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LOWER MULTIPLICITY BOUNDS VIA CLASSIFYING SPACES

(based on joint work with P. Capovilla, K. Li, M. Moraschini, R. Sauer)

I cat_F II CLASSIFYING SPACES

Def. (cat_F). Let F be a class of groups, let X be a top. space.

- An <u>F-subset</u> of X is a subset $U \subset X$ with

$$\forall_{x \in U} \ \underbrace{\mathrm{im}\, \pi_1(U \hookrightarrow X, x)}_{< \pi_1(X,x)} \in F.$$

- An <u>F-cover</u> of X is an open cover of X by F-subsets.

- $$\mathrm{cat}_F(X) := \min\{ n \in \mathbb{N} \mid \exists_{F\text{-cover } U \text{ of } X} \ |U| \leq n \}$$

if X is a CW-complex

$$= \min\{ n \in \mathbb{N} \mid \exists_{F\text{-cover } U \text{ of } X \text{ by connected sets}} \ \mathrm{mult}(U) \leq n \}$$

Remark: · 1: the class of trivial groups
cat_1 studied by Eilenberg-Ganea
If X is an aspherical CW-complex, then $\mathrm{cat}_1 X = \mathrm{cat}_{LS} X$.

· If $1 \subset F$ and if X is a CW-cplx then:

$$? \leq \mathrm{cat}_F(X) \leq \mathrm{cat}_1(X) \leq \mathrm{cat}_{LS}(X) \leq \dim X + 1$$

especially for $F := \underbrace{\mathrm{Am}}$ the class of amenable gps

Def. (amenable gp) A group Γ is <u>amenable</u> if there ex. a Γ-inv. mean on $\ell^\infty(\Gamma, \mathbb{R})$, i.e.: an $\mathbb{R}$-lin. map $m : \ell^\infty(\Gamma,\mathbb{R}) \to \mathbb{R}$ with

- $m(1) = 1$
- $m(f) \geq 0$ for all $f \in \ell^\infty(\Gamma,\mathbb{R})$ with $f \geq 0$
- $\forall_{f \in \ell^\infty(\Gamma,\mathbb{R})} \forall_{\gamma \in \Gamma}$ $m(\underbrace{\gamma \cdot f}_{(x \mapsto f(\gamma^{-1}x))}) = m(f)$.

Examples: · all finite gps are amenable

· Abelian gps are amenable

· Am is closed wrt subgroups, quotients, extensions

⇝ all virtually solvable gps are amenable

· F_2 is <u>not</u> amenable

· open: Thompson F: amenable?!

Examples:

$\Sigma_2 =$  $\qquad 2 \le \mathrm{cat}_{Am}(\Sigma_2) \le 3$

$X = (S^1 \vee S^1)^{\times n} \qquad 2 \le \mathrm{cat}_{Am}(X) \le n+1$

exact values ?

Obstruction for cat_{Am}:

1. · bounded cohomology / simplicial volume [Gromov, Ivanov]

2: for aspherical spaces: L^2-Betti numbers [Gromov, Sauer]

- for aspherical occ mfds: rg, cost, log torsion homology growth [Sauer, Moraschini, L]

goal: unified strategy for 1/2.

II) USING CLASSIFYING SPACES

Ingredients: (1) nerves

(2) classifying spaces

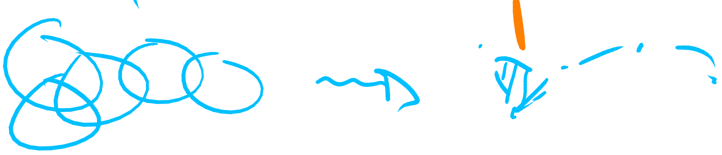
(1) Nerves

$\mathcal{F}$: closed under conj./subgroups

$\mathcal{U}$ $\mathcal{F}$-cover of X (connected)

$\rightsquigarrow$ nerve $N(\mathcal{U})$

$\mathcal{U}$ + nerve map $X \xrightarrow{\nu} |N(\mathcal{U})|$



$\widetilde{\mathcal{U}}$ lifted cover of $\widetilde{X}$ (connected)

$\rightsquigarrow$ $\Gamma \curvearrowright$ nerve $N(\widetilde{\mathcal{U}})$

isotropy is in $\mathcal{F}$

$+ \ \widetilde{X} \xrightarrow{\widetilde{\nu}} |N(\widetilde{\mathcal{U}})|$ Γ-map and: $\dim N(\tilde{\mathcal{U}}) = \mathrm{mult}(\mathcal{U}) - 1$

Prop. [CLM] Let X be conn. CW-complex, $\Gamma = \pi_1(X)$, $\mathcal{F}$ a subgp family of Γ.
If $\mathrm{cat}_{\mathcal{F}}(X) \leq 2$, then Γ is the fund. gp of a graph of gps with vertex/edge stab. in $\mathcal{F}$.

