

Isomorphisms in ℓ^1 -homology

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December 20, 2006

Abstract. Taking the ℓ^1 -completion and the topological dual of the singular chain complex gives rise to ℓ^1 -homology and bounded cohomology respectively. Unlike ℓ^1 -homology, bounded cohomology is very well understood by the work of Gromov and Ivanov. Based on an observation by Matsumoto and Morita, we derive a mechanism linking isomorphisms on the level of homology of Banach chain complexes to isomorphisms on the level of cohomology of the dual Banach cochain complexes and vice versa. Therefore, certain results on bounded cohomology can be transferred to ℓ^1 -homology. For example, we obtain a new proof of the fact that ℓ^1 -homology only depends on the fundamental group and that ℓ^1 -homology with twisted coefficients admits a description in terms of projective resolutions. The latter one in particular fills a gap in Park's approach.

Mathematics Subject Classification (2000) – 55N35, 46M10, 57N99

1 Introduction

Semi-norms on singular homology provide topological invariants such as the simplicial volume. However, for example, singular homology itself turns out not to be very feasible for the study of the ℓ^1 -semi-norm (and hence of the simplicial volume). Only by passing to related theories such as bounded cohomology or ℓ^1 -homology the bigger picture becomes visible.

Unlike ℓ^1 -homology, bounded cohomology is very well understood by the work of Gromov [5] and Ivanov [9]. One of the main features of bounded cohomology is

that it only depends on the fundamental group of the space in question [5, 9; p. 40, Theorem 4.3]. Hence, it is natural to ask whether the same is true for ℓ^1 -homology.

A first step towards a result of this type is the observation by Matsumoto and Morita [10; Corollary 2.4] that the ℓ^1 -homology of a space is trivial if and only if its bounded cohomology is trivial. More generally, the homology of a Banach chain complex vanishes if and only if the cohomology of the dual Banach cochain complex vanishes (Theorem (3.4)).

Based on Matsumoto and Morita's result, the fact that bounded cohomology of spaces with amenable fundamental group is trivial, and an ℓ^1 -version of Brown's theorem, Bouarich [2; Corollaire 6] gave the first proof that ℓ^1 -homology depends only on the fundamental group.

In this article, we present a different, more lightweight, strategy: Applying the generalised version of Matsumoto and Morita's result to mapping cones of morphisms of Banach chain complexes links isomorphisms on homology to isomorphisms in cohomology:

Theorem (1.1). *Let $f: C \rightarrow D$ be a morphism of Banach chain complexes and let $f': D' \rightarrow C'$ be its dual.*

1. *Then the induced homomorphism $H_*(f): H_*(C) \rightarrow H_*(D)$ is an isomorphism of vector spaces if and only if $H^*(f'): H^*(D') \rightarrow H^*(C')$ is an isomorphism of vector spaces.*
2. *Furthermore, if $H^*(f'): H^*(D') \rightarrow H^*(C')$ is an isometric isomorphism, then also $H_*(f): H_*(C) \rightarrow H_*(D)$ is an isometric isomorphism.*

Notice however, that in general the cohomology of the dual complex does *not* coincide with the dual of the homology (see Remark (3.3)).

In the following, the main examples for Banach (co)chain complexes are ℓ^1 -chain complexes and bounded cochain complexes of topological spaces: The ℓ^1 -chain complex $C_*^{\ell^1}(X)$ of a topological space X is the ℓ^1 -completion of the singular chain complex of X with real coefficients and ℓ^1 -homology of X , denoted by $H_*^{\ell^1}(X)$, is defined as the homology of the ℓ^1 -chain complex.

Dually, the bounded cochain complex $C_b^*(X)$ of a topological space X is the topological dual of $C_*^{\ell^1}(X)$ and bounded cohomology, denoted by $H_b^*(X)$, is the cohomology of $C_b^*(X)$.

Applying Theorem (1.1) in this setting thus allows to transfer certain results on bounded cohomology to ℓ^1 -homology. For example, we obtain a new proof of the fact that ℓ^1 -homology depends only on the fundamental group:

Theorem (1.2). *The ℓ^1 -homology of topological spaces depends only on the fundamental group. More precisely: Let $f: X \rightarrow Y$ be a continuous map of connected countable CW-complexes inducing an isomorphism on the level of fundamental groups. Then the induced map*

$$H_*^{\ell^1}(f): H_*^{\ell^1}(X) \rightarrow H_*^{\ell^1}(Y)$$

is an isometric isomorphism with respect to the ℓ^1 -semi-norm.

More generally, the homomorphism $\pi_1(f)$ is allowed to have amenable kernel (Corollary (4.3)).

Another important class of Banach (co)chain complexes is provided by the ℓ^1 -completion $C_*^{\ell^1}(G)$ of the bar resolution of a discrete group G , as well as the dual complex $C_b^*(G)$. These complexes give rise to ℓ^1 -homology and bounded cohomology of discrete groups respectively. Applying Theorem (1.1) in this situation shows that ℓ^1 -homology and bounded cohomology of groups behave similarly (see Subsection 5.3).

Like ordinary group (co)homology ℓ^1 -homology and bounded cohomology of groups can be computed by certain projective and injective resolutions [9, 13]. More astonishingly, Ivanov showed that bounded cohomology of a space coincides with bounded cohomology of the corresponding fundamental group [9; Theorem 4.1] and hence that bounded cohomology of topological spaces also can be computed by certain injective resolutions.

Applying Theorem (1.1) to a suitable morphism $C_*^{\ell^1}(X) \rightarrow C_*^{\ell^1}(\pi_1(X))_{\pi_1 X}$ implies that ℓ^1 -homology of topological spaces admits an analogous description in terms of projective resolutions:

Theorem (1.3). *Let X be a connected countable CW-complex with fundamental group G and let C be a strong relatively projective resolution of the trivial Banach G -module \mathbf{R} . Then there is a canonical isomorphism*

$$H_*^{\ell^1}(X) \cong H_*(C_G) \cong H_*^{\ell^1}(G).$$

If C is the Banach bar resolution of G , then this isomorphism is isometric.

(Park [13] already claimed that this is true. However, due to a gap in her argument, her proof is not complete. This issue is addressed in Caveat (5.9) and (6.4).)

More generally, we prove Theorem (1.3) for ℓ^1 -homology with twisted coefficients (Theorem (6.3)). The corresponding generalisation of Ivanov's result on bounded cohomology is presented in Appendix B.

The motivation for studying not only bounded cohomology but also ℓ^1 -homology is that certain problems concerning the ℓ^1 -semi-norm on singular homology might be more accessible via ℓ^1 -homology than via bounded cohomology. For example, one might hope to get a better understanding of the semi-norms on spaces constructed out of smaller spaces.

This paper is organised as follows: In Section 2, the basic objects of study, i.e., normed and Banach chain complexes, are introduced. A thorough investigation of duality in the context of Banach chain complexes is provided in Section 3, including a proof of Theorem (1.1). In Section 4, the duality results are applied to the case of ℓ^1 -homology thus in particular providing a proof of Theorem (1.2). Analogously, Section 5 deals with applications of Theorem (1.1) to ℓ^1 -homology of discrete groups. The description of ℓ^1 -homology of spaces in terms of projective resolutions (and a proof of Theorem (1.3)) is given in Section 6. The background on homological algebra in the category of Banach G -modules needed in Section 5 and Section 6 is collected in Appendix A. Finally, Appendix B contains the description of bounded cohomology of spaces with twisted coefficients in terms of injective resolutions.

2 Homology of normed chain complexes

In this section, we introduce the basic objects of study, i.e., normed chain complexes and their homology. The homology of normed chain complexes inherits a semi-norm. For example, in the case of the singular chain complex, this semi-norm contains valuable geometric information such as the simplicial volume. In order to understand the semi-norm on homology, it suffices to consider the homology of the completion of the normed chain complex in question (Proposition (2.7)) and hence we can restrict ourselves to the case of Banach chain complexes. For singular homology the corresponding Banach chain complex leads to the definition of ℓ^1 -homology. A concise definition of ℓ^1 -homology and bounded cohomology of topological spaces is given in Subsection 2.3.

2.1 Normed chain complexes

- Definition (2.1).** 1. A **normed chain complex** is a chain complex of normed vector spaces, where all boundary morphisms are bounded linear operators. Analogously, a **normed cochain complex** is a cochain complex of normed vector spaces, where all coboundary morphisms are bounded linear operators.
2. A **Banach (co)chain complex** is a normed (co)chain complex consisting of Banach spaces.
3. A **morphism of normed (co)chain complexes** is a (co)chain map between normed (co)chain complexes consisting of bounded operators. \diamond

In this article, all Banach spaces are Banach spaces over \mathbf{R} and all (co)chain complexes are indexed over \mathbf{N} .

Definition (2.2). Let (C, ∂) be a normed chain complex. Then the **dual cochain complex** (C', ∂') is the normed cochain complex defined by

$$\forall_{n \in \mathbf{N}} \quad (C')^n := (C_n)',$$

where \cdot' stands for taking the (topological) dual normed vector space, together with the coboundary operators

$$\begin{aligned} (\partial')^n &:= (\partial_{n+1})' : (C')^n \longrightarrow (C')^{n+1} \\ f &\longmapsto (c \mapsto f(\partial_{n+1}(c))) \end{aligned}$$

and the norm given by $\|f\|_\infty := \sup\{|f(c)| \mid c \in C_n, \|c\| = 1\}$ for $f \in (C')^n$. \diamond

- Remark (2.3).** 1. If C is a normed (co)chain complex, then the (co)boundary operator can be extended to a (co)boundary operator on the completion \overline{C} that is bounded in each degree. Hence, the completion \overline{C} of C is a Banach (co)chain complex.
2. If C is a Banach chain complex, then its dual C' is also complete and thus a Banach cochain complex. Moreover, if C is a normed chain complex, then $C' = (\overline{C})'$. \square

Examples of Banach (co)chain complexes include the ℓ^1 -chain complexes of topological spaces (Subsection 2.3), the completion of the bar resolution of a discrete group (Subsection 5.2) and more general the resolutions used in the definition of ℓ^1 -homology of discrete groups (Subsection (5.1)). The corresponding dual cochain complexes are the source of the various incarnations of bounded cohomology.

2.2 (Semi)norms on (co)homology

Clearly, the presence of chain complexes calls for the investigation of the corresponding homology. In the case of normed chain complexes, the homology groups carry an additional piece of information – the semi-norm.

- Definition (2.4).** 1. Let (C, ∂) be a normed chain complex and let $n \in \mathbf{N}$. The n -th homology of C is the quotient

$$H_n(C) := \frac{\ker(\partial_n: C_n \rightarrow C_{n-1})}{\text{im}(\partial_{n+1}: C_{n+1} \rightarrow C_n)}.$$

2. Dually, if (C, δ) is a normed cochain complex, then its n -th cohomology is the quotient

$$H^n(C) := \frac{\ker(\delta^n: C^n \rightarrow C^{n+1})}{\text{im}(\delta^{n-1}: C^{n-1} \rightarrow C^n)}.$$

3. Let C be a normed chain complex. Then the norm $\|\cdot\|$ on C induces a semi-norm, also denoted by $\|\cdot\|$, on the homology $H_*(C)$ as follows: If $\alpha \in H_n(C)$, then

$$\|\alpha\| := \inf\{\|c\| \mid c \in C_n, \partial(c) = 0, [c] = \alpha\}.$$

Similarly, we define a semi-norm on the cohomology of normed cochain complexes. \diamond

Because the images of the (co)boundary operators of Banach (co)chain complexes are not necessarily closed, the induced semi-norms on (co)homology need not be norms. Therefore, it is sometimes convenient to look at the corresponding reduced versions instead:

- Definition (2.5).** 1. Let (C, ∂) be a normed chain complex and let $n \in \mathbf{N}$. Then the n -th reduced homology of C is given by

$$\overline{H}_n(C) := \ker \partial_n / \overline{\text{im} \partial_{n+1}},$$

where $\overline{}$ denotes the closure in C .

2. Analogously, if (C, δ) is a normed cochain complex and $n \in \mathbf{N}$, then the n -th reduced cohomology of C is given by

$$\overline{H}^n(C) := \ker \delta^n / \overline{\operatorname{im} \delta^{n-1}}. \quad \diamond$$

Remark (2.6). Any morphism $f: C \rightarrow D$ of normed chain complexes induces linear maps $H_n(f): H_n(C) \rightarrow H_n(D)$. Since f is continuous in each degree, these maps descend to linear maps $\overline{H}_n(f): \overline{H}_n(C) \rightarrow \overline{H}_n(D)$. Moreover, the maps $H_n(f)$ and $\overline{H}_n(f)$ are bounded. \square

In order to understand semi-norms on the homology of normed chain complexes, it suffices to consider the case of Banach chain complexes [17; Lemma 2.9]:

Proposition (2.7). *Let (D, ∂) be a normed chain complex and let (C, ∂) be a dense subcomplex. Then the induced map $H_*(C) \rightarrow H_*(D)$ is isometric. In particular, the induced map $\overline{H}_*(C) \rightarrow \overline{H}_*(D)$ must be injective.*

Proof. In the following, we write $i: C \hookrightarrow D$ for the inclusion and $\|\cdot\|$ for the norm on D .

