Functorial semi-norms in homology and geometry

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The name of the game

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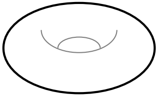

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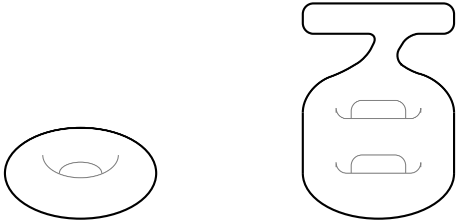

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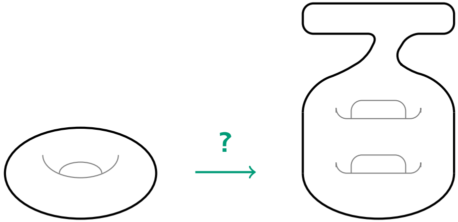

Problem

Describe the set of all continuous maps between two given manifolds of the same dimension!

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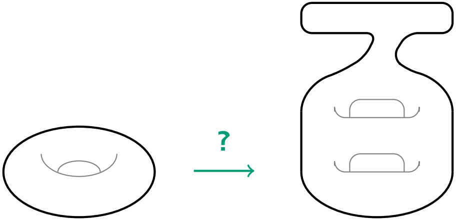

Problem

Describe the set of all continuous maps between two given manifolds of the same dimension!

Problem

Describe the set of all possible mapping degrees for maps between two given manifolds of the same dimension!

The name of the game

$$\text{functorial semi-norms} \quad \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \quad \text{degree theorems}$$

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$$\text{functorial semi-norms} \rightleftarrows \text{degree theorems}$$

Problem A (Gromov)

Show that “most interesting” even-dimensional manifolds cannot be dominated by a product of surfaces!

Problem B (Gromov)

Are all functorial semi-norms on singular homology trivial
on all simply connected spaces?

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...Both problems can be solved by suitable translations into algebra.

Overview

Functorial semi-norms $\longrightarrow$ degree theorems

Degree theorms $\longrightarrow$ functorial semi-norms

Solving Problem A: Negatively curved groups

Solving Problem B: Inflexible manifolds

Functorial semi-norms in singular homology

Definition (functorial semi-norm in homology; Gromov)

Let $d \in \mathbb{N}$. A functorial semi-norm on $H_d(\,\cdot\,;\mathbb{R})$

- consists of a choice of a (possibly infinite) semi-norm $|\cdot|$ on $H_d(X;\mathbb{R})$ for all topological spaces X that is compatible with continuous maps:

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- consists of a choice of a (possibly infinite) semi-norm $|\cdot|$ on $H_d(X;\mathbb{R})$ for all topological spaces X that is compatible with continuous maps:
- For every continuous map $f\colon X \longrightarrow Y$ and all classes $\alpha \in H_d(X;\mathbb{R})$ we have

$$\bigl|H_d(f;\mathbb{R})(\alpha)\bigr| \leq |\alpha|.$$

Why are functorial semi-norms interesting?

Proposition (functorial semi-norms $\longrightarrow$ degree theorems)

Let $d \in \mathbb{N}$, and let $|\cdot|$ be a functorial semi-norm on $H_d(\,\cdot\,;\mathbb{R})$.

- *If $f\colon M \longrightarrow N$ is a map between oriented closed connected d-manifolds, then*

$$|\deg f| \cdot \bigl|[N]_\mathbb{R}\bigr| = \bigl|H_d(f;\mathbb{R})([M]_\mathbb{R})\bigr| \leq \bigl|[M]_\mathbb{R}\bigr|.$$

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- *In particular: In this situation, if $|[N]_{\mathbb{R}}| \neq 0$, then*

$$|\deg f| \leq \frac{\bigl|[M]_{\mathbb{R}}\bigr|}{\bigl|[N]_{\mathbb{R}}\bigr|}.$$

The ℓ^1-semi-norm

Example (Gromov)

Let $d \in \mathbb{N}$.

- If $c = \sum_{\sigma \in S_d(X)} a_\sigma \cdot \sigma$ is a singular chain, then $|c|_1 := \sum_{\sigma \in S_d(X)} |a_\sigma|$.
- If $\alpha \in H_d(X;\mathbb{R})$, then the ℓ^1-semi-norm of α is defined by

$$\|\alpha\|_1 := \inf\bigl\{|c|_1 \bigm| c \in C_d(X;\mathbb{R}) \text{ is a cycle representing } \alpha\bigr\} \in \mathbb{R}_{\geq 0}.$$

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Definition (simplicial volume; Gromov)

The simplicial volume of an oriented closed connected M is defined by

$$\|M\| := \bigl\| [M]_\mathbb{R} \bigr\|_1.$$

By the above, non-vanishing results for the simplicial volume lead to degree theorems.

