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Gradient Invariants of Aspherical Manifolds with Small Amenable Multiplicity

I Uniform vanishing results II The dynamical view

III Computations

I

Setup (vague): I : $[0,\infty]$-valued invariant on groups/spaces

$\rightsquigarrow$ $\hat{I}$: res fin. groups/spaces $\longrightarrow [0,\infty]$

$$\Gamma \longmapsto \hat{I}(\Gamma) := \text{"}\lim_{\substack{\Lambda < \Gamma, \\ [\Gamma:\Lambda]<\infty}}\text{"} \frac{I(\Lambda)}{[\Gamma:\Lambda]}.$$

Ex: · $\hat{b}_k(\cdot\,;F)$ ← field, PID $\left(\hat{b}_k(\cdot\,;\mathbb{Q}) = b_k^{(2)}\ [\text{Lück}]\right)$

· $\hat{t}_k := \overbrace{\log |\text{tors}\, H_k(\cdot\,;\mathbb{Z})|}$

· $t_g := \hat{r}_k$ — min # gens of a group

Theorem ① [LMS] Let M be an occ aspherical mfld of dim $n > 0$, with res. fin. π_1 and $\operatorname{mult}_{Am} M \leq n$.

Then $\forall_k \quad \hat{b}_k(M; \mathbb{R}) = 0$ (PID), $\quad \hat{t}_k(M) = 0$,

$\operatorname{rg} \pi_1(M) = 0$.

min. multiplicity of an amenable open cover of M

($V \subset M$ amenable $:\iff \forall_{x \in V} \operatorname{im}(\pi_1(V, x) \to \pi_1(M, x))$ amenable)

related results: [Sauer], [Okun-Schreve],

$\hat{b}_*$: PD

$\hat{t}_*$: "Soulé"

rg: explicit [Haulmark-Schreve]

Theorem ② [LMS] In the situation of ①, we have

$$\|M\|^{\infty}_{\mathbb{Z}} = 0.$$

$\widehat{\|\cdot\|_{\mathbb{Z}}}$

$\|N\|_{\mathbb{Z}} := \min \{ \sum_j |a_j| \mid \sum_j a_j \sigma_j \in C_n(N; \mathbb{Z}) \text{ fund. cycle} \}$

singular chain complex!

How to prove (2) (?) use ergodic theory!
(similar to rg/cost by Abért–Nikolov)

II THE DYNAMICAL VIEW

Definition [Gromov, M Schmidt] Let M be an occ n-mfd, let $\Gamma := \pi_1(M)$, let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a pmp action (on a std Borel prob space).
Then

$$\|M\|^{\alpha} := \|M\|_{L^\infty(X,\mathbb{Z})}$$

twisted by α

"flexible"

$$:= \inf \Big\{ \sum_j \int |f_j| \, d\mu \;\Big|\; \sum_j f_j \otimes \sigma_j \in L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M};\mathbb{Z}) \text{ is a fund. cycle of } M \Big\}$$

$$\in \mathbb{R}_{\geq 0}.$$

"rigid"

The integral foliated simplicial volume of M:

$$\|M\|_{\mathcal{F},\mathbb{Z}} := \inf_{\alpha} \|M\|^{\alpha}.$$

Theorem [L, Pagliantini] If M is an occ mfd with res. fin. π_1, then

$$\|M\|^{\infty}_{\mathbb{Z}} = \|M\|^{\pi_1(M) \curvearrowright \widehat{\pi_1(M)}}$$

Theorem. If M is an occ mfd and $\alpha : \pi_1(M) \curvearrowright (X,\mu)$ is a pmp action, then

$$\forall_{k \in \mathbb{N}} \quad b_k^{(2)}(M) \leq \|M\|^\alpha \quad \text{[Gromov, Schmidt]}$$

$$\operatorname{cost}(\alpha) \leq \|M\|^\alpha \quad \text{[L]}.$$

III COMPUTATIONS

Theorem [Fauser–L–Moraschini–Quintanilha]
If M is an occ aspherical 3-mfd, then

$$\|M\|^{\widehat{\pi_1(M)}} = \|M\|^\infty_{\mathbb{Z}_p} = \|M\|_{\mathbb{R}} = \frac{\text{hyp vol}(M)}{v_3}.$$

[Gromov, Soma]

Proof. $\geq$: ✓

$\leq$: JSJ-decomp + geometr.

$$\|M\|^{\widehat{\pi_1(M)}} \overset{!}{\leq} \sum_{W \text{ JSJ pieces}} \|W, \partial W\|^{\widehat{\pi_1(\cancel{M}W)}}$$

hyp (ergodic thy!)