Since C is a subcomplex, $\|H_*(i)\| \leq 1$. Conversely, let $z \in C_n$ be a cycle and let $\varepsilon \in \mathbf{R}_{>0}$. Furthermore, let $\bar{z} \in D_n$ be a cycle such that $[\bar{z}] = H_n(i)([z]) \in H_n(D)$. To prove the proposition, it suffices to find a cycle $z' \in C_n$ satisfying

$$[z'] = [z] \in H_n(C) \quad \text{and} \quad \|z'\| \leq \|\bar{z}\| + \varepsilon.$$

By definition of \bar{z} , there must be a chain $\bar{w} \in D_{n+1}$ with $\partial_{n+1}(\bar{w}) = i(z) - \bar{z}$. Since C_{n+1} lies densely in D_{n+1} and since $\|\partial_{n+1}\|$ is finite, there is a chain $w \in C_{n+1}$ such that

$$\|\bar{w} - i(w)\| \leq \frac{\varepsilon}{\|\partial_{n+1}\|}.$$

Then $z' := z + \partial(w) \in C_n$ is a cycle with $[z'] = [z] \in H_n(C)$ and

$$\|\bar{z} - i(z')\| = \|\partial_{n+1}(\bar{w} - i(w))\| \leq \varepsilon.$$

In particular, $\|z'\| \leq \|\bar{z}\| + \varepsilon$. Hence, $H_n(i)$ must be an isometry. \square

2.3 Main examples – ℓ^1 -homology and bounded cohomology

We now introduce ℓ^1 -homology of topological spaces (with trivial coefficients). Other flavours of ℓ^1 -homology like ℓ^1 -homology of discrete groups or ℓ^1 -homology with twisted coefficients are presented in Subsection 5.1 and Subsection 6.1 respectively.

Definition (2.8). Let (X, A) be a pair of topological spaces. The ℓ^1 -norm on the singular chain complex $C_*(X)$ with real coefficients is defined as follows: If $c = \sum_{j=0}^k a_j \cdot \sigma_j \in C_*(X)$, then

$$\|c\|_1 := \sum_{j=0}^k |a_j|.$$

Since $C_*(A)$ is ℓ^1 -closed in $C_*(X)$, the semi-norm on the quotient $C_*(X, A) = C_*(X)/C_*(A)$ induced by $\|\cdot\|_1$ is a norm, which is also denoted by $\|\cdot\|_1$. \diamond

With respect to the ℓ^1 -norm, the boundary operator $\partial_n: C_n(X, A) \rightarrow C_{n-1}(X, A)$ is a bounded operator with operator norm $(n+1)$. Hence, $C_*(X, A)$ is a normed chain complex. Clearly, $C_*(X)$ and $C_*(X, A)$ are in general not complete and thus these complexes are no Banach chain complexes.

Definition (2.9). Let (X, A) be a pair of topological spaces. The ℓ^1 -**chain complex of (X, A)** is the ℓ^1 -completion

$$C_*^{\ell^1}(X, A) := \overline{C_*(X, A)}^{\ell^1}$$

of the normed chain complex $C_*(X, A)$. We write $C_*^{\ell^1}(X) := C_*^{\ell^1}(X, \emptyset)$. \diamond

By Remark (2.3), the completion $C_*^{\ell^1}(X, A)$ is a Banach chain complex. Furthermore, it can be shown that there is an isometric isomorphism

$$C_*^{\ell^1}(X, A) = C_*^{\ell^1}(X)/C_*^{\ell^1}(A)$$

of Banach chain complexes.

Definition (2.10). If (X, A) is a pair of topological spaces, then the Banach cochain complex

$$C_b^*(X, A) := (C_*^{\ell^1}(X, A))' = (C_*(X, A))'$$

is called the **bounded cochain complex of (X, A)** . Moreover, we use the abbreviation $C_b^*(X) := C_b^*(X, \emptyset)$. \diamond

Using the isomorphism $C_*^{\ell^1}(X, A) = C_*^{\ell^1}(X)/C_*^{\ell^1}(A)$, it is not difficult to see that there is for all $n \in \mathbf{N}$ an isometric isomorphism [15; Proposition 2.1.7]

$$C_b^n(X, A) \cong \{f \in C_b^n(X) \mid f|_{C_n^{\ell^1}(A)} = 0\}.$$

If $f: (X, A) \rightarrow (Y, B)$ is a continuous map of pairs of topological spaces, then the induced map $C_*(f): C_*(X, A) \rightarrow C_*(Y, B)$ is a chain map that is bounded in each degree (with operator norm equal to 1), i.e., it is a morphism of normed chain complexes. Its extension $C_*^{\ell^1}(f): C_*^{\ell^1}(X, A) \rightarrow C_*^{\ell^1}(Y, B)$ is a morphism of Banach chain complexes and its dual $C_b^*(f): C_b^*(Y, B) \rightarrow C_b^*(X, A)$ is a morphism of Banach cochain complexes.

Definition (2.11). Let (X, A) be a pair of topological spaces.

1. The ℓ^1 -**homology of (X, A)** is defined as

$$H_*^{\ell^1}(X, A) := H_*(C_*^{\ell^1}(X, A)).$$

Dually, the **bounded cohomology of (X, A)** is given by

$$H_b^*(X, A) := H^*(C_b^*(X, A)).$$

We write $H_*^{\ell^1}(X) := H_*^{\ell^1}(X, \emptyset)$ and $H_b^*(X) := H_b^*(X, \emptyset)$ for short.

2. The semi-norms on $H_*^{\ell^1}(X, A)$ and $H_b^*(X, A)$ are the ones induced by $\|\cdot\|_1$ and $\|\cdot\|_\infty$ respectively and are also denoted by $\|\cdot\|_1$ and $\|\cdot\|_\infty$ respectively.
3. If $f: (X, A) \rightarrow (Y, B)$ is a continuous map of pairs of topological spaces, then the maps on ℓ^1 -homology and bounded cohomology induced by $C_*^{\ell^1}(f)$ and $C_b^*(f)$ are denoted by $H_*^{\ell^1}(f)$ and $H_b^*(f)$ respectively. \diamond

An example of a topological invariant defined in terms of the ℓ^1 -semi-norm on singular homology is the simplicial volume:

Definition (2.12). Let M be an oriented, closed, connected n -dimensional manifold and let $[M] \in H_n(M)$ be the image of the integral fundamental class of M under the change of coefficients homomorphism. Then the **simplicial volume of M** is defined as

$$\|M\| := \|[M]\|_1. \quad \diamond$$

In other words, the simplicial volume measures the complexity of the (real) fundamental class with respect to the ℓ^1 -norm. An important aspect of this invariant is its relation to Riemannian geometry and rigidity phenomena [5].

However, singular homology itself does not seem to be very well suited to compute the simplicial volume. Both ℓ^1 -homology and bounded cohomology provide a systematic way of studying the simplicial volume (see Proposition (2.13) and Theorem (3.6)). For example, bounded cohomology and ℓ^1 -homology share the advantage to vanish in a large number of cases.

Proposition (2.13). *Let (X, A) be a pair of topological spaces. Then the homomorphism $H_*(X, A) \rightarrow H_*^{\ell^1}(X, A)$ induced by the inclusion $C_*(X, A) \subset C_*^{\ell^1}(X, A)$ is isometric with respect to the semi-norms on $H_*(X, A)$ and $H_*^{\ell^1}(X, A)$ induced by the ℓ^1 -norm.*

In particular, if $H_n^{\ell^1}(X, A) = 0$, then $\|\alpha\|_1 = 0$ for all $\alpha \in H_n(X, A)$.

Proof. This follows from Proposition (2.7), because $C_*(X, A) \subset C_*^{\ell^1}(X, A)$ is a dense subcomplex (with respect to $\|\cdot\|_1$). \square

Remark (2.14). From Proposition (2.13) we can deduce that ℓ^1 -homology of topological spaces is not always zero:

For example, it is well-known that the singular homology of oriented connected closed hyperbolic (or, more general, negatively curved) manifolds contains classes whose ℓ^1 -semi-norm does not vanish [5, 8]. Hence, by Proposition (2.13), the image in ℓ^1 -homology of these classes cannot be zero. \diamond

3 Duality

In this section, we investigate the relation induced by the evaluation map between the dual of homology of a Banach chain complex and cohomology of the dual Banach cochain complex. In particular, we discuss Matsumoto and Morita's duality

principle (Theorem (3.4)) stating that the vanishing of $(H_*(C))'$ is equivalent to the vanishing of $H^*(C')$ and the relation of the corresponding semi-norms (Theorem (3.6)).

The key to lifting this duality principle to morphisms, e.g., to prove that $H_*(f)$ is an isomorphism if and only if $H^*(f')$ is an isomorphism, is applying the duality principle to mapping cones of morphisms of Banach chain complexes. These mapping cones are introduced in Subsection 3.2 and the proof of Theorem (1.1) is presented in Subsections 3.3 and 3.4.

3.1 Linking homology and cohomology

Obviously, evaluation links homology of a normed chain complex to cohomology of its dual cochain complex:

Definition (3.1). Let C be a normed chain complex. Evaluation $C^m \otimes C_n \rightarrow \mathbf{R}$ induces linear maps, the so-called **Kronecker products**,

$$\begin{aligned} \langle \cdot, \cdot \rangle: H^*(C') \otimes H_*(C) &\rightarrow \mathbf{R}, \\ \langle \cdot, \cdot \rangle: \bar{H}^*(C') \otimes \bar{H}_*(C) &\rightarrow \mathbf{R}. \end{aligned} \quad \diamond$$

Remark (3.2). Let $f: C \rightarrow D$ be a morphism of normed chain complexes and let $n \in \mathbf{N}$. Then the induced homomorphisms $H_n(f)$ and $H^n(f')$ are adjoint in the sense that

$$\langle \varphi, H_n(f)(\alpha) \rangle = \langle H^n(f')(\varphi), \alpha \rangle$$

for all $\alpha \in H_n(C)$ and all $\varphi \in H^n(D')$. Analogously, $\bar{H}_n(f)$ and $\bar{H}^n(f')$ are adjoint with respect to $\langle \cdot, \cdot \rangle$. \diamond

The algebraic dual of homology of a chain complex of vector spaces coincides with the cohomology of the algebraic dual complex. However, the topological dual fails to satisfy such a property:

Remark (3.3). *There is no obvious duality isomorphism between homology and cohomology of Banach chain complexes:*

Let C be a Banach chain complex. Then we have the following commutative diagram

$$\begin{array}{ccc} H^*(C') & \longrightarrow & \text{hom}_{\mathbf{R}}(H_*(C), \mathbf{R}) \\ \downarrow & \searrow & \uparrow \\ \bar{H}^*(C') & \longrightarrow & (\bar{H}_*(C))' \end{array}$$

where the horizontal arrows are the homomorphisms induced by the Kronecker products (i.e., they are induced by evaluation of elements in C' on elements in C), the left vertical arrow is the canonical projection and the right vertical arrow is the composition $(\bar{H}_*(C))' \hookrightarrow \text{hom}_{\mathbf{R}}(\bar{H}_*(C), \mathbf{R}) \hookrightarrow \text{hom}_{\mathbf{R}}(H_*(C), \mathbf{R})$ of inclusions.

The lower horizontal morphism, and hence also the diagonal morphism, is surjective by the Hahn-Banach theorem. Moreover, Matsumoto and Morita showed

that the diagonal morphism is injective if and only if $H^*(C') = \overline{H}^*(C')$ holds [10; Theorem 2.3].

But this is obviously in general not the case. It is even wrong in the special case $C = C_*^{\ell^1}(X)$ for certain topological spaces X [18, 19]. Hence, there is no obvious duality between ℓ^1 -homology and bounded cohomology.

Even the lower horizontal arrow is in general not injective: The kernel of the evaluation map

$$\ker \partial^{m+1} \longrightarrow (\ker \partial_n / \overline{\text{im } \partial_{n+1}})' = (\overline{H}_n(C))'$$

coincides with $({}^\perp \text{im}(\partial^n))^\perp$, which is the weak*-closure of $\text{im } \partial^n$ [16; Theorem 4.7]. Furthermore, the norm-closure $\overline{\text{im } \partial^n}$ and the weak*-closure $({}^\perp \text{im}(\partial^n))^\perp$ coincide if and only if $\text{im } \partial_{n+1}$ is closed [16; Theorem 4.14]. Thus there is also no obvious duality isomorphism between reduced ℓ^1 -homology and reduced bounded cohomology. \diamond

Surprisingly, there is still the following relation between homology and cohomology of Banach chain complexes, which has been discovered by Matsumoto and Morita [10; Corollary 2.4] (and which was also studied by Grigorchuk [4]):

Theorem (3.4) (Duality principle). *Let C be a Banach chain complex. Then $H_*(C)$ vanishes if and only if $H^*(C')$ vanishes.*

Here, the “*” carries the meaning “All of the $H_n(C)$ are zero iff all the $H^n(C')$ are zero.”

For the sake of completeness we provide a proof of this theorem. The proof is based on the following fact, stating that taking dual Banach spaces is something like an exact functor. (It is not really exact, because the categories involved are not Abelian).

Lemma (3.5). *Let $f: U \longrightarrow V$ and $g: V \longrightarrow W$ be two bounded operators of Banach spaces satisfying $g \circ f = 0$. Then the following two statements are equivalent:*

1. *The image of g is closed and $\text{im } f = \ker g$.*
2. *The image of f' is closed and $\text{im}(g') = \ker(f')$.*

Proof. The various kernels and images are related as follows [16; Theorem 4.7 and Theorem 4.12], where $\overline{\text{im}(g')}^*$ denotes the weak*-closure of $\text{im}(g')$:

$$\begin{aligned} (\text{im } f)^\perp &= \ker(f'), \\ (\ker g)^\perp &= ({}^\perp \text{im}(g'))^\perp = \overline{\text{im}(g')}^*. \end{aligned}$$

Suppose the image of g is closed and $\text{im } f = \ker g$. Then also $\text{im } f$ is closed. Hence, $\text{im}(f')$ and $\text{im}(g')$ are (weak*-)closed by the closed range theorem [16; Theorem 4.14]. Therefore, we obtain $\ker(f') = \overline{\text{im}(g')}^* = \text{im}(g')$.

