The ergodic decomposition theorem

Clara Löh   clara.loeh@uni-muenster.de   12.07.2006

Abstract. In the study of equivariant phenomena in algebraic topology, it is often convenient to enlarge the considered G-spaces by crossing with a probability space carrying a measure preserving G-action. This gives one motivation to study the category of measure preserving G-actions. One fundamental result is the *ergodic decomposition theorem* stating that any measurable G-action can be decomposed (via integration) into simpler pieces – namely into ergodic G-spaces. Therefore, for many applications it is sufficient to consider only ergodic G-spaces, which are tractable by certain tools, like Birkhoff's ergodic theorem or the Rohlin lemma. The goal of this talk is to give a brief introduction into ergodic G-spaces and to provide an overview of the proof of the ergodic decomposition theorem (following Varadarajan's paper [7]).

1 Dreaming of decompositions

In the study of equivariant phenomena in algebraic topology, it is often convenient to enlarge the considered G-spaces by crossing with a parameter space, i.e., a probability space carrying a measure preserving G-action. The product space with the diagonal G-action provides more space for certain constructions, e.g., for equivariant covers with controlled multiplicity [2, 5; p. 300f, Theorem 4.1]. A related application is the study of the integral foliated simplicial volume [2, 6].

Representation theory	*Standard actions*
representations on finite vector spaces	measure preserving actions on probability spaces
irreducible representations	ergodic actions
direct sum decompositions	integral decompositions
complete reducibility	ergodic decomposition

Figure 1: Comparing representation theory and standard actions

Of course, like topological spaces measure spaces (and actions on them) can be quite wild. However, most measure spaces occurring in the realm of equivariant algebraic topology turn out to belong to a particularly nice category of measure spaces, the category of standard Borel spaces. Therefore, in the following only actions on standard Borel spaces will be considered:

Definition (1.1).

- A **standard Borel space** is a measurable space isomorphic (as measurable space) to a Polish space (i.e., a separable, complete metrisable topological space) equipped with its Borel σ-algebra.
- Let G be a countable discrete group. A **standard G-space** is a standard Borel space X with an action of G on X by measurable automorphisms.
- A measure μ on a standard space X is called **G-invariant**, if

  $$\mu(A) = \mu(g \cdot A)$$

  holds for all $g \in G$ and all measurable sets $A \subset X$.
- A **standard G-probability space** is a standard G-space together with a G-invariant probability measure. $\diamond$

In order to gain a better understanding of standard actions of groups, we first have to identify the basic building blocks of standard group actions – the so-called ergodic actions. As a guideline we look at the "divide and conquer" approach in the representation theory of (compact) groups: There, a representation is called irreducible, if it contains no proper subrepresentations. The corresponding definition in the context of standard group actions looks as follows:

Definition (1.2). Let X be a standard G-space. A G-invariant measure μ on X is called **G-ergodic**, if $\mu(A) \in \{0,1\}$ holds for all G-invariant measurable subsets A of X. $\diamond$

(Martin [4] gives an explanation of the meaning and the evolution of the artificial word "ergodic.")

Finite dimensional representations of compact groups are accessible, because they satisfy complete reducibility, i.e., they can (essentially uniquely) be decomposed as direct sums of irreducible representations. Hence the study of representations reduces to the study of irreducible representations.

The obvious question whether a similar decomposition also exists for standard G-actions can be answered affirmatively. However, the decomposition does in general not involve only finitely many ergodic spaces, but needs an integration process:

Theorem (1.3) (Ergodic decomposition theorem). *Let G be a countable discrete group and let X be a standard G-space. If there exists at least one G-invariant probability measure on X, then the set $E^G(X)$ of ergodic probability measures on X is not empty and there exists a map $\beta\colon X \longrightarrow E^G(X)$ with the following properties:*

1. *If $A \subset X$ is measurable, then the map*
$$\begin{aligned} X &\longrightarrow [0,1] \\ x &\longmapsto \beta_x(A) \end{aligned}$$
is measurable.

2. *For all measurable subsets A and all G-invariant probability measures μ on X, we have*
$$\mu(A) = \int_X \beta_x(A)\, d\mu(x).$$

These notes are organised as follows: In Section 2, we collect some basic material concerning ergodic actions (including examples) and present a functional anayltic point of view on the theory of ergodic measures. Using the insights of Section 2, in Section 3, we sketch Varadarajan's proof [7] of the ergodic decomposition theorem.