Sketch proof.

$\Gamma \curvearrowright N(\tilde{\mathcal{U}})$ is a forest

$\mathrm{mult}(\mathcal{U}) \leq 2$

$\rightsquigarrow$ apply Bass-Serre thm. $\square$

(2) Classifying spaces

Def. Let Γ be a gp and let $\mathcal{F}$ be a subgp family of Γ. A <u>model of $E_{\mathcal{F}}\Gamma$</u> is a Γ-CW-complex E with:

- all isotropy of E is in $\mathcal{F}$
- for all Γ-CW complexes X with isotropy in $\mathcal{F}$, there ex., up to Γ-htpy exactly one Γ-map $X \to E_{\mathcal{F}}\Gamma$.

Thm. Such models exist!

Example. · $F = 1 \rightsquigarrow E_F\Gamma$ "=" $E\Gamma$

· F = all subgps of Γ, then the pt is a model for $E_F\Gamma$

⚠ In general: Hard to find "nice" models for $E_F\Gamma$.

Prop. [CLM] Let X be a conn. CW-complex, let $\Gamma := \pi_1(X)$, F be a subgp family of Γ. Then

$$\mathrm{cat}_F(X) - 1 = \text{minimal } k \in \mathbb{N}_{\geq 0}$$

s.t. the classif. map $\tilde{X} \to E_F\Gamma$ is Γ-homotopic to a map to "the" k-skeleton of $E_F\Gamma$.

$$\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[d, "\wr"'] \arrow[dr, dashed] & E_F\Gamma \arrow[loop right, phantom, "\circlearrowleft\Gamma"] \\
{|N(\tilde{\mathcal{U}})|} \arrow[r, dashed] \arrow[ur] & (E_F\Gamma)_k \arrow[u, hook]
\end{tikzcd}$$

Proof: $\leq$ pull back the "open stars cover"

$\geq$ equiv. nerves and cellular approx. □

Context: [Eilenberg–Ganea]: $F = 1$

- characterize $TC(B\Gamma) - 1$ via the classif. map $E(\Gamma \times \Gamma) \longrightarrow E_\Delta(\Gamma \times \Gamma)$

[Farber, Grant, Lupton, Oprea]

Strategy for lower bounds:

Setup: X: conn. CW-complex, $\Gamma := \pi_1(X)$, F a subgrp family of Γ, $k := \mathrm{cat}_F(X) - 1 < \infty$.

Idea: let $H \xrightarrow{T} K : \Gamma\text{-CW} \longrightarrow C$ has a 0-object (contravariant)

Goal: find conditions that ensure that $T_{\tilde{X}} : H(\tilde{X}) \to K(\tilde{X})$ is zero.

We consider:

let $E :=$ " $(E_F\Gamma)_k$ "

$$\begin{tikzcd}
H(\tilde X) \arrow[r, equal] \arrow[d, "c"', "\simeq"] & H(\tilde X) \arrow[d, dashed] \arrow[r, "T_{\tilde X}"] & K(\tilde X) \\
H(E\Gamma) \arrow[d, "c"', "\simeq"] & G \cong \Gamma & \\
H(E_F\Gamma) & H(E) \arrow[l] \arrow[r, "T_E"] & K(E) \arrow[uu]
\end{tikzcd}$$

$K(E) \simeq 0$

If

(0) H is $\simeq_\Gamma$-inv

(1) $H(E\Gamma) \to H(\tilde X)$ is an iso

(2) $H(E_F\Gamma) \to H(E\Gamma)$ is an iso

(3) $K(Y) \simeq 0$ for all G-complexes Y with $\dim Y < \operatorname{cat}_F(X)$

then $\boxed{T_{\tilde X} = 0}$.

Example:

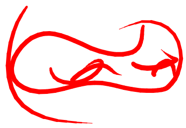
- $F := \mathrm{Am}$
- $H \Longrightarrow K$ comparison map

$$\begin{array}{ccc} H & \Longrightarrow & K \\ \| & & \| \\ H_b^{\cancel{*}\ \geq \ell+1}(\cdot\,;\mathbb{R}) & & H^{\cancel{*}\ \geq \ell+1}(\cdot\,;\mathbb{R}) \end{array}$$

(0) ✓ (3) ✓

(1), (2) ✓ "mapping thm" [Gromov ✓]

$$\rightsquigarrow \quad H_b^{\geq \ell+1}(X;\mathbb{R}) \xrightarrow[0]{\text{comp. map}} H^{\geq \ell+1}(X;\mathbb{R})$$

E.g.: $\mathrm{cat}_{\mathrm{Am}}($  $) = 3$.

Open problems:

- $\mathrm{cat}_{\mathrm{Am}}(X \times X) \leq TC(X)$ for which X ?
- amp-length estimate on $H_b^*(\cdot\,;\mathbb{R})$ for $\mathrm{cat}_{\mathrm{Am}}$?