Simplicial Volume of One-Relator Groups

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http://www.mathematik.uni-r.de/loeh/talk.pdf

I One-Relator Groups II scl III Simplicial Volume

I One-Relator Groups

one-relator group $\langle \overset{\text{set}}{S} \mid \overset{\in F := F(S)}{r} \rangle := F / \langle r \rangle^{\triangleleft}_F$

Examples: • $\langle a \mid a^2 \rangle \cong \mathbb{Z}/2$

• $\langle a, b \mid [a,b] \rangle \cong \mathbb{Z}^2$

• $\langle a_1, \dots, a_n, b_1, \dots, b_n \mid [a_1, b_1] \cdots [a_n, b_n] \rangle \cong \pi_1(\Sigma_n)$

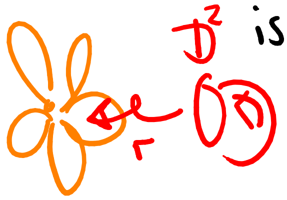
• Baumslag–Solitar groups

⋮

Question. How are properties of $\langle S \mid r \rangle$ related to properties of r ?

Well-known:

$\bigvee_S S^1$



• If r is <u>not</u> a proper power, then $\langle S \mid r \rangle$ is torsion-free and the presentation complex is a classifying space for $\langle S \mid r \rangle$ ($\leadsto \operatorname{cd} \langle S \mid r \rangle \leq 2$).

• if r is a proper power, then $\langle S \mid r \rangle$ is hyperbolic.

• if $r \in [F,F] \setminus \{e\}$, then $\langle S \mid r \rangle$ amenable $\iff$ $\langle S \mid r \rangle \cong \mathbb{Z}^2$.

· $H_2(\langle S|r\rangle;\mathbb{Z}) \underset{\mathbb{Z}}{\cong} \mathbb{Z}$ "gen. by r"

if $r\in[F,F]\setminus\{e\}$. [Hopf formula]

Now: focus on the case $r\in[F,F]\setminus\{e\}$ and the following invariants:

- stable commutator $\mathrm{scl}_S r$ ← invariant of r
- simplicial volume of $\langle S|r\rangle$ ← invariant of $\langle S|r\rangle$

II SCL

Definition. For $g\in F$, we define

$$\mathrm{cl}_S(g) := \inf\{n\in\mathbb{N} \mid \exists_{x_1,\dots,x_n,y_1,\dots,y_n}\ g=[x_1,y_1]\cdots[x_n,y_n]\}$$
$$\in \mathbb{N}\cup\{\infty\}$$

(difficult to compute! [Henar])

$$\mathrm{scl}_S(g) := \inf_{n\in\mathbb{N}_{>0}} \frac{\mathrm{cl}_S(g^n)}{n} \in \cancel{\mathbb{R}_{\geq 0}}\ \mathbb{Q}_{\geq 0} \cup\{\infty\}.$$

(can be computed efficiently [Calegari], i.e., there is a polynomial (in $\lg_S$) algorithm)

Known: · gap: $\forall_{g\in[F,F]\setminus\{e\}}\ \mathrm{scl}_S(g)\geq \frac{1}{2}$

· open problem (second gap):

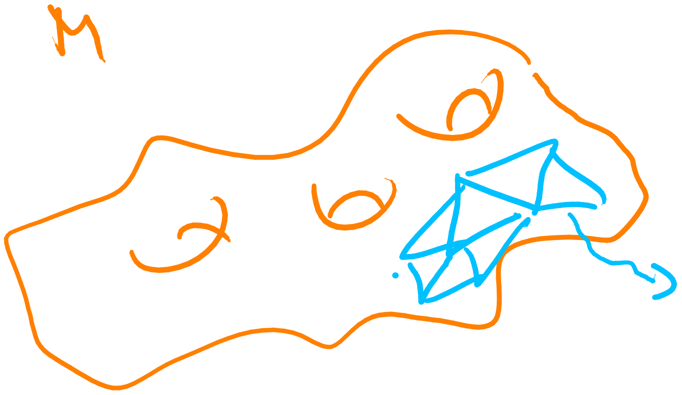
$$\exists_{g\in[F,F]}\ \frac{1}{2} < \mathrm{scl}_S(g) < \frac{7}{12}.$$

· $\mathrm{scl}_{\{a_1,\dots,a_n,b_1,\dots,b_n\}}[a_1,b_1]\cdots[a_n,b_n] = n-\frac{1}{2}.$