2 A closer look at ergodic actions

In this section, we collect some basic material concerning ergodic actions. We start with examples, give a brief discussion of the Birkhoff ergodic theorem and the Rohlin lemma, and then proceed to a functional analytic point of view needed for the proof of the ergodic decomposition theorem.

2.1 Examples of ergodic actions

To get some intuition of ergodic actions, we look at two simple examples:

Example (2.1). For $\alpha \in S^1 \subset \mathbf{C}$ the corresponding $\mathbf{Z}$-action on S^1 by rotation
$$\begin{aligned} \mathbf{Z} \times S^1 &\longrightarrow S^1 \\ (n,z) &\longmapsto \alpha^n \cdot z. \end{aligned}$$
is ergodic if and only if α is not a root of unity [8; Theorem 1.8]. ■

Example (2.2). For a countable discrete group G we can consider the shift action on the product $X := \{0,1\}^G$, the so-called **Bernoulli shift** associated to G. Then the G-fold product of the equidistribution on $\{0,1\}$ is an ergodic measure of the G-space X if and only if G is infinite [6; Lemma 3.37]. ■

2.2 Tools for ergodic actions

Ergodic actions might seem wild from the topological point of view (for example, rotation by an irrational angle around S^1 does not lead to a nice quotient space). However, ergodicity turns out to give rise to nice measure theoretic properties:

Theorem (2.3) (Birkhoff ergodic theorem). *Let X be a standard $\mathbf{Z}$-space, let $f\colon X \longrightarrow \mathbf{R}$ be a bounded measurable function and let μ be a $\mathbf{Z}$-invariant probability measure on X. Let T be a generator of $\mathbf{Z}$. Then the limit*

$$f^* := \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{j=0}^{n-1} f \circ T^j$$

exists μ-almost everywhere and the equality $f^* \circ T = f^*$ holds μ-almost everywhere. Moreover, the function f^* is μ-integrable and

$$\int_X f^* \, d\mu = \int_X f \, d\mu.$$

The ergodic theorem can be interpreted as stating that the time mean (i.e., f^*) and the space mean (i.e., $\int_X f \ d\mu$) of all f coincide μ-almost everywhere if and only if μ is $\mathbf{Z}$-ergodic. A proof of the ergodic theorem can be found for example in Walters's book [8; Theorem 1.14].

Another nice property of $\mathbf{Z}$-ergodic spaces is the Rohlin lemma [1; Theorem 1.5.9], which allows certain "division operations" by partitioning the measure space [6; Proposition 5.30].

Theorem (2.4) (Rohlin lemma). *Let X be a standard $\mathbf{Z}$-space and let μ be a $\mathbf{Z}$-ergodic measure on X without atoms. Let T be a generator of $\mathbf{Z}$. Then for any $\varepsilon \in \mathbf{R}_{>0}$ and any $n \in \mathbf{N}$ there is a measurable set $A \subset X$ such that $A, T \cdot A, \dots, T^n \cdot A$ are pairwise disjoint and*

$$\mu\Bigl(\bigcup_{j=0}^{n} T^j \cdot A\Bigr) > 1 - \varepsilon.$$

2.3 Convex geometry

The proof of the ergodic decomposition theorem relies on functional analytic properties of the space of finite signed measures on a given G-space and the fact that the set of G-invariant probability measures is a compact convex subset of this space. In this functional analytic language, the ergodic measures are the extreme points of this compact convex subset (Proposition (2.9)). Before we study the geometry of those spaces of measures, we first introduce the neccessary notions from convex geometry:

Definition (2.5). Let C be a convex subset of an $\mathbf{R}$-vector space V. A point $x \in C$ is an **extreme point of** C, if there is no representation $x = t \cdot y + (1-t) \cdot z$ with $y \neq z \in C$ and $t \in (0,1)$. $\diamond$

