

# RANK GRADIENT VS. STABLE INTEGRAL SIMPLICIAL VOLUME

CLARA LÖH

ABSTRACT. We observe that stable integral simplicial volume of closed manifolds gives an upper bound for the rank gradient of the corresponding fundamental groups.

## 1. INTRODUCTION

The *residually finite view* on groups or spaces aims at understanding groups and spaces through gradient invariants: If  $I$  is an invariant of groups, then we define the associated gradient invariant  $\widehat{I}$  for groups  $\Gamma$  by

$$\widehat{I}(\Gamma) := \inf_{H \in F(\Gamma)} \frac{I(H)}{[\Gamma : H]},$$

where  $F(\Gamma)$  denotes the set of all finite index subgroups of  $\Gamma$ . For example, the rank gradient is the gradient invariant associated with the minimal number of generators of groups (Section 2), originally introduced by Lackenby [13]. Further well-studied examples are the Betti number gradient and the logarithmic torsion homology gradient.

Stable integral simplicial volume is the gradient invariant associated with integral simplicial volume (Section 3). It is known that stable integral simplicial volume yields upper bounds for Betti number gradients and logarithmic torsion homology gradients [5, Theorem 1.6, Theorem 2.6].

In this note, we observe that stable integral simplicial volume also gives an upper bound for the rank gradient of the corresponding fundamental groups:

**Theorem 1.1.** *Let  $M$  be an oriented closed connected manifold with fundamental group  $\Gamma$  and let  $\Gamma_* = (\Gamma_k)_{k \in \mathbb{N}}$  be a chain of finite index subgroups of  $\Gamma$ . Then*

$$\text{rg}(\Gamma, \Gamma_*) \leq \|M\|_{\mathbb{Z}}^{\Gamma_*}.$$

*In particular,  $\text{rg} \Gamma \leq \|M\|_{\mathbb{Z}}^{\infty}$ .*

The result even holds without any asymptotics (Lemma 4.2), but the gradient invariants seem to be the relevant invariants.

In particular, vanishing results for stable integral simplicial volume imply corresponding vanishing results for the rank gradient. For example, we obtain an alternative argument for the following rather special case of a result by Lackenby [13, Theorem 1.2]:

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**Corollary 1.2.** *Let  $\Gamma$  be a residually finite infinite amenable group that admits an oriented closed connected manifold as model of the classifying space  $K(\Gamma, 1)$ . Then  $\text{rg } \Gamma = 0$ .*

*Proof.* Let  $M$  be such a model of  $K(\Gamma, 1)$ . Then  $\|M\|_{\mathbb{Z}}^{\infty} = 0$  [5, Theorem 1.10]. Hence, Theorem 1.1 gives  $\text{rg } \Gamma = 0$ .  $\square$

However, the bound in Theorem 1.1 in general is far from being sharp. For instance, if  $\Gamma$  is the fundamental group of an oriented closed connected surface  $M$  of genus  $g \in \mathbb{N}_{\geq 1}$ , then it is well known that  $\text{rg } \Gamma = b_1^{(2)} = 2 \cdot g - 2$  [2][6, Proposition VI.9], but  $\|M\|_{\mathbb{Z}}^{\infty} = \|M\| = 4 \cdot g - 4$  [8, 14]. In more positive words, Theorem 1.1 shows that stable integral simplicial volume is a geometric refinement of the rank gradient.

In contrast to the residual point of view, the *dynamical view* on groups or spaces aims at understanding groups and spaces through actions on probability spaces. In the residually finite case, the profinite completion provides a canonical link between the residually finite and the dynamical view.

For example, the  $\mathbb{Q}$ -Betti number gradients coincide with the  $L^2$ -Betti numbers [15], which in turn admit a dynamical description [7]. The dynamical sibling of the rank gradient is  $\text{cost}$  [6, 11], and the  $\text{cost}$  (minus 1) of the profinite completion coincides with the rank gradient [2]. However, it remains an open problem to decide whether the rank gradient and  $\text{cost}$  coincide for all residually finite finitely generated groups.

