau}(\Delta^k)$. Applying the above corollary twice more (on the compact sets $\tilde{\tau}(K)$ and $\tilde{\tau}(e_0)$ and the induced coverings $(V \cap V_j)_{j \in J}$ and $(U \cap V_j)_{j \in J}$), we can find a finite tiny covering $(U_i)_{i \in I}$ of $\tilde{\tau}(\Delta^k)$ and compact subsets $(K_i)_{i \in I}$ of Δ^k such that the intersection

$$\tilde{B} := \bigcap_{i \in I} U_i^{K_i}$$

satisfies

$$\tilde{B} \subset U^{\{e_0\}} \quad \text{and} \quad \tilde{B} \subset V^K.$$

By construction, \tilde{B} is open in $\text{map}(\Delta^k, \tilde{X})$ and $\tilde{B} \subset V^K \cap U^{\{e_0\}} = A$. It therefore suffices to show that $P|_{U^{\{e_0\}}}(\tilde{B}) \subset \text{map}(\Delta^k, X)$ is open. More precisely, we prove that

$$P|_{U^{\{e_0\}}}(\tilde{B}) = B,$$

where

$$B := \bigcap_{i \in I} (p(U_i))^{K_i}.$$

It is clear that $P|_{U^{\{e_0\}}}(\tilde{B}) \subset B$. Conversely, let $\varrho \in B$. It suffices to check that the unique lift $\tilde{\varrho} \in \text{map}(\Delta^k, \tilde{X})$ with $\tilde{\varrho}(e_0) \in U$ lies in \tilde{B} . In the following, we prove that the set

$$D := \{x \in \Delta^k \mid \forall_{i \in I(x)} \tilde{\varrho}(x) \in U_i\}$$

is open and closed and that it contains e_0 , where we used the notation

$$I(x) := \{i \in I \mid x \in K_i\}.$$

The key to proving this claim is the following lemma based on tininess:

Lemma (A.8). *If $x \in \Delta^k$ and if there is a $j \in I(x)$ such that $\tilde{\varrho}(x) \in U_j$, then $\tilde{\varrho}(x) \in U_i$ for all $i \in I(x)$.*

Proof. Let $i \in I(x)$. Because of $\tilde{\tau}(K_i) \subset U_i$ and $\tilde{\tau}(K_j) \subset U_j$, we obtain $U_i \cap U_j \neq \emptyset$. Hence $U_i \cup U_j$ is small (the family $(U_\ell)_{\ell \in I}$ is tiny). In particular, $p^{-1}(\varrho(x)) \cap (U_i \cup U_j)$ contains precisely one element (namely $\tilde{\varrho}(x) \in U_j$). But $i \in I(x)$ implies that $p^{-1}(\varrho(x)) \cap U_i$ also has to contain (exactly) one element. Therefore, $\tilde{\varrho}(x) \in U_i \cap U_j$, which shows $\tilde{\varrho}(x) \in U_i$, as desired. \square

- *The set D is open:* For each $x \in D$,

$$W := \bigcap_{i \in I(x)} \tilde{\varrho}^{-1}(U_i) \cap \bigcap_{i \in I \setminus I(x)} (\Delta^k \setminus K_i)$$

is an open subset of Δ^k with $x \in W$ and $W \subset D$.

- *The set D is closed:* Let $x \in D \setminus \Delta^k$. By the above lemma, $\tilde{\varrho}(x) \notin U_i$ for all $i \in I(x)$. Since $\Delta^k = \bigcup_{i \in I} \overset{\circ}{K}_i$, there is an $i \in I$ such that $x \in \overset{\circ}{K}_i$ and $\tilde{\varrho} \notin U_i$. Let

$$W := \overset{\circ}{K}_i \cap \tilde{\varrho}^{-1}(p^{-1}(p(U_i)) \setminus U_i).$$

Since p is trivial over $p(U_i)$ (with discrete fibre), the preimage

$$\tilde{\varrho}^{-1}(p^{-1}(p(U_i)) \setminus U_i)$$

is open. By construction, $x \in W$ and $W \subset \Delta^k \setminus D$. Therefore, D is closed.

- *The vertex e_0 lies in D :* This is a direct consequence of $\varrho \in B \subset p(U)^{\{e_0\}}$ and Lemma (A.8).

Since Δ^k is connected, this implies $D = \Delta^k$ and hence proves Lemma (A.1). \square

Finally, we are able to conclude that P possesses a Borel section, as claimed in Theorem (4.1):

Proof (of Theorem (4.1)). We can cover $\text{map}(\Delta^k, \tilde{X})$ with countably many open sets $(V_n)_{n \in \mathbf{N}}$ on which P is a homeomorphism and such that $P(V_n)$ is open since \tilde{X} is second countable (e.g., one could take a countable covering of \tilde{X} by small sets U and consider sets of the form $U^{\{e_0\}}$). Setting $W_0 := P(V_0)$ and

$$\forall n \in \mathbf{N} \quad W_{n+1} := P(V_{n+1}) \setminus \bigcup_{j \in \{0, \dots, n\}} W_j,$$

we get a countable family $(W_n)_{n \in \mathbf{N}}$ of mutually disjoint Borel sets in $\text{map}(\Delta^k, X)$ such that the inverse $P^{-1}|_{W_n}$ is well-defined and continuous for each $n \in \mathbf{N}$. Moreover, $\text{map}(\Delta^k, X)$ is covered by the $(W_n)_{n \in \mathbf{N}}$ because P is surjective. Putting all these maps together yields the desired Borel section of P . \square

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