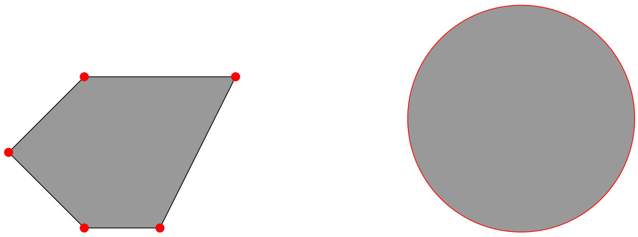

Figure 2: Extreme points (red) of convex sets

Example (2.6). The extreme points of polytopes are exactly the vertices. The set of extreme points of the closed disk D^n is the sphere S^{n-1} (cf. Figure 2). ■

As the examples suggest, the points of any compact convex subset of a finite dimensional real vector space can be obtained as (finite) convex combinations of extreme points. In other words: compact convex subsets of finite dimensional vector spaces are the convex hull of their extreme points. A slightly weaker statement generalises to all dimensions [3; Theorem IV 1.5]:

Theorem (2.7) (Krein, Milman). *Let C be a compact convex subset of a locally convex topological vector space V. Then C is the smallest closed convex subset of V containing all extreme points of C.*

In particular, if C is not empty, then C has at least one extreme point.

2.4 A geometric interpretation of ergodicity

With the language of convex geometry at hand, it is possible to give a geometric interpretation of ergodic measures (Proposition (2.9)).

Definition (2.8). Let X be a standard G-space.

- The set of G-invariant probability measures on X is denoted by $M^G(X)$.
- We write $E^G(X)$ for the set of G-ergodic probability measures on X.
- We write $M(X)$ for the $\mathbf{R}$-vector space of all finite signed measures on X, i.e. (by the Hahn decomposition theorem)

$$M(X) = \operatorname{span}_{\mathbf{R}}\{\mu \mid \mu \text{ is a probability measure on } X\}.$$

- The set of all bounded measurable functions $X \longrightarrow \mathbf{R}$ is denoted by $L^\infty(X, \mathbf{R})$. (Caveat: In this space, we do *not* identify functions that coincide on certain null sets).
- If X is a topological space, we write $C(X, \mathbf{R})$ for the continuous functions of type $X \longrightarrow \mathbf{R}$. ◇

Proposition (2.9) (Ergodic measures as extreme points). *Let X be a standard G-space. Then the set $M^G(X)$ of invariant probability measures is a convex subset of $M(X)$ and the extreme points of $M^G(X)$ are precisely the ergodic probability measures of X.*

Proof. Obviously, the set $M^G(X)$ is convex.

Extreme points are ergodic measures: Let $\mu \in M^G(X)$ be an extreme point of $M^G(X)$ in $M(X)$ and *assume* μ is not ergodic. Then there is a G-invariant measurable subset A with $0 < \mu(A) < 1$. But then

$$\mu = \mu(A) \cdot \frac{1}{\mu(A)} \mu|_A + \bigl(1 - \mu(A)\bigr) \cdot \frac{1}{1 - \mu(A)} \mu|_{X \setminus A}$$

is a non-trivial convex combination of invariant probability measures, contradicting the extremality of μ. Hence μ must be ergodic.

Ergodic measures are extreme points: Conversely, let $\mu \in E^G(X)$. Let

$$\mu = t \cdot \nu + (1 - t) \cdot \nu'$$

be a convex combination with $t \in (0,1)$ and $\nu, \nu' \in M^G(X)$. It suffices to show $\nu = \nu'$.

Since $t \neq 0$ and $1 - t \neq 0$, we have $\nu \ll \mu$ and $\nu' \ll \mu$. Therefore, the Radon-Nikodym derivatives

$$f := \frac{d\nu}{d\mu} \qquad \text{and} \qquad f' := \frac{d\nu'}{d\mu}$$

exist. Because ν and ν' are G-invariant, also the functions f and f' are G-invariant. We now consider the measurable sets

$$\begin{aligned} A_+ &:= \bigl\{ x \in X \bigm| f(x) > f'(x) \bigr\}, \\ A_- &:= \bigl\{ x \in X \bigm| f(x) < f'(x) \bigr\}. \end{aligned}$$

Since both f and f' are G-invariant, the sets A_+ and A_- are also G-invariant. Thus – the measure μ is ergodic – we obtain $\mu(A_+) \in \{0,1\}$ and $\mu(A_-) \in \{0,1\}$. Moreover, A_+ and A_- are clearly disjoint, i.e., at most one of them can have μ-measure equal to 1.