Conversely, suppose the image of f' is closed and $\text{im}(g') = \ker(f')$. Thus, also $\text{im}(g')$ is closed. By the closed range theorem, $\text{im}(g')$ is even weak*-closed and both $\text{im } f$ and $\text{im } g$ are closed. In particular,

$$(\text{im } f)^\perp = (\ker g)^\perp.$$

Since $\text{im } f$ is closed and $\text{im } f \subset \ker g$, the Hahn-Banach theorem shows that $\text{im } f = \ker g$. \square

Proof (of Theorem (3.4)). If the (co)homology of a Banach (co)chain complex vanishes, then the images of all the (co)boundary operators are closed, because they are kernels of bounded operators. Therefore, the theorem follows from Lemma (3.5). \square

Moreover, the semi-norms on homology and cohomology are intertwined in the following way [5, 1; p. 17, Proposition F.2.2]:

Theorem (3.6) (Duality principle for semi-norms). *Let C be a Banach chain complex and let $n \in \mathbf{N}$. Then*

$$\|\alpha\| = \sup \left\{ \frac{1}{\|\varphi\|_\infty} \mid \varphi \in H^n(C') \text{ and } \langle \varphi, \alpha \rangle = 1 \right\}$$

holds for each $\alpha \in H_n(C)$. Here, $\sup \emptyset := 0$.

Proof. If $\alpha \in H_n(C)$ and $\varphi \in H^n(C')$, then

$$|\langle \varphi, \alpha \rangle| \leq \|\alpha\| \cdot \|\varphi\|_\infty.$$

This shows that $\|\alpha\|$ is at least as large as the supremum. Now suppose $\|\alpha\| \neq 0$, i.e., if c is a cycle representing α , then $c \notin \overline{\text{im } \partial_{n+1}}$. Thus, by the Hahn-Banach theorem there exists a functional $f: C_n \rightarrow \mathbf{R}$ satisfying

$$f|_{\text{im } \partial_{n+1}} = 0, \quad f(c) = 1, \quad \|f\|_\infty \leq 1/\|\alpha\|.$$

In particular, $f \in C^n$ is a cocycle. Let $\varphi := [f] \in H^n(C')$ be the corresponding cohomology class. Then, by construction, $\langle \varphi, \alpha \rangle = 1$ and $\|\varphi\|_\infty \leq \|f\|_\infty \leq 1/\|\alpha\|$. Hence, $\|\alpha\|$ is at most the supremum. \square

The discussion in Remark (3.3) shows however that the semi-norm on $H^*(C')$ can in general not be computed by the semi-norm on $H_*(C)$. (Since it might happen that the reduced homology $\overline{H}_*(C)$ is zero, but $\overline{H}^*(C')$ is non-zero).

3.2 Mapping cones

Mapping cones of chain maps are a device translating questions about isomorphisms on homology into questions about the vanishing of homology groups (Lemma (3.9)).

Definition (3.7). 1. Let $f: (C, \partial^C) \rightarrow (D, \partial^D)$ be a morphism of normed chain complexes. Then the **mapping cone** of f , denoted by $\text{Cone}(f)$, is the normed chain complex defined by

$$\text{Cone}(f)_n := C_{n-1} \oplus D_n$$

linked by the boundary operator that is given by the matrix

$$\begin{array}{ccc} \text{Cone}(f)_n = C_{n-1} \oplus D_n & & \\ \begin{pmatrix} -\partial^C & 0 \\ f & \partial^D \end{pmatrix} \downarrow & \begin{matrix} -\partial^C \downarrow & \searrow f \\ & \downarrow \partial^D \end{matrix} & \\ \text{Cone}(f)_{n-1} = C_{n-2} \oplus D_{n-1} & & \end{array}$$

2. Dually, if $f: (D, \delta_D) \rightarrow (C, \delta_C)$ is a morphism of normed cochain complexes, then the **mapping cone** of f , also denoted by $\text{Cone}(f)$, is the normed cochain complex defined by

$$\text{Cone}(f)^n := D^{n+1} \oplus C^n$$

and the coboundary operator determined by the matrix

$$\begin{array}{ccc} \text{Cone}(f)^n = D^{n+1} \oplus C^n & & \\ \begin{pmatrix} -\delta_D & 0 \\ f & \delta_C \end{pmatrix} \downarrow & \begin{matrix} -\delta_D \downarrow & \searrow f \\ & \downarrow \delta_C \end{matrix} & \\ \text{Cone}(f)^{n+1} = D^{n+2} \oplus C^{n+1} & & \end{array}$$

In both cases, we equip the mapping cone with the direct sum of the norms, i.e., the norm given by $\|(x, y)\| := \|x\| + \|y\|$. \diamond

Clearly, if f is a morphism of Banach (co)chain complexes, then the mapping cone $\text{Cone}(f)$ is also a Banach (co)chain complex.

Definition (3.8). If C is a normed chain complex, the normed chain complex ΣC that is derived from C via $(\Sigma C)_n := C_{n-1}$ is called the **suspension** of C . For a normed cochain complex C , the suspension ΣC is defined by $(\Sigma C)^n := C^{n-1}$. \diamond

The main feature of mapping cones is being able to detect isomorphisms on homology in the following sense:

- Lemma (3.9).** 1. Let $f: C \rightarrow D$ be a morphism of normed chain complexes. Then the induced map $H_*(f): H_*(C) \rightarrow H_*(D)$ is an isomorphism (of vector spaces) if and only if all homology groups $H_*(\text{Cone}(f))$ vanish.
2. Dually, let $f: D \rightarrow C$ be a morphism of normed cochain complexes. Then the induced map $H^*(f): H^*(D) \rightarrow H^*(C)$ is an isomorphism if and only if all cohomology groups $H^*(\text{Cone}(f))$ vanish.

Proof. The sequence (where the morphisms are given by the obvious inclusion and projection) $0 \rightarrow D \rightarrow \text{Cone}(f) \rightarrow \Sigma C \rightarrow 0$ of normed chain complexes is exact and hence gives rise to a long exact sequence in homology, whose connecting morphism is easily seen to coincide with $H_*(f)$ [3; Proposition 0.6]:

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_n(\text{Cone}(f)) & \longrightarrow & H_n(\Sigma C) & \longrightarrow & H_{n-1}(D) \longrightarrow H_{n-1}(\text{Cone}(f)) \longrightarrow \dots \\ & & & & \parallel & \nearrow & \\ & & & & H_{n-1}(C) & \xrightarrow{H_{n-1}(f)} & \end{array}$$

This proves the first part. The second part can be shown in the same way, making use of the long exact cohomology sequence corresponding to the short exact sequence $0 \rightarrow C \rightarrow \text{Cone}(f) \rightarrow \Sigma^{-1}D \rightarrow 0$ of normed cochain complexes. \square

In order to understand the relation between the induced maps $H_*(f)$ and $H^*(f')$ it is therefore necessary to relate the mapping cone of f to the one of f' .

Lemma (3.10). *Let $f: C \rightarrow D$ be a morphism of normed chain complexes and $f': D' \rightarrow C'$ the induced morphism between the dual complexes. Then there is a natural isomorphism*

$$\text{Cone}(f)' \cong \Sigma \text{Cone}(f')$$

of normed cochain complexes, relating the mapping cones of f and f' . In particular,

$$H^*(\text{Cone}(f)') \cong H^*(\Sigma \text{Cone}(f')).$$

Proof. For each $n \in \mathbf{N}$, there is an isomorphism of normed vector spaces

$$\begin{aligned} (\text{Cone}(f)')^n &= (C_{n-1} \oplus D_n)' \rightarrow (D_n)' \oplus (C_{n-1})' = \text{Cone}(f')^{n-1} \\ \varphi &\mapsto (d \mapsto \varphi(0, d), c \mapsto \varphi(c, 0)) \\ ((c, d) \mapsto \varphi(c) + \psi(d)) &\longleftarrow (\psi, \varphi). \end{aligned}$$

By definition, the coboundary operator of $\text{Cone}(f)'$ is given by

$$\begin{aligned} (C_{n-1} \oplus D_n)' &\rightarrow (C_n \oplus D_{n+1})' \\ \varphi &\mapsto \left((c, d) \mapsto \varphi(-\partial_C(c), f(c) + \partial_D(d)) \right) \\ &= \left((c, d) \mapsto (\varphi(-\partial_C(c), 0) + \varphi(0, f(c))) + \varphi(0, \partial_D(d)) \right) \end{aligned}$$

which corresponds under the isomorphisms given above exactly to the coboundary operator on $\Sigma \text{Cone}(f')$. Therefore, we obtain an isomorphism $\text{Cone}(f)' \cong \Sigma \text{Cone}(f')$ of normed cochain complexes. \square

3.3 Transferring algebraic isomorphisms

Fusing the properties of mapping cones with the duality principle (Theorem (3.4)) yields a proof of the first part of Theorem (1.1):

Theorem (3.11). *Let $f: C \rightarrow D$ be a morphism of Banach chain complexes. Then the induced homomorphism $H_*(f): H_*(C) \rightarrow H_*(D)$ is an isomorphism of vector spaces if and only if the induced homomorphism $H^*(f'): H^*(D') \rightarrow H^*(C')$ is an isomorphism of vector spaces.*

Proof. By Lemma (3.9), the map $H_*(f)$ is an isomorphism iff $H_*(\text{Cone}(f)) = 0$. In view of the duality principle (Theorem (3.4)) and Lemma (3.10), this is equivalent to

$$0 = H^*(\text{Cone}(f)') = H^*(\Sigma \text{Cone}(f')) = H^{*-1}(\text{Cone}(f')).$$

(The duality principle is applicable, because the cone of a morphism of Banach (co)chain complexes is a Banach (co)chain complex.) On the other hand, the cohomology groups $H^{*-1}(\text{Cone}(f'))$ are all zero if and only if $f': D' \rightarrow C'$ is an isomorphism (Lemma (3.9)). \square

3.4 Transferring isometric isomorphisms

Similarly, combining the properties of mapping cones with the duality principle for semi-norms (Theorem (3.6)) proves the second part of Theorem (1.1):

Theorem (3.12). *Let $f: C \rightarrow D$ be a morphism of Banach chain complexes. If the induced homomorphism $H^*(f'): H^*(D') \rightarrow H^*(C')$ is an isometric isomorphism, then also $H_*(f): H_*(C) \rightarrow H_*(D)$ is an isometric isomorphism.*

Proof. By Theorem (3.11), the map $H_*(f)$ is an isomorphism. That this isomorphism is isometric is a consequence of the duality principle for semi-norms (Theorem (3.6)), namely:

Let $n \in \mathbf{N}$ and let $\alpha \in H_n(C)$. Using the duality principle for semi-norms twice and the fact that $H^*(f')$ is an isometric isomorphism, we obtain

$$\begin{aligned} \|H_n(f)(\alpha)\| &= \sup \left\{ \frac{1}{\|\psi\|_\infty} \mid \psi \in H^n(D') \text{ and } \langle \psi, H_n(f)(\alpha) \rangle = 1 \right\} \\ &= \sup \left\{ \frac{1}{\|\psi\|_\infty} \mid \psi \in H^n(D') \text{ and } \langle H^n(f')(\psi), \alpha \rangle = 1 \right\} \\ &= \sup \left\{ \frac{1}{\|H^n(f')(\psi)\|_\infty} \mid \psi \in H^n(D') \text{ and } \langle H^n(f')(\psi), \alpha \rangle = 1 \right\} \\ &= \sup \left\{ \frac{1}{\|\varphi\|_\infty} \mid \varphi \in H^n(C') \text{ and } \langle \varphi, \alpha \rangle = 1 \right\} \\ &= \|\alpha\|, \end{aligned}$$

as desired. □

As already explained at the end of Subsection 3.1, one cannot expect that a statement of the form “If $H_*(f)$ is an isometric isomorphism, then also $H^*(f')$ is an isometric isomorphism” holds.

4 Applications to ℓ^1 -homology of spaces

Using the translation mechanisms from Section 3, we now derive statements concerning isomorphisms in ℓ^1 -homology of spaces:

Corollary (4.1). *Let $f: (X, A) \rightarrow (Y, B)$ be a continuous map of pairs of topological spaces.*

1. *The induced homomorphism $H_*^{\ell^1}(f): H_*^{\ell^1}(X, A) \rightarrow H_*^{\ell^1}(Y, B)$ is an isomorphism if and only if $H_b^*(f): H_b^*(Y, B) \rightarrow H_b^*(X, A)$ is an isomorphism.*
2. *If $H_b^*(f): H_b^*(Y, B) \rightarrow H_b^*(X, A)$ is an isometric isomorphism, then $H_*^{\ell^1}(f)$ is also an isometric isomorphism.*
3. *In particular, $H_*^{\ell^1}(X, A)$ vanishes if and only if $H_b^*(X, A)$ vanishes.*

Proof. By definition, $C_b^*(X, A) = (C_*^{\ell^1}(X, A))'$ as well as $C_b^*(Y, B) = (C_*^{\ell^1}(Y, B))'$ and the cochain map $C_b^*(f): C_b^*(Y, B) \rightarrow C_b^*(X, A)$ coincides with $(C_*^{\ell^1}(f))'$. Applying Theorem (3.11) and Theorem (3.12) to $C_*^{\ell^1}(f)$ proves the Corollary. \square

Corollary (4.1) allows to transfer certain results from bounded cohomology to ℓ^1 -homology. For example, we obtain a new proof of the fact that ℓ^1 -homology depends only on the fundamental group and that amenable groups are a blind spot of ℓ^1 -homology (Corollary (4.3)):

Definition (4.2). A discrete group A is called **amenable**, if there is a left-invariant mean on the set $B(A, \mathbf{R})$ of bounded functions from A to \mathbf{R} , i.e., if there is a linear map $m: B(A, \mathbf{R}) \rightarrow \mathbf{R}$ satisfying

$$\forall f \in B(A, \mathbf{R}) \quad \forall a \in A \quad m(f) = m(b \mapsto f(a^{-1} \cdot b))$$

and

$$\forall f \in B(A, \mathbf{R}) \quad \inf\{f(a) \mid a \in A\} \leq m(f) \leq \sup\{f(a) \mid a \in A\}. \quad \diamond$$

All finite and all Abelian groups are amenable. Moreover, the class of amenable groups is closed under taking subgroups and quotients. An example of a non-amenable group is the free group $\mathbf{Z} * \mathbf{Z}$. A detailed discussion of amenable groups can be found in Paterson's book [14].

