https://loeh.app.ur.de/talk.pdf

28.11.2022

L^2-Betti Numbers and Computability of Reals

(joint work with Matthias Uschold)

I Computability of invariants in topology/group theory

II L^2-Betti numbers

III Computability of L^2-Betti number values

I

algebraic/geometric topology: homotopy invariants

finite simplicial complex

$$\text{top. spaces} \xrightarrow{I} \text{real numbers}$$

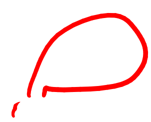
X

$$X \simeq Y \implies I(X) = I(Y)$$

examples:

- Euler characteristic: $\chi(X) = \sum_{j \in \mathbb{N}} (-1)^j \cdot \#\, j\text{-simplices in } X$
- Betti numbers: $b_k(X) = \dim_{\mathbb{R}} H_k(X;\mathbb{R})$

$$\chi(X) = \sum_{j \in \mathbb{N}} (-1)^j \cdot b_j(X)$$



- (triviality of) the fundamental group $\pi_1(X)$

for groups: group $\longrightarrow$ real numbers

$$\Gamma \longmapsto \underbrace{B\Gamma}_{\text{top. space}} \longmapsto I(B\Gamma)$$

Question: Is I computable?

for finite simplicial complexes:

- χ is computable ✓
- b_k is computable ✓
- triviality of π_1: undecidable!

Meta observation: Many (non-computable) invariants have some traces of computability, at least for the values that appear.

Concrete examples:

mfd topology
- degree sets (recursively enumerable)
- simplicial volume (values: right-computable reals)

group thy
- rank gradient "
- stable commutator length "

of fin gen recursively pres. groups

Why? compatibility of values

$\Downarrow$

restrictions on the possible values

$\Downarrow$

non-realisability results.

II L^2-Betti Numbers

usual Betti numbers: $b_k(X) = \dim_{\mathbb{R}} H_k(X;\mathbb{R})$

idea of L^2-Betti numbers:

equivariant version of Betti numbers

classical case: if $V \subset \mathbb{R}^n$ is an $\mathbb{R}$-subspace, then

$$\dim_{\mathbb{R}} V = \underbrace{\mathrm{tr}\,(\text{orth. proj. onto } V)}_{\text{in } L^2\text{-case: take this as def.}}$$

$\sum_{\gamma \in \Gamma} a_\gamma \gamma$, $a_\gamma \in \mathbb{C}$

group ring

Definition. Let Γ be a group.

- Let $A \in M_{n \times n}(\mathbb{C}\Gamma)$. Then

$$\mathrm{tr}_{\mathbb{C}\Gamma} A := \sum_{j=1}^{n} e\text{-coeff of } A_{jj} \quad \in \mathbb{C}$$

- Let $A \in M_{m \times n}(\mathbb{C}\Gamma)$. $\leadsto R_A^{(2)} : (\ell^2\Gamma)^m \to (\ell^2\Gamma)^n$, $x \mapsto x \cdot A$

Then

$$\circledast \quad b^{(2)}(A;\Gamma) := \mathrm{tr}_{\mathbb{C}\Gamma}\left(\text{orth. proj. onto } \ker R_A^{(2)}\right) \in \mathbb{R}_{\geq 0}$$

For groups: $b_k^{(2)}(\Gamma) := b_k^{(2)}(B\Gamma)$ $\in \mathbb{R}_{\geq 0}$.
(of finite type)

Examples: · $b_0^{(2)}(\Gamma) = \dfrac{1}{|\Gamma|}$.

($\leadsto$ $b_0^{(2)}$(groups) is not computable from fin. presentations of groups)

· $b_k^{(2)}(\Gamma) = 0$ if $k \geq 1$ and Γ is abelian

· $b_1^{(2)}$(free group of rk 2) $= 1$

· $\chi(B\Gamma) = \sum_{j \in \mathbb{N}} (-1)^j \cdot b_j^{(2)}(\Gamma)$

Theorem [Grabowski] Every elt of $\mathbb{R}_{\geq 0}$ arises in the form $b^{(2)}(A;\Gamma)$, where Γ is a fin gen group and $A \in M_{m \times n}(\mathbb{Z}\Gamma)$.

III

Theorem (A) [L-Uschold; gen. Groth] Let Γ be fin. gen. group with word problem of Turing degree $\leq a$. Let $m,n \in \mathbb{N}$, $A \in M_{m\times n}(\mathbb{Z}\Gamma)$. Then:

1. $b^{(2)}(A;\Gamma) \in \mathbb{R}$ is a-right computable
2. If $m=n$ and A is self-adjoint and of det-class, then $b^{(2)}(A;\Gamma) \in \mathbb{R}$ is a-computable.

Sketch proof of 1:

- Proposition [Lück] Let $A \in M_{m\times n}(\mathbb{Z}\Gamma)$, $K \geq \|R_A^{(2)}\|$. Then

$$\left(c(A,K)_p := \operatorname{tr}_{\mathbb{C}\Gamma}\left(\left(1 - \frac{1}{K^2}\cdot A\cdot A^*\right)^p\right)\right)_{p\in\mathbb{N}}$$

is mon. decreasing and

$$b^{(2)}(A;\Gamma) = \lim_{p\to\infty} c(A,K)_p.$$

Pf: uses spectral measure

- from A: compute a rough $K \geq \|R_A^{(2)}\|$.
- for $p \in \mathbb{N}$: compute

uses word problem!

$$c(A,K)_p = \operatorname{tr}_{\mathbb{C}\Gamma}\underbrace{\left(1 - \frac{1}{K^2}\cdot AA^*\right)^p}_{\text{comp. from } A}. \quad \square$$

Theorem (B) [L-Uschold; gen Pichot-Schick-Žuk]
Let a be an enumerable Turing degree. Then the set of L^2-Betti numbers arising from fin gen groups of det-class with word problem of degree $\leq a$ is $EC_a \cap \mathbb{R}_{\geq 0}$.

Theorem (C) [L-Uschold; L^2-version of Nabutovsky-Weinberger]
Let a be an enumerable Turing degree. Then: There ex. an algo of deg $\leq a+4$ that:

- given a fin gen gp Γ, a fin gen set S, an algo for the word problem of Γ wrt S of deg $\leq a$, and given $k \in \mathbb{N}$,

- computes the binary expansion of $b_k^{(2)}(\Gamma)$ (or detects that $b_k^{(2)}(\Gamma) = \infty$).