The classical version of stable integral simplicial volume is Gromov's simplicial volume; the dynamical version is integral foliated simplicial volume [9, 17], and integral foliated simplicial volume with respect to the profinite completion coincides with stable integral simplicial volume [5, Theorem 2.6]. Therefore, it is natural to consider the following problem:

**Question 1.3.** Let  $M$  be an oriented closed connected aspherical manifold with (integral foliated) simplicial volume equal to 0. Does this imply that  $\text{Cost } \pi_1(M) = 1$ ? More generally, does (integral foliated) simplicial volume of  $M$  give a linear upper bound for  $\text{Cost}(\pi_1(M)) - 1$ ?

It should be noted that this is conjecturally trivial (but no direct route is known): The Singer conjecture predicts that all  $L^2$ -Betti numbers are 0, except possibly in the middle dimension. Moreover, conjecturally, if the first  $L^2$ -Betti number is 0, the  $\text{cost}$  equals 1. Both conjectures seem bold, but no counterexamples are known.

**Organisation of this article.** We briefly review the rank gradient (Section 2) and stable integral simplicial volume (Section 3). The (elementary) proof of Theorem 1.1 is given in Section 4.

## 2. RANK GRADIENT

The rank gradient of a group is the gradient invariant associated with the rank: For a finitely generated group  $\Gamma$ , we denote the rank of  $\Gamma$ , i.e., the minimal size of a generating set of  $\Gamma$ , by  $d(\Gamma)$ .

**Definition 2.1** (rank gradient [13]). Let  $\Gamma$  be a finitely generated group and let  $\Gamma_* = (\Gamma_k)_{k \in \mathbb{N}}$  be a (descending) chain of finite index subgroups of  $\Gamma$ .

Then the *rank gradient of  $\Gamma$  with respect to  $\Gamma_*$*  is defined as

$$\text{rg}(\Gamma, \Gamma_*) := \inf_{k \in \mathbb{N}} \frac{d(\Gamma_k) - 1}{[\Gamma : \Gamma_k]}.$$

Moreover, the *absolute rank gradient of  $\Gamma$*  is defined as

$$\text{rg} \Gamma := \inf_{H \in F(\Gamma)} \frac{d(H) - 1}{[\Gamma : H]},$$

where  $F(\Gamma)$  denotes the set of all finite index subgroups of  $\Gamma$ .

If  $\Gamma$  is a finitely generated group and  $(\Gamma_k)_{k \in \mathbb{N}}$  is a chain of finite index subgroups, then the sequence

$$\left( \frac{d(\Gamma_k) - 1}{[\Gamma : \Gamma_k]} \right)_{k \in \mathbb{N}}$$

is non-increasing; therefore,

$$\text{rg}(\Gamma, \Gamma_*) = \inf_{k \in \mathbb{N}} \frac{d(\Gamma_k) - 1}{[\Gamma : \Gamma_k]} = \lim_{k \rightarrow \infty} \frac{d(\Gamma_k) - 1}{[\Gamma : \Gamma_k]}.$$

The rank gradient was originally introduced by Lackenby [13]. It is known that finitely generated residually finite (infinite) amenable groups have trivial rank gradient [13, 2] and, more generally, that residually finite groups that contain an infinite amenable normal subgroup have trivial rank gradient [2]. In contrast, free groups of rank  $r \in \mathbb{N}_{\geq 1}$  have rank gradient  $r - 1$  [13] and fundamental groups of oriented closed connected surfaces of genus  $g \in \mathbb{N}_{\geq 1}$  have rank gradient  $2 \cdot g - 2$  [2][6, Proposition VI.9]. Moreover, some inheritance results are known for free products [1], certain free products with amalgamation, and certain HNN extensions [16]. Further classes of groups with known rank gradient are certain Artin groups and their relatives [10] and generalised Thompson groups [12].

In all known cases, the rank gradient coincides with the first  $L^2$ -Betti number and is independent of the chosen chain of (normal) subgroups (with trivial intersection), but it remains an open problem whether this is always the case.