Corollary (4.3) (Mapping theorem for ℓ^1 -homology). *The ℓ^1 -homology of connected countable CW-complexes depends only on the fundamental group. More generally: Let $f: X \rightarrow Y$ be a continuous map of connected countable CW-complexes such that the induced map $\pi_1(f): \pi_1(X) \rightarrow \pi_1(Y)$ is surjective and has amenable kernel. Then the induced homomorphism*

$$H_*^{\ell^1}(f): H_*^{\ell^1}(X) \rightarrow H_*^{\ell^1}(Y)$$

is an isometric isomorphism.

Proof. It is a classical result in the theory of bounded cohomology that in this situation $H_b^*(f): H_b^*(Y) \rightarrow H_b^*(X)$ is an isometric isomorphism [5, 9; p. 40, Theorem 4.3]. Applying Corollary (4.1) completes the proof. \square

Bouarich gave the first proof that ℓ^1 -homology only depends on the fundamental group [2; Corollaire 6] based on Theorem (3.4), the fact that bounded cohomology of simply connected spaces vanishes, and an ℓ^1 -version of Brown's theorem. Moreover, Park [13; Corollary 4.2] already claimed that Corollary (4.3) holds. However, due to a gap in her argument, her proof is not complete. This issue is addressed in Caveat (5.9) and Caveat (6.4).

Corollary (4.3) also gives a new proof of the following result of Bouarich [2; Corollaire 5].

Corollary (4.4). *Let $p: E \rightarrow B$ be a fibration of connected countable CW-complexes with path-connected fibre F . If $\pi_1(F)$ is amenable, then the induced map $H_*^{\ell^1}(p): H_*^{\ell^1}(E) \rightarrow H_*^{\ell^1}(B)$ is an isometric isomorphism.*

Proof. From the portion

$$\cdots \longrightarrow \pi_1(F) \longrightarrow \pi_1(E) \xrightarrow{\pi_1(p)} \pi_1(B) \longrightarrow \pi_0(F) = 0$$

of the long exact sequence associated to the fibration p , we obtain that $\pi_1(p)$ is surjective and that $\ker \pi_1(p)$, as homomorphic image of the amenable group $\pi_1(F)$, must be amenable [14; Proposition 1.12 and 1.13]. Now the result follows from Corollary (4.3). \square

Furthermore, the translation mechanisms of Section 3 enable us to show that ℓ^1 -homology of spaces indeed can be computed by certain projective resolutions (Subsection 6.2), as already claimed by Park [13; Theorem 4.1].

5 Applications to ℓ^1 -homology of groups

In this section, we introduce ℓ^1 -homology and bounded cohomology of discrete groups by means of (relative) homological algebra (Subsection 5.1) as considered by Ivanov, Monod, and Park [9, 11, 13]. Concrete instances of this abstract framework are provided by the standard resolutions (Subsection 5.2), which are the obvious counterparts of their algebraic ancestors.

As in the case of ℓ^1 -homology of topological spaces, we can apply Theorem (1.1) in this setting to transfer results from bounded cohomology to ℓ^1 -homology of discrete groups (Subsection 5.3).

The background on (relative) homological algebra in the category of Banach G -modules is collected in Appendix A.

5.1 ℓ^1 -homology of discrete groups

Analogously to (co)homology of groups, ℓ^1 -homology and bounded cohomology of discrete groups can be defined by choosing the right class of projective and injective resolutions, namely the ones provided by the framework of relative homological algebra.

Definition (5.1). Let G be a discrete group and let V be a Banach G -module.

1. Then ℓ^1 -homology of G with coefficients in V is defined as

$$H_*^{\ell^1}(G; V) := H_*(C_G),$$

where $(C, \varepsilon: C_0 \rightarrow V)$ is any strong relatively projective G -resolution of V . We abbreviate $H_*^{\ell^1}(G) := H_*^{\ell^1}(G, \mathbf{R})$, where \mathbf{R} is the trivial Banach G -module.

2. Dually, bounded cohomology of G with coefficients in V is defined as

$$H_b^*(G; V) := H^*(C^G),$$

where $(C, \varepsilon: V \rightarrow C^0)$ is any strong relatively injective G -resolution of V and we write $H_b^*(G) := H_b^*(G, \mathbf{R})$, where \mathbf{R} is the trivial Banach G -module. \diamond

As in classical homological algebra, these definitions do not depend on the choice of the resolutions (Proposition (A.15)) and appropriate resolutions do always exist (Proposition (5.6)).

Moreover, $H_*^{\ell^1}(\cdot; \cdot)$ and $H_b^*(\cdot; \cdot)$ are functorial in both variables: Let $\varphi: H \rightarrow G$ be a homomorphism of discrete groups, let V be a Banach H -module, and let W be a Banach G -module.

1. If $f: V \rightarrow \varphi^*W$ is a morphism of Banach H -modules, then there is a homomorphism

$$H_*^{\ell^1}(\varphi; f): H_*^{\ell^1}(H; V) \rightarrow H_*^{\ell^1}(G; W).$$

2. Similarly, if $f: \varphi^*W \rightarrow V$ is a morphism of Banach H -modules, then there is a homomorphism

$$H_b^*(\varphi; f): H_b^*(G; W) \rightarrow H_b^*(H; V).$$

Here φ^*W denotes the Banach H -module with underlying Banach space W and the H -action given by $h \cdot w := \varphi(h) \cdot w$ for all $h \in H$ and all $w \in W$.

Namely, let (C, ε) be a strong relatively projective H -resolution of V and let (D, η) be a strong relatively projective G -resolution of W . Then (φ^*D, η) is a strong H -resolution of φ^*W and hence there is a morphism $F: C \rightarrow \varphi^*D$ of Banach H -chain complexes extending f , which is (up to H -homotopy) unique (Proposition (A.11)). This chain map induces a morphism $\bar{F}: C_H \rightarrow \varphi^*D_H$ of Banach chain complexes. Then

$$H_*(\varphi^*D_H \rightarrow D_G) \circ H_*(\bar{F}): H_*(C_H) \rightarrow H_*(D_G)$$

is the desired homomorphism $H_*^{\ell^1}(\varphi; f)$. Since F is unique up to H -homotopy, the composition $H_*(\varphi^*D_H \rightarrow D_G) \circ H_*(\bar{F})$ is independent of the choice of F . Furthermore, one can show that the homomorphism $H_*^{\ell^1}(\varphi; f)$ defined in this way indeed is independent of the chosen projective resolutions (using Proposition (A.15)).

Analogously, $H_b^*(\varphi; f)$ is defined by looking at strong relatively injective resolutions [9, 11; Section 3.7, Section 8].

5.2 Standard resolutions

The ℓ^1 -completion of the ordinary bar resolution of a discrete group gives rise to a strong relatively projective resolution of the trivial Banach module \mathbf{R} . By taking projective tensor products and mapping spaces of this resolution we obtain strong relatively projective and strong relative injective resolutions for arbitrary discrete groups and arbitrary coefficient modules (Proposition (5.6)).

Definition (5.2). Let G be a discrete group. The **Banach bar resolution** of G is the ℓ^1 -completion of the bar resolution of G , i.e., the Banach G -chain complex defined as follows:

1. For each $n \in \mathbf{N}$ let

$$C_n^{\ell^1}(G) := \left\{ \sum_{g \in G^{n+1}} a_g \cdot g_0 \cdot [g_1 | \dots | g_n] \mid \forall_{g \in G^{n+1}} a_g \in \mathbf{R} \text{ and } \sum_{g \in G^{n+1}} |a_g| < \infty \right\}$$

together with the norm $\|\sum_{g \in G^{n+1}} a_g \cdot g_0 \cdot [g_1 | \dots | g_n]\|_1 := \sum_{g \in G^{n+1}} |a_g|$ and the G -action characterised by

$$h \cdot (g_0 \cdot [g_1 | \dots | g_n]) := (h \cdot g_0) \cdot [g_1 | \dots | g_n]$$

for all $g \in G^{n+1}$ and all $h \in G$.

2. The boundary operator is the G -morphism uniquely determined by

$$\begin{aligned} C_n^{\ell^1}(G) &\longrightarrow C_{n-1}^{\ell^1}(G) \\ g_0 \cdot [g_1 | \dots | g_n] &\longmapsto g_0 \cdot g_1 \cdot [g_2 | \dots | g_n] \\ &\quad + \sum_{j=1}^{n-1} (-1)^j \cdot g_0 \cdot [g_1 | \dots | g_{j-1} | g_j \cdot g_{j+1} | g_{j+2} | \dots | g_n] \\ &\quad + (-1)^n \cdot g_0 \cdot [g_1 | \dots | g_{n-1}]. \end{aligned}$$

3. Moreover, we define the augmentation $\varepsilon: C_0^{\ell^1}(G) \longrightarrow \mathbf{R}$ by summation of the coefficients. \diamond

Definition (5.3). Let G be a discrete group and let V be a Banach G -module.

1. Let $C_*^{\ell^1}(G; V)$ be the Banach G -chain complex given by

$$C_*^{\ell^1}(G; V) := C_*^{\ell^1}(G) \bar{\otimes} V.$$

2. Dually, we define the Banach G -cochain complex $C_b^*(G; V)$ by

$$C_b^*(G; V) := B(C_*^{\ell^1}(G), V).$$

(Details on the corresponding norms, G -actions and (co)boundary operators can be found in Example (A.3) and Example (A.9). \diamond)

Remark (5.4). Let G be a discrete group. Then clearly, $C_*^{\ell^1}(G; \mathbf{R}) = C_*^{\ell^1}(G)$. If V is a Banach G -module, then the relation between $\bar{\otimes}$ and \cdot' (see Remark (A.4)) shows that

$$C_b^*(G; V') = (C_*^{\ell^1}(G; V))'. \quad \square$$

Remark (5.5). For any discrete group G , the bijections

$$\begin{aligned} G^{n+1} &\longrightarrow G^{n+1} \\ g_0 \cdot [g_1 | \dots | g_n] &\longmapsto (g_n^{-1}, \dots, g_0^{-1}) \end{aligned}$$

induce an isomorphism $(C_*^{\ell^1}(G))' \longrightarrow C_b^*(G)$ of Banach G -cochain complexes, where $C_b^*(G)$ is the strong relatively injective resolution of \mathbf{R} defined by Ivanov [9; Section 3.4]. \square

Proposition (5.6). *Let G be a discrete group and let V be a Banach G -module.*

1. *Then $C_*^{\ell^1}(G; V)$ together with the augmentation $\varepsilon \otimes \text{id}_V$ is a strong relatively projective resolution of V .*
2. *Dually, $C_b^*(G; V)$ together with the augmentation $B(\varepsilon, \text{id}_V)$ is a strong relatively injective resolution of V .*

Here, $\varepsilon: C_0^{\ell^1}(G) \rightarrow \mathbf{R}$ is the augmentation introduced in Definition (5.2).

Proof. 1. Park showed that $(C_*^{\ell^1}(G), \varepsilon)$ is a strong relatively projective resolution of the trivial Banach G -module \mathbf{R} [13; p. 596f]. In particular, there is a contracting chain homotopy s of norm at most 1 of the concatenated chain complex $C_*^{\ell^1}(G) \square \varepsilon$.

Therefore, $s \otimes \text{id}_V$ is a contracting chain homotopy of $C_*^{\ell^1}(G; V) \square (\varepsilon \otimes \text{id}_V)$, which also has norm at most 1. Hence, $(C_*^{\ell^1}(G; V), \varepsilon \otimes \text{id}_V)$ is a strong resolution of the Banach G -module $\mathbf{R} \otimes V = V$.

