http://www.mathematik.uni-r.de/loeh/talk.pdf

RANK GRADIENT, COST, AND SIMPLICIAL VOLUME

I GROUPS II MANIFOLDS

Context: residually finite view $\longleftrightarrow$ dynamical view

I GROUPS

RESIDUALLY FINITE VIEW

Def. (rank gradient [Lackenby]). Let Γ be fi. gen. res. fin. group.

- If Γ_* is a residual chain of Γ, then

min # of gens of Γ_n — Nielsen-Schreier

$$rg(\Gamma,\Gamma_*) := \inf_{n\in\mathbb{N}} \frac{d(\Gamma_n) - 1}{[\Gamma:\Gamma_n]} \in \mathbb{R}_{\geq 0}.$$

- The <u>rank gradient</u> of Γ is

$$rg\,\Gamma := \inf_{\substack{\Lambda \triangleleft \Gamma \\ \text{fi.}}} \frac{d(\Lambda)-1}{[\Gamma:\Lambda]} \in \mathbb{R}_{\geq 0}.$$

Question.
- How to compute?
- Dependence on the residual chain?

Basic Examples:
- $rg\, F_r = r - 1$
- $rg\, \pi_1(\Sigma_g) = 2g - 2$
- $rg(\Gamma \times \Lambda) = 0$ if $|\Gamma| = \infty = |\Lambda|$.

• $\operatorname{rg} \Gamma = 0$ if Γ is amenable, $|\Gamma| = \infty$

[Pappas] • $\operatorname{rg}(\Lambda_1 \underset{A}{*} \Lambda_2, \Gamma_*) = \operatorname{rg}(\Lambda_1, \Lambda_1 \cap \Gamma_*) + \operatorname{rg}(\Lambda_2, \Lambda_2 \cap \Gamma_*) + \frac{\Lambda}{|A|}$.

amenable

proof uses ergodic theory!

DYNAMICAL VIEW: idea: groups $\rightsquigarrow$ dynamical systems, orbit relations

Def. (cost [Levitt]). Let Γ be a fin gen group.

- Let $\Gamma \curvearrowright (X,\mu)$ be an ess free pmp action on a standard Borel prob. space. Then

$$\operatorname{cost}(\Gamma \curvearrowright (X,\mu)) := \operatorname{cost}(\mathcal{R}_{\Gamma \curvearrowright X}, \mu)$$

orbit relation: $\{(x, \gamma \cdot x) \mid x \in X, \gamma \in \Gamma\}$

$$\operatorname{cost}(\mathcal{R},\mu) := \inf_{\Phi \text{ s.t. } \langle \Phi \rangle = \mathcal{R}} \operatorname{cost} \Phi \qquad \operatorname{cost}\Phi := \sum_{i \in I} \mu(A_i)$$

$\Phi = (\varphi_i : A_i \xrightarrow[\mu]{\cong} B_i)_{i \in I}$

$\forall_{x \in A_i}: \varphi(x) \underset{\mathcal{R}}{\sim} x$

$\langle \Phi \rangle$ = smallest eq rel on X containing $\{(x, \varphi_i(x)) \mid i \in I, x \in A_i\}$

- The *cost* of Γ is

$$\operatorname{cost} \Gamma := \inf_{\Gamma \curvearrowright (X,\mu) \text{ as above}} \operatorname{cost}(\Gamma \curvearrowright (X,\mu)) \in \mathbb{R}_{\geq 0}$$

Questions: How to compute? Dependence on the action?

fixed price problem

Theorem. [Abért, Nikolov]. Let Γ be a f.g. res. fin. infinite group. Then

$$\mathrm{rg}(\Gamma, \Gamma_*) = \mathrm{cost}(\Gamma \curvearrowright \widehat{\Gamma}_*) - 1.$$

Application: fixed price problem and the rk vs Heegaard genus conj are incompatible.

II MANIFOLDS

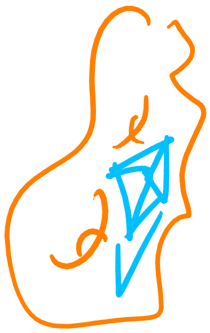
RESIDUALLY FINITE VIEW

$\mathbb{Z}, \mathbb{R}$

Def. (simplicial volume [Gromov]). Let R be a normed ring and let M be an occ n-mfd.