III SIMPLICIAL VOLUME

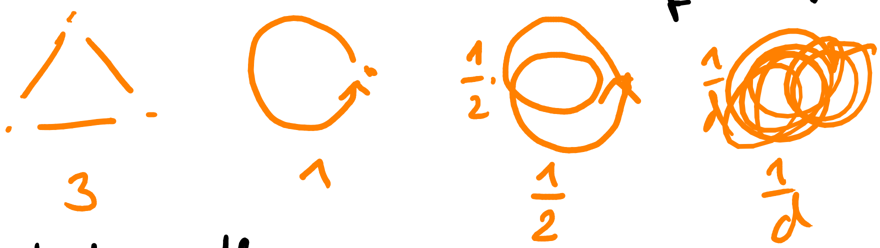
Definition. [Gromov] The simplicial volume of an oriented closed connected n-mfd M is

$$\|M\| := \|[M]_{\mathbb{R}}\|_1 := \inf\Big\{ \sum_j |a_j| \;\Big|\; \sum_j \overset{\mathbb{R}}{a_j} \cdot \overset{\Delta^n \to M}{\sigma_j} \in C_n(M;\mathbb{R}) \text{ fundamental cycle of } M \Big\} \in \mathbb{R}_{\geq 0}.$$



Some properties:

- If M is triangulable, then $\|M\| \leq \min \#$ n-simplices in a triangulation of M.
- $\|S^1\| = 0$



- if M is hyperbolic, then
  $$\|M\| = \frac{\operatorname{vol} M}{v_n} > 0 \qquad \text{[Gromov, Thurston]}$$
  ⇝ rigidity! $\|\Sigma_g\| = 4\cdot(g-1)$
- If $\pi_1(M)$ is amenable (and $\dim > 0$), then
  $$\|M\| = 0 \qquad \text{[Gromov, Ivanov]}$$

in general: hard to compute!

Definition [Heuer–L] Let $r \in [F,F]\setminus\{e\}$. Then the <u>simplicial volume</u> of $\langle S|r\rangle$ is def'd as

$$\|\langle S|r\rangle\| := \|[\langle S|r\rangle]_{\mathbb{R}}\|_1 \in \mathbb{R}_{\geq 0}$$

$\rightsquigarrow [\langle S|r\rangle]_{\mathbb{R}} \in H_2(\langle S|r\rangle;\mathbb{R}) \cong \mathbb{R}$ gen.

Example.

$$\|\langle \underbrace{a_1,\dots,a_n,b_1,\dots,b_n}_{S} \mid \overbrace{\underbrace{[a_1,b_1]\cdots[a_n,b_n]}_{r}}^{\mathrm{scl}:\ n-\frac{1}{2}}\rangle\| = \|\Sigma_n\| = 4\cdot(n-1)$$

$$= 4\cdot\left(\mathrm{scl}_S\, r - \frac{1}{2}\right).$$

Question:
- When do we have

$$(*) \quad \|\langle S|r\rangle\| = 4\cdot\underbrace{\left(\mathrm{scl}_S\, r - \frac{1}{2}\right)}_{\geq 0} \ ?$$

- (How) Can $\|\langle S|r\rangle\|$ be computed?

Some answers: [Heuer–L]
- if r is a proper power, then $\langle S|r\rangle$ is hyperbolic and so $\|\langle S|r\rangle\| > 0$ (uses results of Mineyev)
- $(*)$ holds if r is decomposable:

$$\underbrace{r = r_1\cdot r_2}_{\text{different gen's}} \quad \text{or} \quad r = \underbrace{r_1\cdot t\cdot r_2}_{\text{do not contain } t}\cdot t^{-1}$$

$\rightsquigarrow$ mod 1 all of $\mathbb{Q}_{\geq 0}$ occur as $\|\langle S|r\rangle\|$

- always: $\|\langle S|r\rangle\| < 4\cdot \mathrm{scl}_S\, r$

- asymptotically $\boxed{*}$ holds:

$$\lim_{N\to\infty} \frac{\|\langle S|r^N\rangle\|}{N} = 4\cdot \mathrm{scl}_S\, r.$$

- $\|\langle S|r\rangle\|$ and $4\cdot \mathrm{scl}_S\, r$ have similar distributions

(uses scl-results by Calegari-Walker)

- There is a function lallop: $[F,F]\to\mathbb{Q}_{\geq 0}$

(similar to Calegari's scallop for scl)

with

- lallop$(r) \leq \|\langle S|r\rangle\|$
- lallop$(r) \leq 4\cdot(\mathrm{scl}_S\, r - \frac{1}{2})$
- admits a polynomial algorithm
- in "many" examples: lallop$(r) = \|\langle S|r\rangle\|$

⇝ Example that does not satisfy $\boxed{*}$:

$$r = a^4 b ABABba BA^2 bAB$$

$$\mathrm{scl}_{\{a,b\}}\, r = \frac{5}{8} \quad \text{but} \quad \|\langle S|r\rangle\| = 0.$$