The ℓ^1-semi-norm, examples

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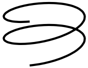
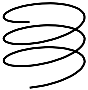
$1 \cdot$  $\frac{1}{2} \cdot$  $\frac{1}{3} \cdot$  $\dots$

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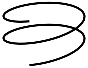
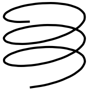
  $1 \cdot$  $\frac{1}{2} \cdot$  $\frac{1}{3} \cdot$  $\dots$

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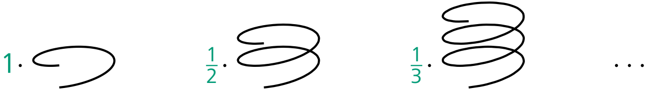

- Tori. Similarly, $\|(S^1)^d\| = 0$.
- Simply connected spaces. If X is simply connected and $\alpha \in H_d(X;\mathbb{R})$, then (!)

$$\|\alpha\|_1 = 0.$$

  More generally, the same holds also for spaces with amenable fundamental group. [Gromov]

The ℓ^1-semi-norm, examples

- Hyperbolic manifolds. If M is an oriented closed connected hyperbolic d-manifold, then
$$\|M\| = \frac{\operatorname{vol} M}{v_d} > 0,$$
where v_d is the supremum of volumes of geodesic d-simplices in hyperbolic d-space. [Gromov, Thurston]

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- Compact locally symmetric spaces of non-compact type.
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- Generalisations to the non-compact case [Löh, Sauer]

The surface semi-norm

Example (Gromov)

If $\alpha \in H_{2d}(X;\mathbb{R})$, then the surface semi-norm of α is

$$\|\alpha\|_S := \inf\Bigl\{ \sum_{j=1}^{k} |a_j| \cdot |\chi(S_j)| \Bigm| k \in \mathbb{N},\ a_1, \dots, a_k \in \mathbb{R} \setminus \{0\},$$

$$S_1, \dots, S_k \text{ are products of oriented closed connected surfaces,}$$

$$f_1 \colon S_1 \to X, \dots, f_k \colon S_k \to X \text{ continuous}$$

$$\text{with } \sum_{j=1}^{k} a_j \cdot H_d(f_j;\mathbb{R})[S_j]_{\mathbb{R}} = \alpha \Bigr\}.$$

Functorial semi-norms on manifolds

Definition (functorial semi-norm on manifolds)

Let $d \in \mathbb{N}$, let Mfd_d be the class of oriented closed connected d-manifolds, and let $S \subset \mathrm{Mfd}_d$.

- A functorial semi-norm on S is a map $v \colon S \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that:
- For all continuous maps $f \colon M \longrightarrow N$ with $M, N \in S$ we have

$$|\deg f| \cdot v(N) \leq v(M).$$

Functorial semi-norms out of mapping degrees

Proposition (Crowley, Löh)

Let $d \in \mathbb{N}$, and let $v \colon S \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a functorial semi-norm on a class $S \subset \mathrm{Mfd}_d$.

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$$|\alpha|_v := \inf\biggl\{ \sum_{j=1}^{k} |a_j| \cdot v(M_j) \biggm| \begin{array}{l} k \in \mathbb{N},\ a_j \in \mathbb{R} \setminus \{0\}, M_j \in S, \\ f_j \colon M_j \to X \text{ continuous} \\ \text{with } \displaystyle\sum_{j=1}^{k} a_j \cdot H_d(f_j;\mathbb{R})[M_j]_{\mathbb{R}} = \alpha \end{array} \biggr\};$$

- *Then $|\cdot|_v$ is a functorial semi-norm on $H_d(\,\cdot\,;\mathbb{R})$, the functorial semi-norm associated with v.*

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- *For all $M \in S$ we have $|[M]_\mathbb{R}|_v = v(M)$.*

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- *Then $|\cdot|_v$ is a functorial semi-norm on $H_d(\,\cdot\,;\mathbb{R})$, the functorial semi-norm associated with v.*
- *For all $M \in S$ we have $\bigl|[M]_{\mathbb{R}}\bigr|_v = v(M)$.*
- *If v is finite and $S = \mathrm{Mfd}_d$, then also $|\cdot|_v$ is finite. [Thom]*

Functorial semi-norms out of mapping degrees

Example (ℓ^1-semi-norm, revisited)

If $d \in \mathbb{N} \setminus \{3\}$, then the functorial semi-norm on $H_d(\,\cdot\,;\mathbb{R})$ associated with the simplicial volume is the ℓ^1-semi-norm.

Functorial semi-norms out of mapping degrees

Example (ℓ^1-semi-norm, revisited)

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Example (surface semi-norm, revisited)

The Euler characteristic is functorial on the class of all products of oriented closed connected surfaces of genus at least 1 [Gromov].
The induced semi-norm on $H_*(\,\cdot\,;\mathbb{R})$ is the surface semi-norm.