$\in R$ $\quad \Delta^n \xrightarrow{\text{cont.}} M$

• Then



$$\|M\|_R := \inf\Big\{ \sum_j |a_j| \;\Big|\; \sum_j a_j \cdot \sigma_j \in C_n(M;R) \text{ R-fund. cycle of } M \Big\} \in \mathbb{R}_{\geq 0}$$

homotopy invariant

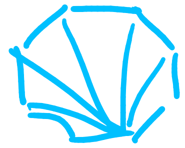
• The stable integral simplicial volume of M

$$b_j^{(2)}(M) \leq \|M\|_{\mathbb{Z}}^{\infty} := \inf_{\substack{p: N \to M \\ \text{fin. cov.}}} \frac{\|N\|_{\mathbb{Z}}}{|\deg p|} \in \mathbb{R}_{\geq 0}.$$

Basic examples: • $\|S^1\|_{\mathbb{Z}}^{\infty} = 0$

[FLPS] more gen: $\|M\|_{\mathbb{Z}}^{\infty} = 0$ if $\pi_1(M)$ amenable and $\widetilde{M} \simeq *$.

[Gromov] • $\|\Sigma_g\|^\infty_{\mathbb{Z}} = 4g-4$

$\leq$: 

$\geq$: hyp geom.

[FLPS] • $\|M\|^\infty_{\mathbb{Z}} = \dfrac{\mathrm{vol}\, M}{v_3}$ if M occ hyp 3-mfd

v_3: vol ideal regular 3-simplex in $\mathbb{H}^3$

uses ergodic theory

DYNAMICAL VIEW

Def. (integral foliated simplicial volume [Gromov, Schmidt])

Let M be an occ n-mfd with $\Gamma := \pi_1(M)$,

- let $\Gamma \curvearrowright (X,\mu)$ be a pmp action on a standard Borel prob space. Then

$$\|M\|^{\Gamma \curvearrowright (X,\mu)} := \|M\|^{\Gamma \curvearrowright}_{L^\infty(X;\mathbb{Z})}$$

twisted coeff's.

- The <u>integral foliated simplicial volume</u> of M is

$$\|M\|_{\mathcal{F},\mathbb{Z}} := \inf_{\Gamma \curvearrowright (X,\mu)} \|M\|^{\Gamma \curvearrowright (X,\mu)}.$$

Theorem [L, Pagliantini] Let M be an occ mfd with res fin. fund. group Γ. Then

$$\|M\|^\infty_{\mathbb{Z}} = \|M\|^{\Gamma \curvearrowright \widehat{\Gamma}}.$$

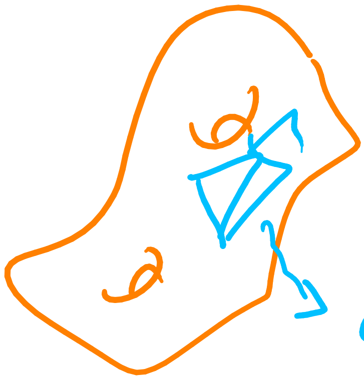
$\rightsquigarrow$ Can we ergodic theory to compute $\|M\|^\infty_{\mathbb{Z}}$?

Theorem [L] Let M be an occ mfd.

① If $\pi_1(M)$ is res. fin, then

$$\operatorname{rg} \pi_1(M) \le \|M\|^\infty_\mathbb{Z}.$$

inf over fin. cov.



$d(\pi_1(M)) \le \|M\|_\mathbb{Z}$

② $$\operatorname{cost}(\pi_1(M)) - 1 \le \|M\|_{\mathcal{F},\mathbb{Z}}.$$

(dynamical version of ①)

In general: very non-sharp:

$$\operatorname{rg} \pi_1(\Sigma_2 \times \Sigma_2) = 0 < \|\Sigma_2 \times \Sigma_2\|^\infty_\mathbb{Z}.$$

$$\operatorname{cost} \Gamma - 1 \le \operatorname{rg} \Gamma$$

$<$ if M hyp 4-mfd

$$\|M\|_\mathbb{R} \le \|M\|_{\mathcal{F},\mathbb{Z}} \le \|M\|^\infty_\mathbb{Z} \le \|M\|_\mathbb{Z}$$