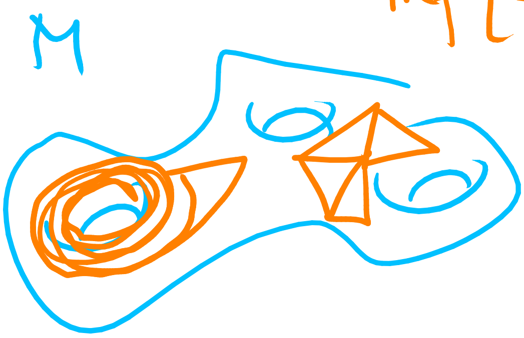
# SIMPLICIAL VOLUME: VALUES & APPLICATIONS

**Definition** [Gromov] Let $M$ be an oriented compact $n$-mfd. The <u>simplicial volume</u> of $M$ is defined as

$$\|M,\partial M\| := \|[M,\partial M]\|_1 \in \mathbb{R}_{\geq 0}$$

$$= \inf\Big\{\sum_j |a_j| \;\Big|\; \sum_j a_j \sigma_j \in C_n(M,\partial M;\mathbb{R}) \text{ is a (rel.) } \mathbb{R}\text{-fund. cycle of } (M,\partial M)\Big\}.$$




**Prop.** (degree estimate) Let $f: M \to N$ be a map of occ $n$-mfds. Then

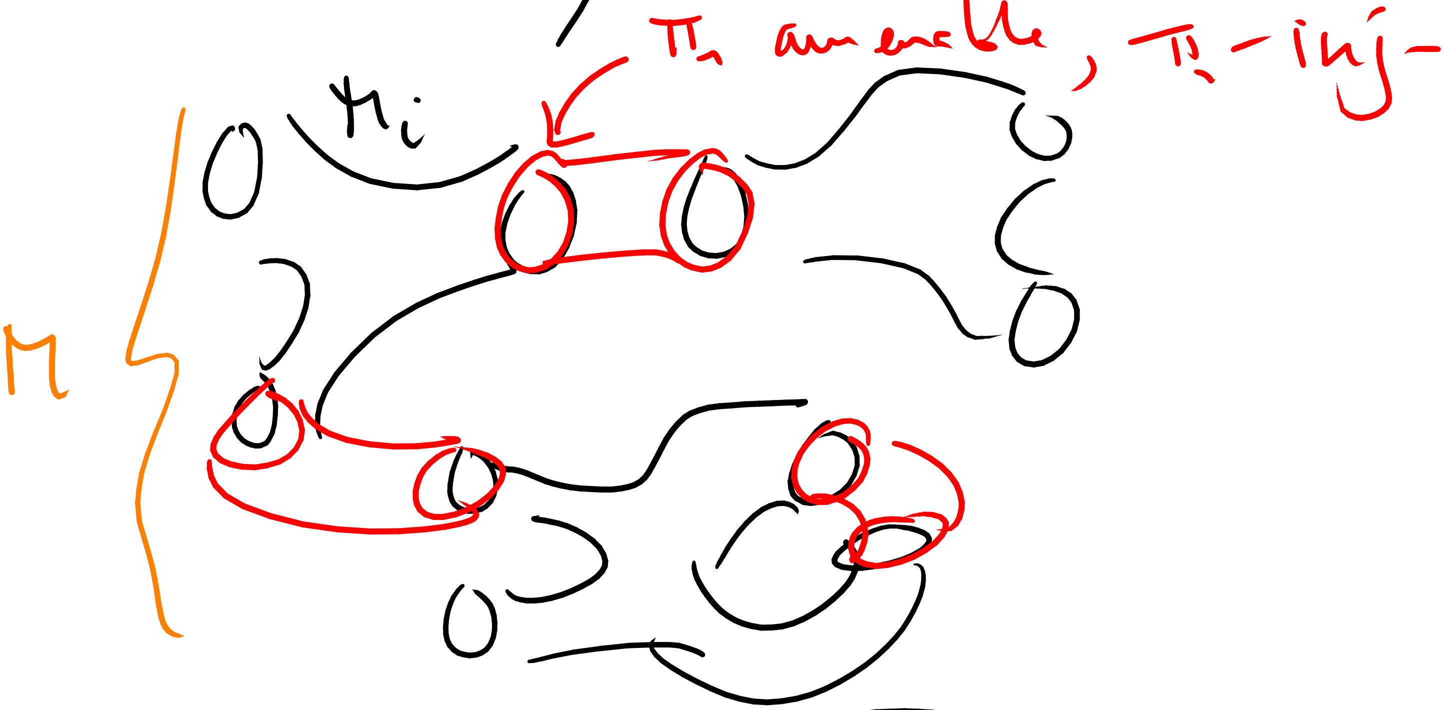
$$\|M\| \geq |\deg f| \cdot \|N\|.$$

$\leadsto$ · simvol is a homotopy inv.

· $|\deg f| \leq \dfrac{\|M\|}{\|N\|} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$

**Example.** $\|S^n\| = 0$ if $n \geq 1$.

Theorem (additivity [Gromov, BBFIPP])




Then: $\|M, \partial M\| = \sum_i \|M_i, \partial M_i\|$.

$\leadsto$ · applies to JSJ-decomp.