For each $n \in \mathbf{N}$, the Banach G -module $C_n^{\ell^1}(G; V) = C_n^{\ell^1}(G) \otimes V$ is relatively projective, because any mapping problem (in the sense of Definition (A.1)) of the form

$$\begin{array}{ccc} & C_n^{\ell^1}(G; V) & \\ & \swarrow \sigma & \downarrow \alpha \\ U & \xrightarrow{\pi} & W \longrightarrow 0 \end{array}$$

is solved by the G -morphism given by

$$\begin{aligned} C_n^{\ell^1}(G; V) &\longrightarrow U \\ g_0 \cdot [g_1 | \dots | g_n] \otimes v_g &\longmapsto g_0 \cdot \sigma \circ \alpha(1 \cdot [g_1 | \dots | g_n] \otimes (g_0^{-1} \cdot v_g)). \end{aligned}$$

2. Clearly, $B(s, \text{id}_V)$ is a contracting cochain homotopy of $B(\varepsilon, \text{id}_V) \square B(C_*^{\ell^1}(G), V)$. Furthermore, for each $n \in \mathbf{N}$ the Banach G -module $C_b^n(G; V) = B(C_n^{\ell^1}(G), V)$ is relatively injective, because any mapping problem (in the sense of Definition (A.1)) of the form

$$\begin{array}{ccc} & C_b^n(G; V) & \\ & \uparrow \alpha & \swarrow \kappa \\ 0 & \longrightarrow U & \xrightarrow{i} W \end{array}$$

σ

can be solved by the G -morphism

$$\begin{aligned} W &\longrightarrow C_b^n(G; V) = B(C_n^{\ell^1}(G), V) \\ w &\longmapsto \left(g_0 \cdot [g_1 | \dots | g_n] \mapsto (\alpha(g_0 \cdot \sigma(g_0^{-1} \cdot w)))(g_0 \cdot [g_1 | \dots | g_n]) \right) \end{aligned}$$

If we were only interested in the case of $V = W'$ for some Banach G -module W , then we could just apply Proposition (A.12) to the first part. \square

5.3 Isomorphisms in ℓ^1 -homology of groups

Similarly to the results in Section 4, we can now derive statements concerning isomorphisms in ℓ^1 -homology of discrete groups:

Corollary (5.7). *Let $\varphi: H \rightarrow G$ be a homomorphism of discrete groups, let V be a Banach H -module, let W be a Banach G -module and suppose there is a morphism $f: V \rightarrow \varphi^*W$ of Banach H -modules.*

1. *Then the homomorphism $H_*^{\ell^1}(\varphi; f): H_*^{\ell^1}(H; V) \rightarrow H_*^{\ell^1}(G; W)$ is an isomorphism if and only if $H_b^*(\varphi; f'): H_b^*(G; W') \rightarrow H_b^*(H; V')$ is an isomorphism.*
2. *If $H_b^*(\varphi; f')$ is an isometric isomorphism, then so is $H_*^{\ell^1}(\varphi; f)$.*

Here, the semi-norms on $H_*^{\ell^1}(\cdot; \cdot)$ and $H_b^*(\cdot; \cdot)$ are the ones induced by the standard resolutions defined in the previous subsection (Definition (5.3), Proposition (5.6)). These semi-norms coincide with the canonical semi-norms in the sense of Ivanov [9, 13, 11; Corollary 3.6.1, Corollary 2.3, Corollary 7.4.7].

Proof. Let $F: C_*^{\ell^1}(H; V) \rightarrow \varphi^*C_*^{\ell^1}(G; W)$ be an extension of $f: V \rightarrow \varphi^*W$. Then we obtain a commutative diagram of the form

$$\begin{array}{ccc} (\varphi^*C_*^{\ell^1}(G; W)_H)' & \xrightarrow{\bar{F}'} & (C_*^{\ell^1}(H; V)_H)' \\ \text{(A.16)} \parallel & & \parallel \text{(A.16)} \\ \varphi^*C_b^*(G; W')^H \xrightarrow{(5.4)} (\varphi^*C_*^{\ell^1}(G; W)')^H & \xrightarrow{F'|} & (C_*^{\ell^1}(H; V)')^H \xrightarrow{(5.4)} C_b^*(G; V')^H. \end{array}$$

The lower row is the restriction (to the subcomplex of invariants) of an extension of $f': \varphi^*W' \rightarrow V'$ to the H -resolution $\varphi^*C_b^*(G; W')$. Hence, the result follows by applying Theorem (3.11) and Theorem (3.12) to \bar{F} . \square

In particular: If G is a discrete group, then $H_*^{\ell^1}(G; V) \cong H_*^{\ell^1}(1; V)$ if and only if $H_b^*(G; V') \cong H_b^*(1; V')$.

Additionally, Corollary (5.7) enables us to carry over many results on bounded cohomology with coefficients to ℓ^1 -homology. A small example of this procedure is the following:

Corollary (5.8). *Let G be a discrete group, let $A \subset G$ be an amenable normal subgroup and let V be a Banach G -module. Then the projection $G \rightarrow G/A$ induces an (isometric) isomorphism*

$$H_*^{\ell^1}(G; V) \cong H_*^{\ell^1}(G/A; V_A).$$

Proof. The corresponding homomorphism

$$H_b^*(G \twoheadrightarrow G/A; V'^A \hookrightarrow V'): H_b^*(G/A; V'^A) \rightarrow H_b^*(G; V')$$

is an isometric isomorphism [12; Theorem 1] (the case with \mathbf{R} -coefficients was already treated by Ivanov [9; Section 3.8]). Since the inclusion $V'^A \hookrightarrow V'$ is the dual of the projection $V \twoheadrightarrow V_A$ (which follows from Proposition (A.16)), we can apply Corollary (5.7). \square

Caveat (5.9). Let G be a discrete group and let $A \subset G$ be an amenable normal subgroup.

Ivanov proved that the cochain complex $C_b^*(G/A)$ is a strong relatively injective G -resolution of the trivial G -module \mathbf{R} [9; Theorem 3.8.4] by showing that the G -morphisms $C_b^*(G/A) \rightarrow C_b^*(G)$ induced by the projection $G \rightarrow G/A$ are split injective [9; Lemma 3.8.1 and Corollary 3.8.2].

Analogously, Park claimed that the G -morphisms $C_n^{\ell^1}(G) \rightarrow C_n^{\ell^1}(G/A)$ are split surjective [13; Lemma 2.4 and Lemma 2.5] and concluded that the $C_n^{\ell^1}(G/A)$ are relatively projective G -modules. Unfortunately, Park's proof [13; proof of Lemma 2.4] contains an error: the A -invariant mean on $B(A, \mathbf{R})$ provided by amenability of A in general is *not* σ -additive.

In fact, $C_n^{\ell^1}(G/A)$ in general is *not* a relatively projective G -module as the following example shows: Let G be an infinite amenable group (e.g., $G = \mathbf{Z}$) and let $A := G$. Then the G -action on $G/A = 1$ is trivial. However, since G is infinite, the G -modules $C_n^{\ell^1}(G)$ do not contain any non-zero G -invariant elements. Therefore, any G -morphism of type $C_n^{\ell^1}(G/A) \rightarrow C_n^{\ell^1}(G)$ must be trivial. We now consider the mapping problem

$$\begin{array}{ccc} C_n^{\ell^1}(G/A) = \mathbf{R} & & \\ \swarrow \text{?} & \downarrow \text{id} & \\ C_n^{\ell^1}(G) & \xrightarrow{\pi} \mathbf{R} & \longrightarrow 0 \end{array}$$

with the G -morphism π given by $g_0 \cdot [g_1 | \dots | g_n] \mapsto 1$, which obviously admits a (non-equivariant) split of norm 1. The argument above shows that this mapping problem cannot have a solution, and hence that $C_n^{\ell^1}(G/A)$ cannot be a relatively projective G -module.

This problem also affects several other results of Park, e.g., her proof of the fact that ℓ^1 -homology only depends on the fundamental group [13; Theorem 4.1] and the equivalence theorem [13; Theorem 3.7 and 4.4]. \diamond

6 ℓ^1 -homology of spaces via projective resolutions

Similarly to singular homology and singular cohomology there are also versions of ℓ^1 -homology and bounded cohomology with twisted coefficients (Subsection 6.1). By applying the translation techniques of Section 3 to an appropriate chain map

$$C_*^{\ell^1}(X; V) \rightarrow C_*^{\ell^1}(\pi_1(X); V)_{\pi_1(X)},$$

we can deduce that ℓ^1 -homology of a space with twisted coefficients coincides with the ℓ^1 -homology of the fundamental group with the corresponding coefficients (Theorem (6.3)). Hence, ℓ^1 -homology of spaces admits also a description in terms of projective resolutions.

6.1 ℓ^1 -homology with twisted coefficients

The same definition as for singular (co)homology gives rise to ℓ^1 -homology and bounded cohomology with twisted coefficients:

Definition (6.1). Let X be a connected topological space with universal covering \tilde{X} and fundamental group G , and let V be a Banach G -module.

1. The ℓ^1 -chain complex of X with twisted coefficients in V is defined as the Banach chain complex of coinvariants

$$C_*^{\ell^1}(X; V) := (C_*^{\ell^1}(\tilde{X}) \otimes V)_G.$$

2. The ℓ^1 -homology of X with twisted coefficients in V , denoted by $H_*^{\ell^1}(X; V)$, is the homology of the Banach chain complex $C_*^{\ell^1}(X; V)$.
3. The bounded cochain complex of X with twisted coefficients in V is defined as the Banach cochain complex of invariants

$$C_b^*(X; V) := B(C_*^{\ell^1}(\tilde{X}), V)^G.$$

4. Bounded cohomology of X with twisted coefficients in V is the cohomology of the Banach cochain complex $C_b^*(X; V)$ and is denoted by $H_b^*(X; V)$.

(Details on the definition of the Banach G -(co)chain complexes $C_*^{\ell^1}(\tilde{X}) \otimes V$ and $B(C_*^{\ell^1}(\tilde{X}), V)$ can be found in Example (A.9)). \diamond

The ℓ^1 -chain complex and the bounded cochain complex of X (as defined in Subsection 2.3) can be recovered from this definition by taking trivial coefficients: Namely, as Park [13; proof of Theorem 4.1] stated, the ℓ^1 -chain complex of X can be viewed as the coinvariants of the ℓ^1 -chain complex of \tilde{X} :

Proposition (6.2). *If X is a connected topological space admitting a universal covering $\pi: \tilde{X} \rightarrow X$, then the morphism $C_*^{\ell^1}(\pi): C_*^{\ell^1}(\tilde{X}) \rightarrow C_*^{\ell^1}(X)$ induces an isometric isomorphism*

$$\varphi: C_*^{\ell^1}(\tilde{X})_{\pi_1(X)} \rightarrow C_*^{\ell^1}(X)$$

of Banach chain complexes.

Therefore, $C_*^{\ell^1}(X; \mathbf{R}) = C_*^{\ell^1}(X)$, and we obtain from Proposition (A.16) that

$$C_b^*(X; \mathbf{R}) = (C_*^{\ell^1}(\tilde{X})')^{\pi_1(X)} = (C_*^{\ell^1}(\tilde{X})_{\pi_1(X)})' = (C_*^{\ell^1}(X))' = C_b^*(X).$$

Proof (of Proposition (6.2)). For brevity, we write $G := \pi_1(X)$ and W for the subcomplex $\text{span}\{g \cdot c - c \mid c \in C_*^{\ell^1}(\tilde{X}), g \in G\}$.

Since $C_*^{\ell^1}(\pi)$ is continuous (with norm 1) and since $C_*^{\ell^1}(\pi)$ clearly vanishes on W , it also vanishes on the closure \bar{W} . In particular, it induces a morphism

$$\varphi: C_*^{\ell^1}(\tilde{X})_G \rightarrow C_*^{\ell^1}(X).$$

of Banach chain complexes with norm equal to 1 [15; Proposition 2.1.7].

We now construct an inverse to φ : To this end, for each $\tau \in \text{map}(\Delta^*, X)$ we choose a π -lift $\tilde{\tau} \in \text{map}(\Delta^*, \tilde{X})$. Then

$$\begin{aligned} \psi: C_*^{\ell^1}(X) &\longrightarrow C_*^{\ell^1}(\tilde{X}) \\ \sum_{j \in \mathbf{N}} a_j \cdot \tau_j &\longmapsto \sum_{j \in \mathbf{N}} a_j \cdot \tilde{\tau}_j + \overline{W} \end{aligned}$$

is a linear map, which satisfies $\|\psi\| \leq 1$. (As we will see in the following paragraph, ψ is the inverse of φ and thus is also compatible with the boundary operators).

Clearly, $\varphi \circ \psi = \text{id}$. Conversely, let $c = \sum_{j \in \mathbf{N}} a_j \cdot \sigma_j + \overline{W} \in C_*^{\ell^1}(\tilde{X})_G$. For any $j \in \mathbf{N}$ there exists a $g_j \in G$ such that $(\pi \circ \sigma_j)^\sim = g_j \cdot \sigma_j$. Therefore, we obtain

$$\begin{aligned} (\psi \circ \varphi)(c) - c &= \left(\sum_{j \in \mathbf{N}} a_j \cdot \widetilde{\pi \circ \sigma_j} - \sum_{j \in \mathbf{N}} a_j \cdot \sigma_j \right) + \overline{W} \\ &= \sum_{j \in \mathbf{N}} a_j \cdot (g_j \cdot \sigma_j - \sigma_j) + \overline{W}. \end{aligned}$$

Since the series $\sum_{j \in \mathbf{N}} |a_j|$ converges, the sum $\sum_{j \in \mathbf{N}} a_j \cdot (g_j \cdot \sigma_j - \sigma_j)$ lies in the ℓ^1 -closure of W , i.e., in \overline{W} . This implies $(\psi \circ \varphi)(c) - c = 0$ and hence $\psi \circ \varphi = \text{id}$. This proves the lemma. \square

6.2 Computing ℓ^1 -homology with twisted coefficients via projective resolutions

Finally, we are able to identify ℓ^1 -homology of topological spaces with ℓ^1 -homology of the associated fundamental groups:

Theorem (6.3). *Let X be a countable connected CW-complex with fundamental group G and let V be a Banach G -module.*

1. *There is a canonical isometric isomorphism*

$$H_*^{\ell^1}(X; V) \cong H_*(C_*^{\ell^1}(G; V)_G).$$

2. *In particular: If C is a strong relatively projective resolution of V , then there is a canonical isomorphism (degreewise isomorphism of semi-normed vector spaces)*

$$H_*^{\ell^1}(X; V) \cong H_*(C_G) \cong H_*^{\ell^1}(G; V).$$

3. *If C is a strong relatively projective resolution of the trivial Banach G -module \mathbf{R} , then there is a canonical isomorphism (degreewise isomorphism of semi-normed vector spaces)*

$$H_*^{\ell^1}(X; V) \cong H_*((C \otimes V)_G).$$

Therefore, the results of Subsection 5.3 are also valid for ℓ^1 -homology with twisted coefficients and hence provide generalisations of the results presented in Section 4.