Assume $\mu(A_+) = 1$ (and hence $\mu(X \setminus A_+) = 0$). Then, by construction of A_+,

$$\begin{aligned} 1 = \nu(X) &= \int_X f \, d\mu = \int_{A_+} f \, d\mu \\ &> \int_{A_+} f' \, d\mu = \int_{A_+} f' \, d\mu = \int_X f' \, d\mu = \nu'(X) \\ &= 1, \end{aligned}$$

which cannot hold. Therefore, $\mu(A_+) = 0$ and similarly, $\mu(A_-) = 0$. In particular, $f = d\nu/d\mu$ and $f' = d\nu'/d\mu$ coincide μ-almost everywhere. But then the measures ν and ν' must be equal. ☐

If X is a nice space, then this point of view and the Krein-Milman theorem can be used to show the existence of ergodic measures. However, to apply the Krein-Milman theorem we need a locally convex structure on $M(X)$. The following description of $M(X)$ in terms of continuous functionals on $C(X, \mathbf{R})$ provides a hint how to define a topology on $M(X)$:

Theorem (2.10) (Riesz representation theorem). *Let X be a compact Hausdorff space. Then the map*

$$\begin{aligned} M_{\text{reg}}(X) &\longrightarrow C(X, \mathbf{R})^* \\ \mu &\longmapsto \Bigl(f \mapsto \int_X f \, d\mu\Bigr) \end{aligned}$$

is a bijection, where $C(X, \mathbf{R})^*$ denotes the space of all continuous functionals $C(X, \mathbf{R}) \longrightarrow \mathbf{R}$ (with respect to the supremum norm on $C(X, \mathbf{R})$), the space X is endowed with the Borel σ-algebra, and $M_{\text{reg}}(X)$ is the space of regular signed finite measures on X.

Moreover, the image under this map of the set of all regular probability measures on X is the set of continuous functionals $\varphi \in C(X, \mathbf{R})^*$ with $\varphi(1) = 1$.

A proof is, for example, given in Lang's book [3; Section IX 2]

Carrying the weak*-topology on $C(X, \mathbf{R})^*$ over to the space of finite signed measures leads to the following definition of a topology on $M(X)$ if X is a compact metric space:

Definition (2.11).

1. Let X be a compact metric space. The **weak*-topology on** $M(X)$ is the smallest topology making the maps

$$\begin{aligned} M(X) &\longrightarrow \mathbf{R} \\ \mu &\longmapsto \int_X f \, d\mu \end{aligned}$$

continuous for all $f \in C(X, \mathbf{R})$.

2. If X is a measurable space, then we equip $M(X)$ with the smallest σ-algebra making the maps

$$\begin{aligned} M(X) &\longrightarrow \mathbf{R} \\ \mu &\longmapsto \mu(A) \end{aligned}$$

measurable for all measurable sets $A \subset X$. $\diamond$

If X is a compact metric space, then the σ-algebra on $M(X)$ in the above definition is the Borel σ-algebra corresponding to the weak*-topology on $M(X)$.

Corollary (2.12) (Existence of ergodic measures). *Let X be a compact metric G-space. If X admits a regular G-invariant probability measure, then X admits also a G-ergodic probability measure.*

Proof. Since X is a compact metric G-space, the set $M^G(X)$ is a weak*-closed compact subset of the locally convex topological vector space $M(X)$.

Now the corollary is a direct consequence of Proposition (2.9) and the Krein-Milman theorem (Theorem (2.7)). □

Using the theorem of Choquet (an integral version of the Krein-Milman theorem), it is possible to prove a slightly weaker ergodic decomposition, namely:

Theorem (2.13) (Ergodic decomposition – weaker version). *Let X be a compact metric G-space and let $\mu \in M^G(X)$. Then there exists a measure σ_μ on $E^G(X)$ such that*

$$\mu(A) = \int_X \eta(A)\, d\sigma_\mu(\eta)$$

holds for all measurable subsets $A \subset X$.