### 3. STABLE INTEGRAL SIMPLICIAL VOLUME

Similarly, stable integral simplicial volume is the gradient invariant associated with integral simplicial volume: The *integral simplicial volume* of an oriented closed connected  $n$ -manifold  $N$  is defined by

$$\|N\|_{\mathbb{Z}} := \min \left\{ \sum_{j=1}^m |a_j| \mid \sum_{j=1}^m a_j \cdot \sigma_j \in C_n(N; \mathbb{Z}) \text{ is a fundamental cycle of } N \right\}.$$

**Definition 3.1** (stable integral simplicial volume). Let  $M$  be an oriented closed connected  $n$ -manifold with fundamental group  $\Gamma$ , let  $\Gamma_* = (\Gamma_k)_{k \in \mathbb{N}}$  be a chain of finite index subgroups of  $\Gamma$ , and for  $k \in \mathbb{N}$  let  $M_k = \widetilde{M}/\Gamma_k$  be the covering manifold of  $M$  associated with the subgroup  $\Gamma_k$ . Then the *stable integral simplicial volume of  $M$  with respect to  $\Gamma_*$*  is defined as

$$\|M\|_{\mathbb{Z}}^{\Gamma_*} := \inf_{k \in \mathbb{N}} \frac{\|M_k\|_{\mathbb{Z}}}{[\Gamma : \Gamma_k]}.$$

The *stable integral simplicial volume* of  $M$  is defined as

$$\|M\|_{\mathbb{Z}}^{\infty} := \inf_{H \in F(\Gamma)} \frac{\|\widetilde{M}/H\|_{\mathbb{Z}}}{[\Gamma : H]}.$$

In the situation of the definition, the sequence

$$\left( \frac{\|M_k\|_{\mathbb{Z}}}{[\Gamma : \Gamma_k]} \right)_{k \in \mathbb{N}}$$

is non-increasing; therefore,

$$\|M\|_{\mathbb{Z}}^{\Gamma_*} = \inf_{k \in \mathbb{N}} \frac{\|M_k\|_{\mathbb{Z}}}{[\Gamma : \Gamma_k]} = \lim_{k \rightarrow \infty} \frac{\|M_k\|_{\mathbb{Z}}}{[\Gamma : \Gamma_k]}.$$

As in the case for the rank gradient, it is unknown whether stable integral simplicial volume is independent of the chosen chain of subgroups (with trivial intersection) or not.

For aspherical oriented closed connected surfaces, for closed hyperbolic 3-manifolds, for closed Seifert manifolds with infinite fundamental group, and for aspherical closed manifolds with residually finite amenable fundamental group the stable integral simplicial volume coincides with ordinary simplicial volume [5]. However, for closed hyperbolic manifolds of dimension at least 4, stable integral simplicial volume is uniformly bigger than ordinary simplicial volume [4].

#### 4. PROOF OF THEOREM 1.1

We will first give a simple geometric proof of Theorem 1.1 with a worse multiplicative constant (Subsection 4.1) and then we will give a more algebraic proof with the improved constant (Subsection 4.2).

**4.1. A geometric proof.** We will prove the following version of Theorem 1.1 (with a slightly worse multiplicative constant): If  $M$  is an oriented closed connected  $n$ -manifold (with  $n > 0$ ) and if  $(\Gamma_k)_{k \in \mathbb{N}}$  is a chain of finite index subgroups of  $\Gamma$ , then

$$\text{rg}(\Gamma, \Gamma_*) \leq n \cdot \|M\|_{\mathbb{Z}}^{\Gamma_*}.$$

This result follows from the following observation, by passing to finite coverings.

**Lemma 4.1** (rank vs. integral simplicial volume). *Let  $M$  be an oriented closed connected  $n$ -manifold with fundamental group  $\Gamma$  (and  $n > 0$ ). Then*

$$d(\Gamma) \leq n \cdot \|M\|_{\mathbb{Z}}.$$

*Proof.* Let  $c = \sum_{j=1}^m a_j \cdot \sigma_j \in C_n(M; \mathbb{Z})$  be an integral fundamental cycle of  $M$  in reduced form; without loss of generality, we may assume that all vertices of  $\sigma_1, \dots, \sigma_m$  are mapped to the basepoint of  $M$  [3, Chapter 9.5]. Out of the combinatorics of  $c$  we can then construct a connected CW-complex  $X_c$ , a homology class  $\alpha_c \in H_n(X_c; \mathbb{Z})$  and a continuous map  $f_c: X_c \rightarrow M$  with