Solving Problem A

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Idea

Translate this problem into a question about fundamental groups.

Presentability by products

Definition

An oriented closed connected manifold M is presentable by a product if there exist oriented closed connected manifolds S_1, S_2 of *non-zero dimension* that admit a map $S_1 \times S_2 \longrightarrow M$ of *non-zero degree*.

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Definition

A group Γ is presentable by a product if there exist groups Γ_1, Γ_2 that admit a homomorphism $\varphi \colon \Gamma_1 \times \Gamma_2 \longrightarrow \Gamma$ that has *finite index image* and such that $\varphi(\Gamma_1)$ and $\varphi(\Gamma_2)$ are *infinite*.

Manifolds/groups not presentable by products

Theorem (Kotschick, Löh)

Let M be a rationally essential oriented closed connected manifold (i.e., the classifying map $M \longrightarrow B\pi_1(M)$ sends $[M]_\mathbb{R}$ to a non-zero class). If $\pi_1(M)$ is not presentable by a product, then M is not presentable by a product.

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Proof.

- Proceed by contraposition.
- Classifying spaces (and maps between them) are compatible (up to homotopy) with products.
- Homomorphisms induced by maps of non-zero degree have finite index image.
- Rational homology of finite groups is concentrated in degree 0. □

Groups not presentable by products

Theorem (Kotschick, Löh; based on results from geometric group theory)

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- More geometrically: Commuting subgroups of infinite order form a too big “flat” part of the group.

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- More geometrically: Commuting subgroups of infinite order form a too big "flat" part of the group.
- However, in all the above classes of groups, there are "many" elements of infinite order that have "small" centralisers. □

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Theorem (Kotschick, Löh; based on results from geometric group theory)

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- *Mapping class groups of oriented closed surfaces of genus at least 2.*
- $\mathrm{Aut}(F_n)$ *and* $\mathrm{Out}(F_n)$ *are for* $n \in \mathbb{N}_{>1}$ *not presentable by products.*
- *...*

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Manifolds not presentable by products

Corollary (Kotschick, Löh)

Oriented closed connected hyperbolic manifolds (of dimension at least 3*) are not presentable by a product (of surfaces).*

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Theorem (Kotschick, Löh)

Oriented closed connected irreducible locally symmetric spaces of non-compact type (of dimension at least 3*) are not presentable by a product (of surfaces).*

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Definition (inflexible manifold)

An oriented closed connected manifold M is inflexible if

$$\bigl\{\deg f \bigm| f\colon M \longrightarrow M \text{ continuous}\bigr\} \subset \{-1, 0, 1\}.$$

Inflexibility and functorial semi-norms

Proposition (Crowley, Löh)

Let N be an oriented closed connected inflexible d-manifold. Then there is a functorial semi-norm $|\cdot|$ on $H_d(\,\cdot\,;\mathbb{R})$ with $\bigl|[N]_\mathbb{R}\bigr| = 100 \notin \{0,\infty\}$.

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- The map

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is a functorial semi-norm on Mfd_d.

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  is a functorial semi-norm on Mfd_d.
- Because N is inflexible we have $v(N) = 100 \cdot 1$.
- Now the functorial semi-norm on $H_d(\,\cdot\,;\mathbb{R})$ associated with v has the desired properties. $\square$

Simply connected inflexible manifolds?

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- Low dimensions. Oriented closed simply connected manifolds of dimension at most 6 are *not* inflexible. [Shiga]

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- Hyperbolic manifolds are inflexible (simplicial volume!), but *not* simply connected (when oriented, closed, connected).

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Idea

Use rational homotopy theory.

Rational homotopy theory in a nutshell

Topology	$\longrightarrow$	Topology$_{\mathbb{Q}}$	$\longleftrightarrow$	Algebra
simply connected spaces		rational spaces		dga's
continuous maps				dga morphisms

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Problem B$''$

Find an inflexible dga that is the model of a simply connected manifold!

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continuous maps				dga morphisms

Problem B''

Find an inflexible dga that is the model of a simply connected manifold!

By the work of Sullivan and Barge this is a purely algebraic problem.

Simply connected inflexible manifolds!

Example (Crowley, Löh; based on work by Arkowitz, Lupton)

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- There is an oriented closed simply connected manifold with minimal model A. [Barge, Sullivan]
- The dga A is inflexible.

A strange functorial semi-norm in singular homology

Corollary (Crowley, Löh)

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- In fact, we construct infinitely many examples of simply connected inflexible manifolds in each of infinitely many dimensions.
- More recently, further examples have been constructed by Costoya and Viruel, as well as by Amann.