3 The ergodic decomposition theorem

In this section, we give sketch of Varadarajan's proof of the ergodic decomposition theorem. In the first subsection, we introduce some notation needed for a concise formulation of the ergodic decomposition theorem. We then present the notion of uniform conditional expectations, which lie at the heart of Varadarajan's proof. In the last subsection, we conclude with the sketch proof of the decomposition theorem.

3.1 Ergodic decompositions

As a first step, we provide a precise definition of what we mean by a decomposition of a G-space:

Definition (3.1). Let X be a standard G-space with $M^G(X) \neq 0$. A **decomposition map of** X is a measurable map $\beta \colon X \longrightarrow E^G(X)$ satisfying the following properties:

1. The map β is G-invariant, i.e., for all $x \in X$ and all $g \in G$ we have
$$\beta_{g \cdot x} = \beta_x.$$
2. For all $\eta \in E^G(X)$, the set
$$X_\eta := \beta^{-1}(\eta) = \{x \in X \mid \beta_x = \eta\}$$
is a measurable subset of X and $\eta(X_\eta) = 1$.
3. For all $\mu \in M^G(X)$ and all measurable subsets $A \subset X$, the integral formula
$$\mu(A) = \int_X \beta_x(A)\, d\mu(x)$$
holds. ◇

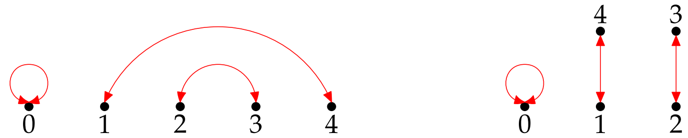

Figure 3: Ergodic decomposition of a $\mathbf{Z}/2$-action

If β is a decomposition map of X, then the supports $(X_\eta)_{\eta\in E^G(X)}$ are uniquely ergodic G-spaces, i.e., the sets $M^G(X_\eta)$ and $E^G(X_\eta)$ contain only exactly one element [7; p. 204].

In terms of decomposition maps, the ergodic decomposition theorem can be phrased as follows (a slightly more general version can be found in Varadarajan's paper [7; Theorem 4.2]):

Theorem (3.2) (Ergodic decomposition theorem). *Let G be a countable discrete group and let X be a standard G-space admitting at least one G-invariant probability measure. Then also the set $E^G(X)$ of ergodic probability measures on X is not empty and there exists an essentially unique decomposition map of X. In particular, there exists a measurable map*

$$\beta\colon X \longrightarrow E^G(X)$$

such that

$$\mu(A) = \int_X \beta_x(A)\, d\mu(x)$$

holds for all $\mu \in M^G(X)$ and all measurable subsets $A \subset X$.

Example (3.3). The $\mathbf{Z}/2$-action on $\mathbf{Z}/5$ given by

$$\begin{aligned} \mathbf{Z}/5 &\longrightarrow \mathbf{Z}/5 \\ [x] &\longmapsto [-x] \end{aligned}$$

equipped with the equidistribution has the following ergodic decomposition (see also Figure 3): The sets "X_η" are the sets $\{0\}$, $\{1,4\}$, $\{2,3\}$ and the corresponding $\mathbf{Z}/2$-ergodic measures are the equidistribution measures. ■

Example (3.4). Let $n \in \mathbf{N}$ and let

$$\alpha := e^{2\pi\cdot i/n},$$

i.e., α is an n-th root of unity. The ergodic decomposition of the $\mathbf{Z}$-action

$$\begin{aligned} \mathbf{Z} \times S^1 &\longrightarrow S^1 \\ (m,z) &\longmapsto \alpha^m \cdot z \end{aligned}$$

looks as follows:

For $z \in S^1$ let

$$X_{\beta_z} := z \cdot \{1, \alpha, \dots, \alpha^{n-1}\}$$

and let β_z be the equidistribution on X_{β_z}. Then the map $z \mapsto \beta_z$ is the ergodic decomposition of the above $\mathbf{Z}$-action on S^1. ■

Example (3.5). Let $\alpha \in S^1 \subset \mathbf{C}$ be no root of unity. Then the ergodic decomposition of the rotation action

$$\begin{aligned} \mathbf{Z} \times S^1 \times [0,1] &\longrightarrow S^1 \times [0,1] \\ (n,z,t) &\longmapsto (\alpha^n \cdot z, t) \end{aligned}$$

on the cylinder (with the Borel σ-algebra) is given by Fubini's theorem: Namely, the sets "X_η" are the fibres $S^1 \times \{t\}$ and the ergodic measure on each fibre is the (normalised) Lebesgue measure. ■

As the examples clearly show, in general, the null sets of the ergodic parts do not have much in common with the null sets of the decomposed invariant measure. Hence, some care has to be taken when using the ergodic decomposition theorem for reduction of general problems to the simpler case of ergodic actions.