Seifert fibred $\rightsquigarrow 0$

□

Theorem (3) [LMS] Let M be an occ aspherical mfd of dim $n > 0$, let $\Gamma := \pi_1(M)$, let $\mathrm{mult}_{\mathrm{Am}} M \leq n$. If $\alpha : \Gamma \curvearrowright (X, \mu)$ is an ess. free pmp action, then

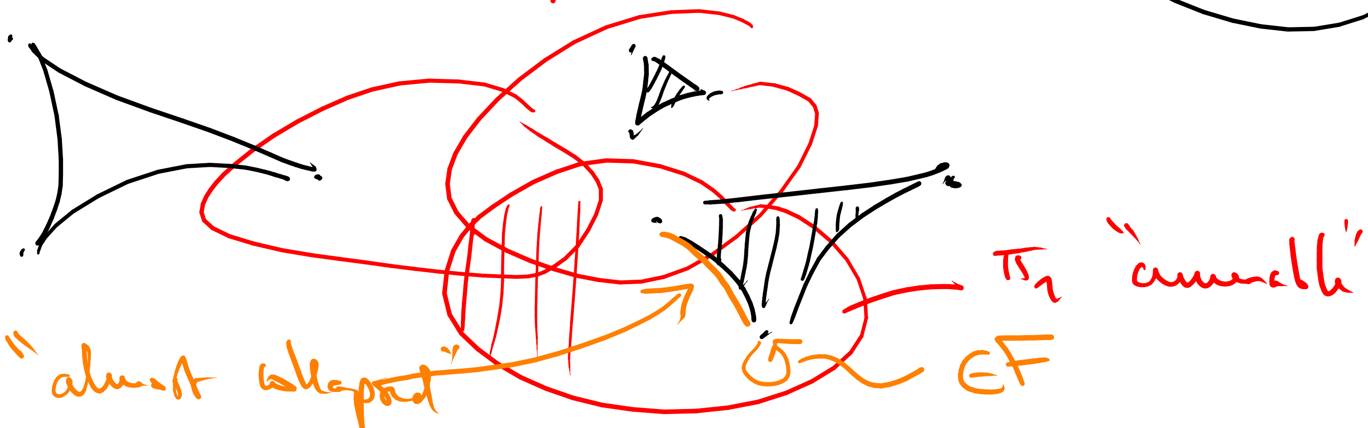
$$\|M\|^{\alpha} = 0 .$$

$\Longrightarrow$ Thm (2) $\Longrightarrow$ Thm (1).

$\alpha : \pi_1(M) \curvearrowright \widehat{\pi_1(M)}$ PD...

Sketch of proof: "dynamical version of the amenable reduction lemma" [Gromov, Alpert–Katz].

$\mathcal{U}$: amenable open cover of M of mult $\leq n$



fund. cycle c of M with:

$\forall \underbrace{\sigma \in c}_{n+1 \text{ vertices}}$ $\exists_{V \in \mathcal{U}}$ two vertices/edge of σ lie in V

(M: occ hyp 3-mfd $\Longrightarrow$ $\mathrm{mult}_{\mathrm{Am}}\, M = 4$)

(Ex: M occ aspherical

∇

A normal, amenable,

?! $\longrightarrow$ $\mathrm{cd}(\pi_1(M)/A) < \dim M$

$\Longrightarrow \mathrm{mult}_{\mathrm{Am}} M \leq \dim M$)

$M = B\pi_1(M)$ $\quad$ small amenable cover of M

$\downarrow$ $\quad$ $\Uparrow$

$B(\pi_1(M)/A)$ $\quad$ $\dim < \dim M$

"open stars cover"