Caveat (6.4). Ivanov proved the corresponding theorem for bounded cohomology with \mathbf{R} -coefficients [9; Theorem 4.1] by verifying that $C_b^*(\tilde{X})$ is a strong relatively injective resolution of the trivial Banach G -module \mathbf{R} [9; Theorem 2.4].

The proof that the resolution $C_b^*(\tilde{X})$ is strong relies heavily on the fact that certain chain maps are split injective (see Lemma (B.4)). However, for the same reasons as explained in Caveat (5.9), it is not possible to translate these arguments into the language of ℓ^1 -chain complexes. Hence, it seems impossible to prove that the chain complex $C_*^{\ell^1}(\tilde{X})$ is a *strong* resolution. In particular, Park's proof [13; proof of Theorem 4.1] of Theorem (6.3) (with \mathbf{R} -coefficients) is not complete. \diamond

Using the techniques developed in Section 3, we can derive Theorem (6.3) from the corresponding result in bounded cohomology (see Theorem (6.6), which is proved in Appendix B).

Proof (of Theorem (6.3)). 1. In order to prove the first part of Theorem (6.3), we proceed as follows:

1. We establish a connection between $C_*^{\ell^1}(\tilde{X}; V)$ and the strong relatively projective resolution $C_*^{\ell^1}(G; V)$.
2. We show that the dual of this morphism when restricted to the invariants induces an isometric isomorphism on the level of cohomology of the invariants (Theorem (6.6)).
3. Finally, we apply Theorem (3.12) to translate this isometric isomorphism back to ℓ^1 -homology.

First step. Park [13; proof of Theorem 4.1] constructed the following map ("dually" to Ivanov's construction [9; proof of Theorem 4.1]):

Let $F \subset \tilde{X}$ be a (set-theoretic) fundamental domain of the $\pi_1(X)$ -action on \tilde{X} . In the following, the vertices of the standard n -simplex Δ^n are denoted by v_0, \dots, v_n . For a singular simplex $\sigma \in \text{map}(\Delta^n, \tilde{X})$ let $g_0(\sigma), \dots, g_n(\sigma) \in G$ be the group elements uniquely characterised by

$$\begin{aligned} g_0(\sigma)^{-1} \cdot \sigma(v_0) &\in F \\ g_1(\sigma)^{-1} \cdot g_0(\sigma)^{-1} \cdot \sigma(v_1) &\in F \\ &\vdots \\ g_n(\sigma)^{-1} \cdot \dots \cdot g_1(\sigma)^{-1} \cdot g_0(\sigma)^{-1} \cdot \sigma(v_n) &\in F. \end{aligned}$$

Then the map $\eta: C_*^{\ell^1}(\tilde{X}) \longrightarrow C_*^{\ell^1}(G)$ given by

$$\begin{aligned} C_n^{\ell^1}(\tilde{X}) &\longrightarrow C_n^{\ell^1}(G) \\ \sigma &\longmapsto g_0(\sigma) \cdot [g_1(\sigma) \mid \dots \mid g_n(\sigma)] \end{aligned}$$

is a morphism of Banach G -chain complexes. Hence,

$$\eta_V := \eta \otimes \text{id}_V: C_*^{\ell^1}(\tilde{X}; V) \longrightarrow C_*^{\ell^1}(G; V)$$

is also a morphism of Banach G -chain complexes.

Let $\overline{\eta_V}: C_*^{\ell^1}(\tilde{X}; V)_G \rightarrow C_*^{\ell^1}(G; V)_G$ denote the morphism of Banach chain complexes induced by η_V . We now show that a different choice of fundamental domain $F^* \subset \tilde{X}$ leads to a map chain homotopic to $\overline{\eta_V}$:

Homological algebra shows that there is up to G -homotopy only one morphism $C_*^{\ell^1}(\tilde{X}) \rightarrow C_*^{\ell^1}(G)$ (Proposition (A.11)), because $C_*^{\ell^1}(\tilde{X})$ is a Banach G -complex consisting of relatively projective G -modules [13; p. 611] and $C_*^{\ell^1}(G)$ is a strong relatively projective resolution of \mathbf{R} (Proposition (5.6)). But η and η^* , the map obtained via F^* , are such morphisms and hence are G -homotopic. Therefore, also $\eta \otimes \text{id}_V$ and $\eta_V^* := \eta^* \otimes \text{id}_V$ must be G -homotopic, which implies that the induced maps $\overline{\eta_V}$ and $\overline{\eta_V^*}$ must be homotopic. In particular,

$$H_*(\overline{\eta_V}): H_*(C_*^{\ell^1}(\tilde{X}; V)_G) \rightarrow H_*(C_*^{\ell^1}(G; V)_G)$$

does not depend on the choice of fundamental domain.

Second step. The dual of η_V coincides under the natural isomorphisms

$$(C_*^{\ell^1}(\tilde{X}; V))' = C_b^*(\tilde{X}; V') \quad \text{and} \quad (C_*^{\ell^1}(G; V))' = C_b^*(G; V')$$

of Banach G -modules (see Remark (A.4)) with the morphism $\vartheta_{V'}: C_b^*(G; V') \rightarrow C_b^*(\tilde{X}; V')$ of Banach G -cochain complexes given by

$$\begin{aligned} C_b^n(G; V') &\rightarrow C_b^n(\tilde{X}; V') \\ f &\mapsto (\sigma \mapsto f(g_0(\sigma), \dots, g_n(\sigma))). \end{aligned} \tag{6.5}$$

In other words, the diagram

$$\begin{array}{ccc} (C_*^{\ell^1}(G; V))' & \xrightarrow{(\eta_V)'} & (C_*^{\ell^1}(\tilde{X}; V))' \\ \text{(A.4)} \parallel & & \parallel \text{(A.4)} \\ C_b^*(G; V') & \xrightarrow{\vartheta_{V'}} & C_b^*(\tilde{X}; V') \end{array}$$

is commutative. Taking G -invariants of this diagram yields the following commutative diagram of morphisms of Banach cochain complexes:

$$\begin{array}{ccc} (C_*^{\ell^1}(G; V)_G)' & \xrightarrow{\overline{\eta_V}'} & (C_*^{\ell^1}(\tilde{X}; V)_G)' \\ \text{(A.16)} \parallel & & \parallel \text{(A.16)} \\ (C_*^{\ell^1}(G; V)')^G & \xrightarrow{(\eta_V)'} & (C_*^{\ell^1}(\tilde{X}; V)')^G \\ \text{(A.4)} \parallel & & \parallel \text{(A.4)} \\ C_b^*(G; V')^G & \xrightarrow{\vartheta_{V'}|} & C_b^*(\tilde{X}; V')^G \end{array}$$

The restriction $\vartheta_{V'}|$ to the subcomplexes of G -invariants induces an isometric isomorphism on the level of cohomology (a proof of this theorem is given in Appendix B):

Theorem (6.6). *Let X be a countable connected CW-complex with fundamental group G and let V be a Banach G -module.*

1. *The morphism $\vartheta_{V'} : C_b^*(G; V') \longrightarrow C_b^*(\tilde{X}; V')$ of Banach G -cochain complexes (defined in (6.5)) induces an isometric isomorphism*

$$H_b^*(X; V') = H^*(C_b^*(\tilde{X}; V')^G) \cong H^*(C_b^*(G; V')^G).$$

Moreover, this isometric isomorphism does not depend on the choice of the fundamental domain F used in the definition of the $g_j(\sigma)$.

2. *In particular: If C is a strong relatively injective resolution of V' , then there is a canonical isomorphism (degree-wise isomorphism of semi-normed vector spaces)*

$$H_b^*(X; V) \cong H^*(C^G) \cong H_b^*(G; V').$$

Hence, also the top row of the diagram (i.e. $\overline{\eta}_{V'}$) must induce an isometric isomorphism on the level of cohomology.

Third step. Therefore, we can derive from Theorem (3.12) that

$$\overline{\eta}_V : C_*^{\ell^1}(X; V) = C_*^{\ell^1}(\tilde{X}; V)_G \longrightarrow C_*^{\ell^1}(G; V)_G$$

induces a (canonical) isometric isomorphism on the level of homology. This proves the first part.

2. Because $C_*^{\ell^1}(G; V)$ is a strong relatively projective resolution of V (Proposition (5.6)), standard methods from homological algebra (Proposition (A.15)) provide us with a canonical isomorphism

$$H_*(C_G) \cong H_*(C_*^{\ell^1}(G; V)_G).$$

Thus the first part yields $H_*^{\ell^1}(X; V) \cong H_*(C_G)$, as was to be shown.

3. If $(C, \eta : C_0 \rightarrow V)$ is a strong relatively projective resolution of V , there exist (mutually G -homotopy inverse) G -chain homotopy equivalences $\varphi : C \square \eta \simeq C_*^{\ell^1}(G) \square \varepsilon : \psi$ (Proposition (A.15)). But then $\varphi \overline{\otimes} \text{id}_V$ and $\psi \overline{\otimes} \text{id}_V$ clearly are (mutually G -homotopy inverse) G -chain homotopy equivalences

$$(C \overline{\otimes} V) \square (\eta \overline{\otimes} \text{id}_V) \simeq (C_*^{\ell^1}(G) \overline{\otimes} V) \square (\varepsilon \overline{\otimes} \text{id}_V) = C_*^{\ell^1}(G; V) \square \varepsilon_V.$$

In particular, we obtain an isomorphism

$$H_*((C \overline{\otimes} V)_G) \cong H_*(C_*^{\ell^1}(G; V)_G),$$

which is in each degree an isomorphism of semi-normed vector spaces. Therefore, we may deduce from the first part that

$$H_*^{\ell^1}(X; V) \cong H_*(C_*^{\ell^1}(G; V)_G) \cong H_*((C \overline{\otimes} V)_G). \quad \square$$

A Homological algebra for Banach G -modules

Ivanov [9] adapted (relative) homological algebra in the sense of Hochschild [7] to fit the needs of bounded cohomology of discrete groups. In this section, we introduce the basic objects of this theory and investigate their compatibility with taking (topological) duals. The key concept are strong relatively injective resolutions, which lead to the desirable fundamental lemma (Proposition (A.11)). Concrete examples of these concepts are studied in Subsection 5.2.

A more detailed account of the material collected in this section is, for example, presented in the work of Ivanov [9] and Monod [11], as well as (for the non-Banach case) in the book of Guichardet [6].

A.1 Banach G -modules

The atoms of the variant of (relative) homological algebra presented in this section are Banach G -modules with a suitable notion of projectivity and injectivity.

Definition (A.1). Let G be a discrete group.

1. A **Banach G -module** is a Banach space V with a G -action $G \times V \rightarrow V$ such that for each $g \in G$ the linear map $v \mapsto g \cdot v$ is an isometry.
2. A **G -morphism** is a bounded linear map between Banach G -modules that is G -equivariant.
3. A G -morphism $\pi: U \rightarrow W$ is called **relatively projective**, if there is a (not necessarily equivariant) linear map $\sigma: W \rightarrow U$ satisfying $\pi \circ \sigma = \text{id}_W$ and $\|\sigma\| \leq 1$.
4. A G -morphism $i: U \rightarrow W$ is called **relatively injective**, if there is a (not necessarily equivariant) linear map $\sigma: W \rightarrow U$ satisfying $\sigma \circ i = \text{id}_U$ and $\|\sigma\| \leq 1$.
5. A Banach G -module V is called **relatively projective**, if for each relatively projective G -morphism $\pi: W \rightarrow U$ and each G -morphism $\alpha: V \rightarrow W$ there is a G -morphism $\beta: V \rightarrow U$ such that

$$\pi \circ \beta = \alpha \quad \text{and} \quad \|\beta\| \leq \alpha.$$

6. A Banach G -module V is called **relatively injective**, if for each relatively injective G -morphism $i: U \rightarrow W$ and for each G -morphism $\alpha: U \rightarrow V$ there is a G -morphism $\beta: W \rightarrow V$ such that

$$\beta \circ i = \alpha \quad \text{and} \quad \|\beta\| \leq \alpha. \quad \diamond$$



Figure (A.2): Mapping problems for relatively projective and relatively injective Banach G -modules respectively

The mapping problems arising in the definition of relatively projective and relatively injective Banach G -modules are depicted in Figure (A.2).

Sometimes, “relatively injective” and “relatively projective” morphisms are also called “admissible monomorphisms” and “admissible epimorphisms” respectively.

The most basic example of a Banach G -module with non-trivial group action is $\ell^1(G)$, the set of all ℓ^1 -functions $G \rightarrow \mathbf{R}$ with the G -action given by shifting the argument. Obviously, any Banach G -module is a module over $\ell^1(G)$. However, the homological algebra we use does not coincide with the homological algebra in the category of $\ell^1(G)$ -modules. Even worse, the category of Banach G -modules is (like the category of Banach spaces) not Abelian.

Example (A.3). Let G be a discrete group and let U and V be two Banach G -modules.