$$H_n(f_c; \mathbb{Z})(\alpha_c) = [c] = [M]_{\mathbb{Z}} \in H_n(M; \mathbb{Z})$$

and

$$d(\pi_1(X_c)) \leq n \cdot m.$$

In detail, this is done as follows: We set

$$X_c := (\{1, \dots, m\} \times \Delta^n) / \sim,$$

where the equivalence relation  $\sim$  is generated by the following gluing conditions: For all  $j, j' \in \{1, \dots, m\}$  and all  $k, k' \in \{0, \dots, n\}$  with  $\sigma_j \circ i_{k'} = \sigma_{j'} \circ i_k$  we let

$$\forall t \in \Delta^{n-1} \quad (j, i_k(t)) \sim (j', i_{k'}(t));$$

here,  $i_k: \Delta^{n-1} \rightarrow \Delta^n$  denotes the inclusion of the  $k$ -th face of  $\Delta^n$ . We endow  $X_c$  with the obvious cellular structure; moreover, without loss of generality we may assume that  $X_c$  is connected (otherwise, we glue as many vertices as needed). Moreover, looking at the description of  $\pi_1(X_c)$  in terms of 1- and 2-cells, shows that  $d(\pi_1(X_c)) \leq n \cdot m$ .

By construction, the maps  $\sigma_1, \dots, \sigma_m$  glue to give a well-defined continuous map  $f_c: X_c \rightarrow M$ .

As next step, we construct  $\alpha_c$ : For  $j \in \{1, \dots, m\}$  we consider the singular simplex  $\tau_j: \Delta^n \rightarrow X_c$  induced by the  $j$ -inclusion  $\Delta^n \rightarrow \{1, \dots, m\} \times \Delta^n$ . Because  $c$  is a cycle in  $M$  the gluing relation ensures that  $z_c := \sum_{j=1}^m a_j \cdot \tau_j \in C_n(X_c; \mathbb{Z})$  is a cycle on  $X_c$ . We write  $\alpha_c := [z_c] \in H_n(X_c; \mathbb{Z})$  for the corresponding class. By construction, we then have  $H_n(f_c; \mathbb{Z})(\alpha_c) = [c]$ , as desired. This concludes the construction of the model space  $X_c$  and its related objects.

We show that  $\pi_1(f_c): \pi_1(X_c) \rightarrow \Gamma$  is surjective: Let  $H := \text{im } \pi_1(f_c) \subset \Gamma$ , let  $\pi: \overline{M} \rightarrow M$  be the covering of  $M$  associated with the subgroup  $H$ , and let  $\overline{f}_c: X_c \rightarrow \overline{M}$  be the corresponding  $\pi$ -lift of  $f_c$ . Then

$$H_n(\pi; \mathbb{Z}) \circ H_n(\overline{f}_c; \mathbb{Z})(\alpha_c) = H_n(f_c; \mathbb{Z})(\alpha_c) = [M]_{\mathbb{Z}}.$$

In particular,  $[M]_{\mathbb{Z}} \in \text{im } H_n(\pi; \mathbb{Z})$ , and so  $|\deg \pi| = 1$  and  $H = \Gamma$ .

Because  $\pi_1(f_c)$  is surjective, we obtain

$$d(\Gamma) \leq d(\pi_1(X_c)) \leq n \cdot m \leq n \cdot |c|_1.$$

Taking the minimum over all fundamental cycles of  $M$  proves the claim.  $\square$

**4.2. A more algebraic proof.** We will now prove an improved version of Lemma 4.1, which implies Theorem 1.1 by passing to finite coverings and the infimum.