3.2 Uniform conditional expectations

When integral formulas as in the ergodic decomposition theorem need to be satisfied it is not too far-fetched to look at conditional expectations. However, since the integral formulas should hold for *all* invariant measures, we need a special kind of conditional expectations:

Definition (3.6). Let X be a standard G-space. A **uniform conditional expectation of** X is a map

$$U \colon L^\infty(X,\mathbf{R}) \longrightarrow L^\infty(X,\mathbf{R})$$

such that for each $\mu \in M^G(X)$ and all $f \in L^\infty(X,\mathbf{R})$ the map Uf is a version of the conditional expectation $E_\mu(f \mid B^G(X))$, i.e., Uf is $B^G(X)$-measurable and

$$\forall_{A \in B^G(X)} \quad \int_A f \, d\mu = \int_A Uf \, d\mu. \qquad \diamond$$

Proposition (3.7) (Properties of conditional expectations). *Let (X,μ) be a probability space, let $f \in L^\infty(X,\mathbf{R})$ and let B be a σ-subalgebra of the σ-algebra of X. Then the following properties hold:*

1. *If Uf and $U'f$ both are versions of the conditional expectation $E_\mu(f \mid B')$, then Uf and $U'f$ coincide μ-almost everywhere.*

2. *If $f \in L^\infty(X,\mathbf{R})$ is a constant function and Uf is a version of the conditional expectation $E_\mu(f \mid B')$, then Uf and f coincide μ-almost everywhere.*

3. *If $f \in L^\infty(X,\mathbf{R})$ and Uf is a version of the conditional expectation $E_\mu(f \mid B')$, then*

$$\|Uf\|_\infty \leq \|f\|_\infty .$$

Like ordinary conditional expectations, in the ergodic universe even uniform conditional expectations exist and are essentially unique:

Theorem (3.8) (Existence and uniqueness of uniform conditional expectations). *Let X be a standard G-space. Then there exists a uniform conditional expectation U of X. If U' is another uniform conditional expectation of X, then for all $\mu \in M^G(X)$ and for all $f \in L^\infty(X, \mathbf{R})$, the functions Uf and $U'f$ coincide μ-almost everywhere.*

Proof. The uniqueness part follows directly from the uniqueness of ordinary conditional expectations.

The existence of ordinary conditional expectations is a consequence of the Radon-Nikodym theorem. However, this theorem is not constructive enough to yield existence for uniform conditional expectations.

To prove existence, Varadarajan [7; Theorem 4.1] provides a cunning inductive (on the "size" of the group) construction of a uniform conditional expectation, where the induction start "**Z**" is taken care of by the Birkhoff ergodic theorem (Theorem (2.3)). □

3.3 Proving the ergodic decomposition theorem

The idea behind Varadarajan's proof of the ergodic decomposition theorem (Theorem (3.2)) is to establish a relation between decomposition maps and uniform conditional expectations. Then the uniqueness and existence of uniform conditional expectations translates into the uniqueness and existence of decomposition maps.

Proof (of the ergodic decomposition theorem). We first prove the *essential uniqueness* of decomposition maps: If $\beta \colon X \longrightarrow E^G(X)$ is a decomposition map, then

$$\begin{aligned} U \colon L^\infty(X, \mathbf{R}) &\longrightarrow L^\infty(X, \mathbf{R}) \\ f &\longmapsto \Bigl(x \mapsto \int_X f \, d\beta_x \Bigr) \end{aligned}$$

is a uniform conditional expectation [7; Lemma 4.3].