1. The **projective tensor product** $U \bar{\otimes} V$ is the Banach G -module whose underlying Banach space is the projective tensor product $U \bar{\otimes} V$ of Banach spaces, i.e., the completion of the tensor product $U \otimes V$ of \mathbf{R} -vector spaces with respect to the norm

$$\forall c \in U \bar{\otimes} V \quad \|c\| := \inf \left\{ \sum_j \|u_j\| \cdot \|v_j\| \mid \sum_j u_j \otimes v_j \text{ represents } c \in U \otimes V \right\}.$$

The G -action on $U \bar{\otimes} V$ is the G -action uniquely determined by

$$\forall g \in G \quad \forall u \in U \quad \forall v \in V \quad g \cdot (u \otimes v) := (g \cdot u) \otimes (g \cdot v).$$

2. The Banach space $B(U, V)$ of all bounded linear functions from U to V (with the operator norm) is a Banach G -module with respect to the G -action

$$\begin{aligned} G \times B(U, V) &\longrightarrow B(U, V) \\ (g, f) &\longmapsto (u \mapsto g \cdot (f(g^{-1} \cdot u))). \end{aligned}$$

In particular, U' is a Banach G -module (where \mathbf{R} is regarded as the trivial Banach G -module). \diamond

The functors $\bar{\otimes}$ and B are adjoint in the following sense:

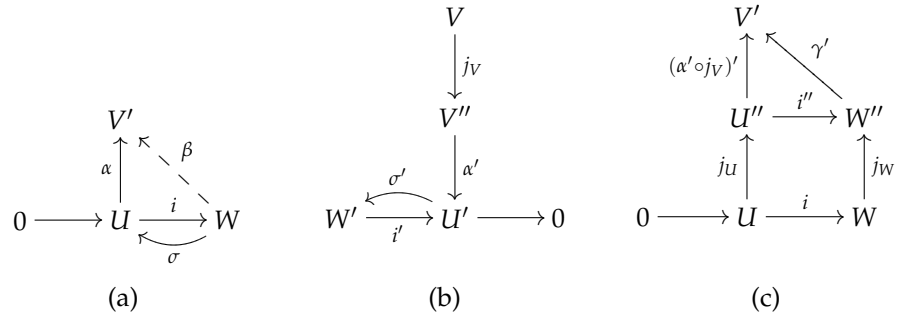


Figure (A.6): Diagrams occurring in the proof of Proposition (A.5)

Remark (A.4). Let G be a discrete group and let U, V , and W be Banach G -modules. Then

$$\begin{aligned} B(U \overline{\otimes} V, W) &\longrightarrow B(U, B(V, W)) \\ f &\longmapsto (u \mapsto (v \mapsto f(u \otimes v))) \\ (u \otimes v \mapsto f(u)(v)) &\longleftarrow f \end{aligned}$$

is an isometric isomorphism of Banach G -modules. \square

Taking duals transforms relatively projective modules into relatively injective modules:

Proposition (A.5). Let V be a relatively projective Banach G -module. Then its dual V' is a relatively injective Banach G -module.

Proof. In order to show that V' is a relatively injective Banach G -module we have to find a G -morphism $\beta: W \rightarrow V'$ fitting into the diagram Figure (A.6)(a) whenever $\alpha: U \rightarrow V'$ is a G -morphism and $i: U \rightarrow W$ is a G -morphism admitting a (not necessarily equivariant) split $\sigma: W \rightarrow U$ satisfying $\sigma \circ i = \text{id}_U$ and $\|\sigma\| \leq 1$.

There is an isometric embedding [15; 2.3.7]

$$\begin{aligned} j_V: V &\longrightarrow V'' \\ v &\longmapsto (f \mapsto f(v)), \end{aligned}$$

which is G -equivariant, of V into its double dual V'' . (However, this embedding is not surjective in general). Taking the dual of the solid part of diagram Figure (A.6)(a) thus gives rise to Figure (A.6)(b). Clearly, $i' \circ \sigma' = \text{id}_{W'}$ and $\|\sigma'\| \leq \|\sigma\| \leq 1$. Since V is relatively projective, we find a G -morphism $\gamma: V \rightarrow W'$ such that $i' \circ \gamma = \alpha' \circ j_V$ and $\|\gamma\| \leq \|\alpha' \circ j_V\| \leq \|\alpha\|$.

Dualising a second time thus yields the commutative diagram Figure (A.6)(c). Unfolding the various definitions shows that $(\alpha' \circ j_V)' \circ j_U = \alpha$. Hence, $\beta := \gamma' \circ j_W$ is a G -morphism with $\beta \circ i = \alpha$ and $\|\beta\| \leq \|\gamma'\| \cdot 1 \leq \|\alpha\|$. \square

Since not all Banach spaces are reflexive, it seems unlikely that the converse of this proposition holds.

A.2 Banach G -chain complexes

In order to get the machinery of homological algebra running, we of course need a suitable universe of chain complexes.

Definition (A.7). Let G be a discrete group.

1. A **Banach G -(co)chain complex** is a (co)chain complex of Banach G -modules, whose (co)boundary operators all are G -morphisms (in the sense of Definition (A.1)).
2. A **morphism of Banach G -(co)chain complexes** is a chain map of Banach G -(co)chain complexes consisting of G -morphisms.
3. Two morphisms of Banach G -(co)chain complexes are **G -homotopic**, if there exists a (co)chain homotopy between them that consists of G -morphisms. \diamond

Fundamental examples of Banach G -(co)chain complexes are the standard resolutions (see Subsection 5.2) and the ones stemming from geometry:

Example (A.8). If a discrete group G acts continuously on a topological space X , then the induced action of G on $C_*^{\ell^1}(X)$ and $C_b^*(X)$ turns these complexes into Banach G -(co)chain complexes. \diamond

Of course, the essential operations $\overline{\otimes}$ and B also have a pendant on the level of G -chain complexes:

Example (A.9). Let G be a discrete group. Let (C, ∂) be a Banach G -chain complex and let V be a Banach G -module.

1. The **projective tensor product** $C \overline{\otimes} V$ is the Banach G -chain complex given by

$$(C \overline{\otimes} V)_n := C_n \overline{\otimes} V$$

with the boundary operator $\partial \overline{\otimes} \text{id}_V$.

2. The Banach G -cochain complex $B(C, V)$ is defined by

$$B(C, V)^n := B(C_n, V),$$

equipped with the coboundary operator

$$\begin{aligned} B(C, V)^n &\longrightarrow B(C, V)^{n+1} \\ f &\longmapsto (c \mapsto f(\partial_{n+1}(c))). \end{aligned}$$

(The Banach G -module structure on $C_n \overline{\otimes} V$ and $B(C_n, V)$ is introduced in Example (A.3). \diamond)

Applying these constructions to the ℓ^1 -chain complexes of universal covering spaces gives rise to ℓ^1 -chain complexes with twisted coefficients and bounded cochain complexes with twisted coefficients (Definition (6.1)).

A.3 Relatively injective and relatively projective resolutions

The key concept of homological algebra is the adequate notion of projective and injective resolutions leading to the fundamental lemma of homological algebra (Proposition (A.11)). In our case, the special form of the mapping problems occurring in the definition of relatively projective G -modules forces us to consider so-called “strong” resolutions.

Definition (A.10). Let G be a discrete group and let V be a Banach G -module.

1. Let (C, ∂) be a normed chain complex of Banach G -modules. An **augmentation of C** with respect to V is a G -morphism $\varepsilon: C_0 \rightarrow V$ satisfying $\partial_1 \circ \varepsilon = 0$. If ε is an augmentation of C , then the concatenation of C and $\varepsilon: C_0 \rightarrow V$ is a Banach G -chain complex, which will be denoted by $C \square \varepsilon$.
2. Dually, an **augmentation of a Banach G -cochain complex (C, δ)** is a G -morphism $\varepsilon: C_0 \rightarrow V$ satisfying $\varepsilon \circ \delta_1 = 0$. The concatenation of $\varepsilon: V \rightarrow C_0$ and C is then a Banach G -cochain complex, which will be denoted by $\varepsilon \square C$.
3. A **(left) resolution of V** is a Banach G -chain complex C together with an augmentation $\varepsilon: C_0 \rightarrow V$ such that $H_*(C \square \varepsilon) = 0$.
4. A **(right) resolution of V** is Banach G -cochain complex C together with an augmentation $\varepsilon: V \rightarrow C_0$ such that $H^*(\varepsilon \square C) = 0$.
5. A resolution of V by Banach G -modules is called **strong**, if the concatenated Banach G -(co)chain complex admits a (not necessarily equivariant) chain contraction of norm at most 1.
6. A resolution of V is called **relatively projective** (or **relatively injective**) if it consists of relatively projective Banach G -modules (or relatively injective Banach G -modules respectively). \diamond

Now the fundamental lemma reads as follows:

Proposition (A.11). Let G be a discrete group, let $f: V \rightarrow W$ be a morphism of Banach G -modules.

1. If $(C, \varepsilon: C_0 \rightarrow V)$ is an augmented Banach G -chain complex consisting of relatively projective G -modules and $(D, \eta: D_0 \rightarrow W)$ is a strong resolution of W , then f can be extended to a morphism $C \square \varepsilon \rightarrow D \square \eta$ of Banach G -chain complexes. Moreover, this morphism is unique up to G -homotopy.
2. Dually, if $(D, \eta: W \rightarrow D^0)$ is an augmented Banach G -cochain complex consisting of relatively injective G -modules and if $(C, \varepsilon: V \rightarrow C^0)$ is a strong resolution of V , then f can be extended to a morphism $\varepsilon \square C \rightarrow \eta \square D$ of Banach G -cochain complexes and this morphism is unique up to G -homotopy.

Proof. This can be proved using standard techniques from homological algebra [6, 11; Proposition 2.2, Lemma 7.2.4]. For example, in order to find an extension of f in the first part, we inductively solve mapping problems of the form

$$\begin{array}{ccc} & & C_{n+1} \\ & \nearrow f_{n+1} & \downarrow f_n \circ \partial_{n+1}^C \\ D_{n+1} & \xrightarrow{\partial_{n+1}^D} & \text{im } \partial_{n+1}^D \end{array}$$

(where $f_{-1} := f$). This is a mapping problem in the sense of Definition (A.1), because $\text{im } \partial_{n+1}^D = \ker \partial_n^D$ is closed – and hence indeed a Banach G -module – and any contracting homotopy of D provides a (non-equivariant) split of the G -morphism $\partial_{n+1}^D: D_{n+1} \rightarrow \text{im } \partial_{n+1}^D$ of norm at most 1. Therefore, the relative projectivity of C_{n+1} ensures the existence of a solution f_{n+1} . \square

Proposition (A.5) extends to resolutions and thus dualising transforms (strong) relatively projective resolutions into (strong) relatively injective ones:

Proposition (A.12). *Let G be a discrete group and let $(C, \varepsilon: C_0 \rightarrow V)$ be a relatively projective resolution of the Banach G -module V . Then its dual $(C', \varepsilon': V' \rightarrow C'^0)$ is a relatively injective resolution of the Banach G -module V' .*

If the resolution (C, ε) is strong, then so is (C', ε') .

Proof. By Proposition (A.5) the Banach G -cochain complex C' consists of relatively injective Banach G -modules. Since (C, ε) is a resolution, $H_*(C \square \varepsilon) = 0$. Because the Banach G -cochain complexes $(C \square \varepsilon)'$ and $\varepsilon' \square C'$ are isomorphic, we obtain $H_*(\varepsilon' \square C') = 0$ from the duality principle (Theorem (3.4)). Hence, (C', ε') is a resolution of V' .

If (C, ε) is strong, then the dual of a chain contraction of $C \square \varepsilon$ with norm at most 1 is a cochain contraction of the dual $\varepsilon' \square C'$ with norm at most 1, i.e., (C', ε') is a strong resolution of V' . \square

A.4 Invariants and coinvariants

The last missing piece en route to (co)homology of groups in the setting of Banach G -modules is an appropriate definition of invariants and coinvariants.

Definition (A.13). Let G be a group and let V be a Banach G -module. The set of **invariants** of V is defined by

$$V^G := \{v \in V \mid \forall_{g \in G} \quad g \cdot v = v\}.$$

The set of **coinvariants** of V is the quotient

$$V_G := V/\overline{W},$$

where $W \subset V$ is the subspace generated by the set $\{g \cdot v - v \mid v \in V, g \in G\}$. \diamond

Clearly, if V is a Banach G -module, then V^G is a Banach space with respect to the restricted norm and V_G is a Banach space with respect to the quotient norm [15; Proposition 2.1.5] – because a *closed* subspace is quotiented out. However, notice that the space W itself used in the previous definition in general is not closed.

Any G -morphism $f: V \rightarrow W$ induces a bounded linear operator $\bar{f}: V_G \rightarrow W_G$ satisfying $\bar{f} \circ (V \rightarrow V_G) = (W \rightarrow W_G) \circ f$ [15; Proposition 2.1.7]. In particular, we can apply this to Banach G -(co)chain complexes:

Definition (A.14). Let G be a discrete group.

1. If (C, δ) is a Banach G -cochain complex, then C^G is the Banach cochain complex given by $(C^G)^n := (C^n)^G$ with the coboundary operator $\delta|_{C^G}$.
2. If (C, ∂) is Banach G -chain complex, then C_G is the Banach chain complex given by $(C_G)_n := (C_n)_G$ and the boundary operator induced by ∂ . \diamond

The following consequence of the fundamental lemma (Proposition (A.11)) lies at the heart of the definition of group (co)homology in this Banach-flavoured setting (see Definition (5.1)).

Proposition (A.15). Let G be a discrete group and let V be a Banach G -module.