· monoid hom

$$\left( {}^{\text{occ}}_{n\text{-mfds}} / \text{homeo}, \# \right) \to (\mathbb{R}_{\geq 0}, +)$$

if $n \geq 3$.

VALUES:

Some known examples:

1. If $\pi_1(M)$ is amenable, then $\|M\| = 0$ (and dim $> 0$) [Gromov, Ivanov]

2. If $M$ admits a hyp. Riem. metric, then
$$\|M\| = \frac{\operatorname{vol}(M)}{v_{\dim M}} \neq 0$$
[Gromov, Thurston]

2a. If $M$ is a closed loc. symm. space of non-compact type, then
$$\|M\| > 0$$
[Lafont, Schmidt]

2b. If $M$ is rat. essential and $\pi_1(M)$ is non-elem. hyp., then
$$\|M\| > 0$$
[Mineyev].

3. $$\|\Sigma_g \times \Sigma_h\| = \frac{3}{2} \cdot \|\Sigma_g\| \cdot \|\Sigma_h\|$$
[Bucher]

$\leadsto$ degree thms for targets of type 2, 2a, 2b, 3.
$$|\deg f| \leq \frac{\|M\|}{\|N\|} \quad \|N\| > 0.$$

What can be said about

$SV(n) := \{\|M\| \mid M \text{ occ } n\text{-mfd}\} \subset \mathbb{R}_{\geq 0}$

countable set

- $SV(2) = 4 \cdot \mathbb{N}$ gap at 0, discrete
- $SV(3)$ gap at 0, not discrete

Theorem [Heuer, L]

1. For all $n \geq 4$: $SV(n)$ is dense in $\mathbb{R}_{\geq 0}$.

2. There ex. a seq. in $SV(4)$ that is lin. indep. over $\mathbb{Q}$.

Idea of proof. (for $n=4$)

- Relate scl to $\|\cdot\|_1$ on $H_2(\cdot;\mathbb{R})$
- Construct mfds from $\alpha \times [\Sigma_2]$
- Use known computations for scl. $\square$

1. $\rightsquigarrow$ (occ $n$-mfds/homeo, #) is not fin. gen. if $n \geq 3$.

# AN APPLICATION TO COBORDISM CATEGORIES

jt w/ M. MORASCHINI, G. RAPTIS

**Definition.** Let $G$ be a class of groups that is closed under isos - let $n \in \mathbb{N}$. Then $\mathrm{Cob}_n^G$ is the following category:

- objects: occ $(n-1)$-mfds s.t. all components have $\pi_1$ in $G$.
- morphisms: (or) cobordisms with $\pi_1$-inj. boundaries


- composition: gluing of cobordisms

additivity $\Longrightarrow$ $\|\cdot\|$ is a functor

$$\mathrm{Cob}_n^G \longrightarrow \mathbb{R}$$

mor: $(W; M, N) \longmapsto \|W, \partial W\|$

if $G \subset$ amenable groups

$$\pi_1(B\mathrm{Cob}_n^G, \emptyset) \cong \pi_0(\Omega_\emptyset B\, \mathrm{Cob}_n^G)$$

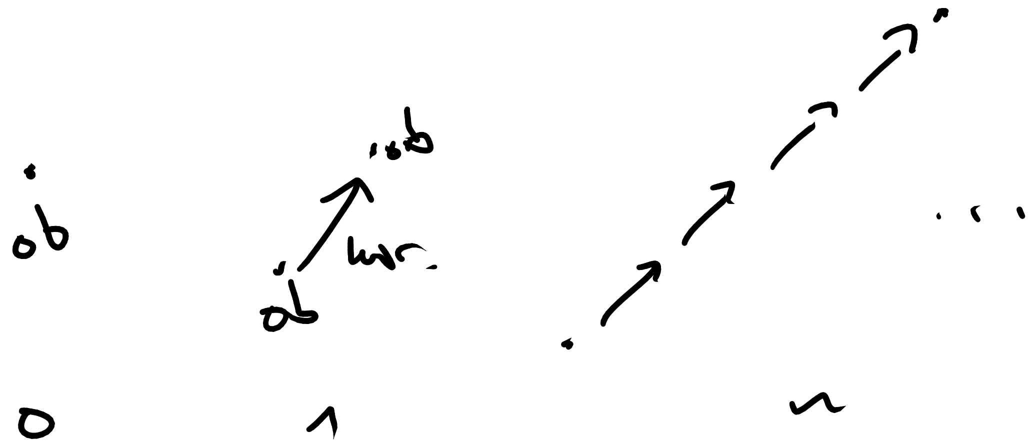
$$\begin{array}{ccc} \pi_1(B\mathrm{Cob}_n^G, \emptyset) & \cong & \pi_0(\Omega_\emptyset B\, \mathrm{Cob}_n^G) \\ \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \|\cdot\|} \\ \mathbb{R} & \cong & \pi_0(\Omega B\, \mathbb{R}) \end{array}$$

$\rightsquigarrow$ im $\varphi_4 \supset SV(4)$ $\leftarrow$ has $\infty$ many $\mathbb{Q}$-lin. indep. elts.

$\rightsquigarrow$ im $\varphi_4$ not fin. gen.

$BC$ : geometric realisation of

cat.




Theorem. [Moraschini, Raptis, L]

$\pi_1(B\mathrm{Cob}_4^G, \emptyset)$ is not fin. gen. if $G \subset \mathrm{Am}$. □