Let $\beta' \colon X \longrightarrow E(X)$ be another decomposition map. Since the σ-algebra of X is countably generated (X is a standard Borel space), the uniqueness (in the sense of Theorem (3.8)) of uniform conditional expectations shows that there is a measurable set $N \subset X$ with $\mu(N) = 0$ for all $\mu \in M^G(X)$ and

$$\beta|_{X \setminus N} = \beta'|_{X \setminus N}.$$

This proves the uniqueness statement.

It remains to show *existence* of decomposition maps: By Theorem (3.8), there is a uniform conditional expectation $U \colon L^\infty(X, \mathbf{R}) \longrightarrow L^\infty(X, \mathbf{R})$. In order to obtain the measures $(\beta_x)_{x \in X}$ out of U we first construct certain continuous linear functionals

$$t_x \colon C(X, \mathbf{R}) \longrightarrow \mathbf{R}$$

and then apply the Riesz representation theorem (Theorem (2.10)) to $(t_x)_{x \in X}$:

However, in order to apply the Riesz representation theorem, we first need to replace X be a suitable compact metric G-space. This is possible, because the category of standard G-spaces is tame enough [7; p. 208f].

Construction of the functionals $(t_x)_{x \in X}$: Since X is a compact metric space, there is a countable dense subset C of $C(X, \mathbf{R})$ with

1. The constant function 1 is in C.
2. For all $a, a' \in \mathbf{Q}$ and all $f, f' \in C$ also the linear combination $a \cdot f + a' \cdot f'$ is an element of C.
3. For all $g \in G$ and all $f \in C$ also the left translation
$$L_g(f) := \bigl(x \mapsto f(g^{-1} \cdot x)\bigr)$$
lies in C.

Since (uniform) conditional expectations are essentially unique (Theorem (3.8)), the (measurable) set

$$\begin{aligned} N := &\bigcup_{a,a' \in \mathbf{Q}} \bigcup_{f,f' \in C} \{x \in X \mid a \cdot (Uf)(x) + a' \cdot (Uf')(x) \neq U(a \cdot f + a' \cdot f')(x)\} \\ &\cup \bigcup_{g \in G} \bigcup_{f \in C} \{x \in X \mid (Uf)(x) \neq U(L_g f)(x)\} \end{aligned}$$

is G-invariant and satisfies $\mu(N) = 0$ for all $\mu \in M^G(X)$.

Let $x \in X \setminus N$. Then the map

$$\begin{aligned} C &\longrightarrow \mathbf{R} \\ f &\longmapsto (Uf)(x) \end{aligned}$$

has a unique extension to a continuous linear functional

$$t_x \colon C(X, \mathbf{R}) \longrightarrow \mathbf{R},$$

because $\|Uf\|_\infty \leq \|f\|_\infty$ holds for all $f \in L^\infty(X, \mathbf{R})$. Clearly, this functional t_x satisfies $\|t_x\| \leq 1$ and $t_x(1) = 1$. Moreover, we have

$$\begin{aligned} t_x(L_g f) &= t_x(f), \\ t_{g \cdot x}(f) &= t_x(f) \end{aligned}$$

for all $g \in G$ and all $f \in C(X, \mathbf{R})$.

It is possible to enlarge N to a measurable set N' satisfying $\mu(N) = 0$ for all $\mu \in M^G(X)$ such that $t_x \colon C(X, \mathbf{R}) \longrightarrow \mathbf{R}$ is a positive functional for all $x \in X \setminus N'$ [7; p. 209f].

By the Riesz representation theorem (Theorem (2.10)), for any $x \in X \setminus N'$ we can find a probability measure β_x on X satisfying

$$t_x(f) = \int_X f \, d\beta_x$$

for all $f \in C(X, \mathbf{R})$.

Using separability arguments it is now quite straightforward to check that the measures $(\beta_x)_{x \in X \setminus N'}$ are G-invariant, that $\beta_{g \cdot x} = \beta_x$ and that they give rise to an integral formula [7; p. 210].

However, to see that $M^G(X)$-almost all $x \in X \setminus N'$ are mapped to *ergodic* measures one needs a subtle Krein-Milman type argument to verify the following [7; Lemma 4.5]: If $A \subset X$ is measurable, then

$$\forall_{\eta \in E^G(X)} \quad \eta(A) = 0 \Longrightarrow \forall_{\mu \in M^G(X)} \quad \mu(A) = 0. \qquad \square$$

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