1. If $(C, \varepsilon: C_0 \rightarrow V)$ and $(D, \eta: D_0 \rightarrow V)$ are two strong relatively projective (left) resolutions of V , then there is a canonical isomorphism (degreewise isomorphism of semi-normed vector spaces)

$$H_*(C_G) \cong H_*(D_G).$$

2. Dually, if $(C, \varepsilon: V \rightarrow C^0)$ and $(D, \eta: V \rightarrow D^0)$ are two strong relatively injective (right) resolutions of V , then there is a canonical isomorphism (degreewise isomorphism of semi-normed vector spaces)

$$H^*(C^G) \cong H^*(D^G).$$

However, the canonical isomorphisms mentioned in the proposition need not be isometric.

Proof. Clearly, any morphism $\varphi: C \square \varepsilon \rightarrow D \square \eta$ of Banach G -chain complexes induces a morphism $C_G \rightarrow D_G$ of Banach chain complexes. Similarly, G -homotopies descend to (bounded) homotopies on the coinvariants. Hence, Proposition (A.11) applied to the G -morphism $\text{id}: V \rightarrow V$ proves the first part.

In the same way the second part can be derived from Proposition (A.11). \square

Proposition (A.16). For all Banach G -modules V the map

$$\begin{aligned} \varphi: (V_G)' &\longrightarrow (V')^G \\ f &\longmapsto f \circ \pi \end{aligned}$$

is a natural, where $\pi: V \rightarrow V_G$ is the canonical projection.

Proof. It is not hard to see that φ is well-defined and $\|\varphi\| \leq 1$. Conversely, we consider the map

$$\begin{aligned} \psi: (V')^G &\longrightarrow (V_G)' \\ f &\longmapsto \bar{f}, \end{aligned}$$

where $\bar{f}: V_G \longrightarrow \mathbf{R}$ is the unique continuous functional satisfying $\bar{f} \circ \pi = f$. Moreover, $\|\bar{f}\|_\infty \leq \|f\|_\infty$. Again, it is not difficult to check that ψ is well-defined and that $\|\psi\| \leq 1$.

By construction, $\varphi \circ \psi = \text{id}$ and $\psi \circ \varphi = \text{id}$, which implies that φ must be an isometric isomorphism. \square

B Bounded cohomology with twisted coefficients

Ivanov proved that bounded cohomology of topological spaces (with \mathbf{R} -coefficients) can be computed in terms of strong relatively injective resolutions of \mathbf{R} [9; Theorem 4.1]. This section is devoted to the following generalisation of Ivanov's result:

Theorem (B.1). *Let X be a countable connected CW-complex with fundamental group G and let V be a Banach G -module.*

1. *The morphism $\vartheta_{V'}: C_b^*(G; V') \longrightarrow C_b^*(\tilde{X}; V')$ of Banach G -cochain complexes (defined in (6.5)) induces an isometric isomorphism*

$$H_b^*(X; V') = H^*(C_b^*(\tilde{X}; V')^G) \cong H^*(C_b^*(G; V')^G).$$

Moreover, this isometric isomorphism does not depend on the choice of fundamental domain used in the definition of the $\vartheta_{V'}$.

2. *In particular: If C is a strong relatively injective resolution of V' , then there is a canonical isomorphism (degree-wise isomorphism of semi-normed vector spaces)*

$$H_b^*(X; V) \cong H^*(C^G) \cong H_b^*(G; V').$$

The proof of the first part relies on the following observation:

Lemma (B.2). *Let X be a countable connected CW-complex with fundamental group G and let V be a Banach G -module. The cochain complex $C_b^*(\tilde{X}; V') = B(C_*^{\ell^1}(\tilde{X}), V')$ together with the augmentation $\varepsilon_X: V' \longrightarrow C_b^0(\tilde{X}; V')$ given by the obvious inclusion is an approximate strong relatively injective resolution of V' .*

Definition (B.3). Let G be a discrete group.

1. If C is a Banach G -cochain complex and $n \in \mathbf{N}$, we define the **truncated cochain complex** $C|_n$ to be the Banach G -cochain complex derived from C by keeping only the modules (and the corresponding coboundary operators) in degree $0, \dots, n$ and defining all modules in higher degrees to be 0.
2. An augmented Banach G -cochain complex $(C, \varepsilon: V \rightarrow C^0)$ is an **approximate strong resolution** of the Banach G -module V , if for any $n \in \mathbf{N}$, the truncated complex $C|_n$ admits a **partial contracting cochain homotopy**, i.e., linear maps $(K_j: C^j \rightarrow C^{j-1})_{j \in \{1, \dots, n\}}$ and $K_0: C^0 \rightarrow V$ of norm at most 1 satisfying

$$\forall_{j \in \{1, \dots, n-1\}} \quad \delta^{j-1} \circ K_j + K_{j+1} \circ \delta^j = \text{id}_{C^j}$$

as well as $K_0 \circ \varepsilon = \text{id}_V$. ◇

The proof of Lemma (B.2) is a straightforward generalisation of Ivanov's proof of the fact that $C_b^*(\tilde{X})$ is a strong relatively injective $\pi_1(X)$ -resolution of \mathbf{R} , one of the main steps being the following splitting:

Lemma (B.4). *Let X and Y be simply connected spaces, let $p: X \rightarrow Y$ be a principal bundle whose structure group is an Abelian topological group G , and let V be a Banach space. Then for each $n \in \mathbf{N}$ there is a partial split of $C_b^*(p; V')|_n$, i.e., a cochain map*

$$A|_n: C_b^*(X; V')|_n \rightarrow C_b^*(Y; V')|_n$$

of truncated complexes satisfying for all $j \in \{0, \dots, n\}$

$$A|_n^j \circ C_b^j(p; V') = \text{id} \quad \text{and} \quad \|A|_n^j\| \leq 1.$$

Proof (of Lemma (B.4)). This can be shown in exactly the same way as the corresponding statement for \mathbf{R} -coefficients [9; Theorem 2.2]:

Let $n \in \mathbf{N}$. Then the group $G_n := \text{map}(\Delta^n, G)$ is Abelian and hence amenable (when regarded as discrete group). Therefore, there exists a G_n -equivariant mean

$$m: B(G_n, V') \rightarrow V',$$

where V' is endowed with the trivial G_n -action. Such a mean can, for example, be constructed via

$$m: B(G_n, V') \rightarrow V' \\ f \mapsto \left(v \mapsto m_{\mathbf{R}}(g \mapsto (f(g))(v)) \right),$$

where $m_{\mathbf{R}}: B(G_n, \mathbf{R}) \rightarrow \mathbf{R}$ is a G_n -invariant mean provided by amenability of G_n .

Now the same construction as in Ivanov's proof [9; proof of Theorem 2.2] gives rise to the partial split $A|_n$. (Perhaps there is no total split $C_b^*(X; V') \rightarrow C_b^*(Y; V')$, because – unlike in the case with \mathbf{R} -coefficients [9; p. 1094] – the theorem of Banach-Alaoglu cannot be applied directly to the space $B(B(G_n, V'), V')$. But for our applications the partial splits suffice.) □

Using Lemma (B.4), we can construct the required partial contracting homotopies of the bounded chain complex with twisted coefficients as in Ivanov's proof for \mathbf{R} -coefficients:

Proof (of Lemma (B.2)). Since \tilde{X} is a simply connected countable CW-complex, there is a sequence

$$\cdots \xrightarrow{p_n} X_n \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} X_2 \xrightarrow{p_1} X_1 := \tilde{X}$$

of principal bundles $(p_n)_{n \in \mathbf{N}_{>0}}$ with Abelian structure groups such that

$$\forall_{j \in \{0, \dots, n\}} \pi_j(X_n) = 0 \quad \text{and} \quad \forall_{j \in \mathbf{N}_{>n}} \pi_j(X_n) = \pi_j(\tilde{X})$$

holds for all $n \in \mathbf{N}_{>0}$ [9; p. 1096]. In particular, all the X_n are simply connected.

Let $n \in \mathbf{N}$. Since X_n is n -connected, one can explicitly construct a partial chain contraction

$$\mathbf{R} \xrightarrow{L_0} C_0(X_n) \xrightarrow{L_1} \cdots \xrightarrow{L_n} C_n(X_n)$$

with $\|L_j\| \leq 1$ for all $j \in \{0, \dots, n\}$ [9; p. 1097]. Because L is bounded, it can be extended to a partial cochain contraction

$$\mathbf{R} \xrightarrow{\bar{L}_0} C_0^{\ell^1}(X_n) \xrightarrow{\bar{L}_1} \cdots \xrightarrow{\bar{L}_n} C_n^{\ell^1}(X_n),$$

which also satisfies $\|\bar{L}_j\| \leq 1$. But then the induced maps

$$V' = B(\mathbf{R}, V') \xleftarrow{B(\bar{L}_0, \text{id}_{V'})} C_b^0(X_n; V') \xleftarrow{B(\bar{L}_1, \text{id}_{V'})} \cdots \xleftarrow{B(\bar{L}_n, \text{id}_{V'})} C_b^n(X_n; V')$$

clearly form a partial cochain contraction with norm at most 1. Using the splits from Lemma (B.4), we can transfer this partial contracting cochain map of X_n to one of X : By Lemma (B.4), for $j \in \{1, \dots, n\}$ we find partial splits

$$A(j)|_n: C_b^*(X_{j+1}; V')|_n \longrightarrow C_b^*(X_j; V')|_n$$

of $C_b^*(p_j)|_n$. We now consider the maps

$$V' \xleftarrow{K_0} C_b^0(\tilde{X}; V') \xleftarrow{K_1} \cdots \xleftarrow{K_n} C_b^n(\tilde{X}; V')$$

defined by

$$K_j := A(1)|_n \circ \cdots \circ A(n-1)|_n \circ B(\bar{L}_j, \text{id}_{V'}) \circ C_b^j(p_{n-1}; V') \circ \cdots \circ C_b^j(p_1; V')$$

for all $j \in \{0, \dots, n\}$. By construction, $\|K_j\| \leq 1$ and K_0, \dots, K_n form a partial cochain contraction [9; p. 1096].

It remains to show that the Banach G -modules $C_b^n(\tilde{X}; V')$ are relatively injective: Let $F \subset \tilde{X}$ be a fundamental domain for the G -action on \tilde{X} . For $n \in \mathbf{N}$, we write

$F_n \subset C_n^{\ell^1}(\tilde{X})$ for the Banach subspace generated by all singular simplices mapping the zeroth vertex of Δ^n into F . Then

$$C_n^{\ell^1}(\tilde{X}) = \ell^1(G) \overline{\otimes} F_n$$

(as Banach G -modules). In particular, we obtain (cf. Remark (A.4))

$$C_b^n(\tilde{X}; V') = B(C_n^{\ell^1}(\tilde{X}), V') = B(\ell^1(G) \overline{\otimes} F_n, V') = B(\ell^1(G), B(F_n, V')).$$

Since the Banach G -module $B(\ell^1(G), B(F_n, V'))$ is relatively injective [11; Proposition 4.4.1], it follows that $C_b^n(\tilde{X}; V')$ is relatively injective.

Hence, the cochain complex $(C_b^*(\tilde{X}; V'), \varepsilon_X)$ is an approximate strong relatively injective resolution of V' . \square

Theorem (B.1) can now be deduced from Lemma (B.2) by means of homological algebra:

Proof (of Theorem (6.6)). 1. The pair $(C_b^*(\tilde{X}; V'), \varepsilon_X: V' \rightarrow C_b^0(\tilde{X}; V'))$ is an approximate strong relatively injective resolution of V' by Lemma (B.2).

The morphism $\vartheta_{V'}: C_b^*(G; V') \rightarrow C_b^*(\tilde{X}; V')$ of Banach G -cochain complexes clearly satisfies (where $\varepsilon: C_0^{\ell^1}(G) \rightarrow \mathbf{R}$ is the augmentation of Definition (5.2))

$$\varepsilon_X \circ \vartheta_{V'}^0 = B(\varepsilon, \text{id}_V) \circ \text{id}_{V'}.$$

I.e., $\text{id}_{V'} \square \vartheta_{V'}: B(\varepsilon, \text{id}_V) \square C_b^*(G; V') \rightarrow \varepsilon_X \square C_b^*(\tilde{X}; V')$ is a morphism of Banach G -cochain complexes.

The inductive proof of Proposition (A.11) depends only on finite initial parts of the resolutions in question. Since $(C_b^*(G; V'), B(\varepsilon, \text{id}_V))$ is a strong relatively injective resolution (Proposition (5.6)), it follows that $\vartheta_{V'}$ is the (up to G -homotopy) unique morphism of Banach G -cochain complexes from $C_b^*(G; V')$ to $C_b^*(\tilde{X}; V')$ and that $\vartheta_{V'}$ admits a G -homotopy inverse.

In particular, the restriction of $\vartheta_{V'}$ to the G -invariants induces an isomorphism

$$H^*(C_b^*(\tilde{X}; V')) \cong H^*(C_b^*(G; V')^G),$$

which is independent of the choice of fundamental domain used in the definition of $\vartheta_{V'}$.

Furthermore, this isomorphism is even isometric: By construction, $\|\vartheta_{V'}\| \leq 1$. Conversely, it is known that the semi-norm on $H_b^*(G; V')$ induced by the norm on the standard resolution $C_b^*(G; V')$ is “minimal” [11; Corollary 7.4.7, Theorem 7.3.1]. Therefore, the isomorphism on cohomology induced by $\vartheta_{V'}$ must be isometric.

2. Because $C_b^*(G; V')$ is a strong relatively injective resolution (Proposition (5.6)), standard methods from homological algebra (Proposition (A.15)) show that there is a canonical isomorphism $H^*(C^G) \cong H^*(C_b^*(G; V')^G)$. \square

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