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use averaging over Følner sets F:

$$c \rightsquigarrow \frac{1}{|F|} \sum_{\gamma \in F} \dots$$

$\underbrace{\qquad\qquad}_{|\cdot|_{\mathrm{ess},1} \longrightarrow 0}$

$F \to \infty$

$\notin \mathbb{Z}$

$$L^\infty(X;\mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_*(\tilde{M};\mathbb{Z})$$

Replace the averaging by a Rokhlin lemma:

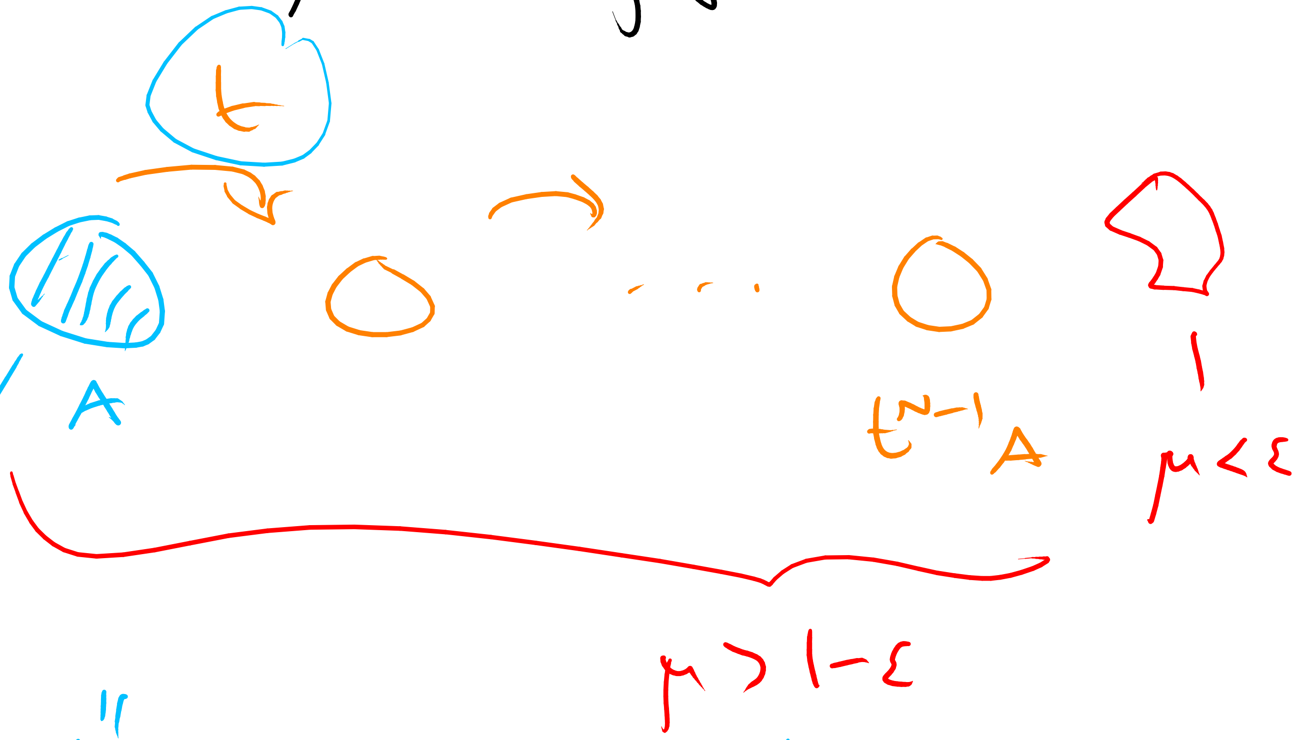
Basic version: (for $\mathbb{Z}$ $= \langle t \rangle$).

Let $\mathbb{Z} \curvearrowright (X,\mu)$ be pmp action, ess. free.

Let $\varepsilon \in \mathbb{R}_{>0}$, $N \in \mathbb{N}$. Then: there ex. a measurable subset $A \subset X$ s.t.

$$\forall_{\substack{j,k \in \{0,\dots,N-1\} \\ j \neq k}} \quad t^jA \cap t^kA = \emptyset$$

and $\mu\left(X \setminus \bigcup_{j=0}^{N-1} t^jA\right) < \varepsilon$.



"$\frac{1}{N} \cdot X$" in an "equivariant" way.