**Lemma 4.2** (rank vs. integral simplicial volume, improved bound). *Let  $M$  be an oriented closed connected  $n$ -manifold with fundamental group  $\Gamma$ . Then*

$$d(\Gamma) \leq \|M\|_{\mathbb{Z}}.$$

*Proof.* Let  $c = \sum_{j=1}^m a_j \cdot \sigma_j \in C_n(M; \mathbb{Z})$  be an integral fundamental cycle of  $M$  in reduced form. We will now consider the corresponding situation on the universal covering: Let  $\pi: \widetilde{M} \rightarrow M$  be the universal covering of  $M$  and let  $D \subset \widetilde{M}$  be a set-theoretic fundamental domain of the deck transformation action of  $\Gamma$  on  $\widetilde{M}$ . We then take the lift  $\tilde{c} = \sum_{j=1}^m a_j \cdot \tilde{\sigma}_j \in C_n(\widetilde{M}; \mathbb{Z})$  of  $c$  such that for every  $j \in \{1, \dots, m\}$  the 0-th vertex of  $\tilde{\sigma}_j$  lies in  $D$ .

For each  $j \in \{1, \dots, m\}$  we let  $g_j \in \Gamma$  be the group element that maps the 0-th vertex of  $\tilde{\sigma}_j \circ i_0$  (i.e., the first vertex of  $\tilde{\sigma}_j$ ) to  $D$ . Then we set

$$S := \{g_1, \dots, g_m\} \quad \text{and} \quad H := \langle S \rangle_{\Gamma} \subset \Gamma.$$

By construction  $d(H) \leq |S| = m \leq |c|_1$ .

We will now show that  $H = \Gamma$  (which completes the proof of the lemma): Let  $\pi_H: \widetilde{M} \rightarrow \widetilde{M}/H =: \overline{M}$  be the upper covering associated with the subgroup  $H \subset \Gamma$  and let  $\bar{c} := C_n(\pi_H; \mathbb{Z})(\tilde{c}) \in C_n(\overline{M}; \mathbb{Z})$ . As first step we show that  $\bar{c}$  is a cycle. In

$$C_*(\overline{M}; \mathbb{Z}) \cong \mathbb{Z} \otimes_{\mathbb{Z}H} C_*(\widetilde{M}; \mathbb{Z})$$

with the trivial  $H$ -action on  $\mathbb{Z}$ , we calculate (using  $g_1, \dots, g_m \in H$ ) that

$$\begin{aligned} \partial(\bar{c}) &= \sum_{j=1}^m a_j \cdot g_j^{-1} \otimes g_j \cdot (\tilde{\sigma}_j \circ i_0) + \sum_{j=1}^m \sum_{k=1}^n (-1)^k \cdot a_j \otimes \tilde{\sigma}_j \circ i_k; \\ &= \sum_{j=1}^m a_j \otimes g_j \cdot (\tilde{\sigma}_j \circ i_0) + \sum_{j=1}^m \sum_{k=1}^n (-1)^k \cdot a_j \otimes \tilde{\sigma}_j \circ i_k \\ &= 1 \otimes \left( \sum_{j=1}^m a_j \cdot g_j \cdot (\tilde{\sigma}_j \circ i_0) + \sum_{j=1}^m \sum_{k=1}^n (-1)^k \cdot a_j \cdot \tilde{\sigma}_j \circ i_k \right); \end{aligned}$$

because the submodule of  $C_{n-1}(\overline{M}; \mathbb{Z})$  generated by simplices with 0-th vertex in  $D$  is isomorphic (through the covering projection) with  $C_{n-1}(M; \mathbb{Z})$  and because  $\partial c = 0$ , it follows that the right hand term is 0. Therefore,  $\bar{c}$  is a cycle.

Let  $p_H: \overline{M} \rightarrow M$  be the lower covering associated with  $H \subset \Gamma$ . Then

$$H_n(p_H; \mathbb{Z})([\bar{c}]) = [c] = [M]_{\mathbb{Z}},$$

and so  $[M]_{\mathbb{Z}} \in \text{im } H_n(p_H; \mathbb{Z})$ . Therefore,  $|\deg p_H| = 1$  and  $H = \Gamma$ .  $\square$

**Remark 4.3.** Of course, the statement of Lemma 4.2 also generalises to the case of general spaces and homology classes in  $H_*(M; \mathbb{Z})$  that do not lie in the image of the covering map of a finite (connected) covering of  $M$ .

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*Clara Löh*

Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg  
clara.loeh@mathematik.uni-r.de,  
<http://www.mathematik.uni-r.de/loeh>