Normal forms for letterless fragments of GLP

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In the following we will investigate the letterless sentences of the polymodal calculus GLP and a certain subset S of the letterless sentences. For both of them we will derive a normal form theorem. The normal forms for S are related to the Cantor normal forms for ordinals below ε_0. Furthermore, the normal forms for the whole fragment will yield an algorithm to decide whether a letterless sentence is provable in GLP or not. Our presentation is based on [Boolos, Chapter 16], [Ignatiev, Section 10] and [Beklemishev, Section 3].

1 The name of the game – The calculus GLP

Definition. The **language** of the polymodal calculus GLP is the language of the system GL, i.e. propositional variables $p, q, \ldots$, boolean connectives $\rightarrow, \perp$, and the modal operators $[0]$ (or $\Box$), $[1], \ldots$. We use the standard abbreviations for $\wedge, \vee, \neg, \leftrightarrow$ and

$$\forall_{n \in \mathbf{N}} \quad \langle n \rangle := \neg [n] \neg.$$

The **axioms** of GLP are all tautologies and all sentences of the following form (for all $n \in \mathbf{N}$):

$$\begin{aligned}
&[n](A \rightarrow B) \rightarrow ([n]A \rightarrow [n]B)\\
&[n]([n]A \rightarrow A) \rightarrow [n]A\\
&[n]A \rightarrow [n+1]A\\
&\langle n \rangle A \rightarrow [n+1]\langle n \rangle A.
\end{aligned}$$

The **rules of inference** are modus ponens and $[0]$-necessitation, i.e.

$$\mathrm{GLB} \vdash A \implies \mathrm{GLB} \vdash [0]A.$$

Remark. Equivalently, the **axioms** of GLP can be given as in [Beklemishev]: Axioms are all boolean tautologies and for every $n, m \in \mathbf{N}$

$$\begin{aligned}
&\langle n\rangle(A \vee B) \to (\langle n\rangle A \vee \langle n\rangle B) \\
&\neg\langle n\rangle\neg\top \\
&\langle n\rangle A \to \langle n\rangle(A \wedge \neg\langle n\rangle A) \\
&\langle n\rangle A \to \langle m\rangle A \text{ if } m \leq n \\
&\langle m\rangle A \to [n]\langle m\rangle A \text{ if } m < n.
\end{aligned}$$

The **rules of inference** are then modus ponens and (for every $n \in \mathbf{N}$)

$$A \to B \vdash \langle n\rangle A \to \langle n\rangle B.$$

Remark. As with GL we have for every $n \in \mathbf{N}$

$$\mathrm{GLB} \vdash [n]A \to [n][n]A$$

Proof. The proofs are completely analoguous to the proof for GL and are described in [Boolos]. □

Theorem. *The system* GLP *is sound and complete with respect to provability in* PA. *The associated truth system* GLSP *is sound and complete with respect to truth in the standard model* $\mathbf{N}$.

Proof. Proofs can be found in [Boolos, chapter 15] (only for the bimodal subsystem GLB) and in [Ignatiev]. □

In this talk our main attention is on letterless sentences:

Definition. A modal sentence (in the language of GLP) is called **letterless** if it contains no propositional variables.

For example $\top$ is letterless, but the equivalent sentence $p \to p$ is not.

Definition. Every ordinal $\alpha > 0$ can be written in **Cantor normal form**, that is

$$\begin{aligned}
\alpha &= \omega^{\lambda_1} + \cdots + \omega^{\lambda_n} \\
&=: \langle \lambda_1, \ldots, \lambda_n\rangle_\omega,
\end{aligned}$$

where $\lambda_1, \ldots, \lambda_n$ are ordinals with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

2 A provability logical view of ε_0

This section is based on Section 3 of [Beklemishev]. We introduce a certain subset S of the letterless sentences of GLP and will prove a normal form theorem for S – these normal forms correspond to Cantor normal forms for ordinals $< \varepsilon_0$.

Definition. For $n \in \mathbf{N}$, let S_n denote the set of all words over the alphabet $\mathbf{N}_{\geq n}$, including the empty word Λ, and let $S := S_0$. With an element $\alpha = n_k \cdots n_1 \in S$ we associate its **modal interpretation**, that is

$$\langle n_k \rangle \cdots \langle 1 \rangle \top$$

(where $\Lambda = \top$.)

In the following we will not distinguish between a word in S and its modal interpretation.

Notation. Below we use $\vdash$ for provability in GLP. For all $\alpha, \beta \in S$ we write

$$\alpha \sim \beta :\Longleftrightarrow \vdash \alpha \to \beta$$

and $\alpha = \beta$ denotes graphical identity.

Definition. For each $n \in \mathbf{N}$ we define $<_n \subset S \times S$ via

$$\forall_{\alpha,\beta \in S} \quad \alpha <_n \beta :\Longleftrightarrow \vdash \beta \to \langle n \rangle \alpha.$$

Lemma 1. *This relation $<_n$ is an ordering on S, i.e. it is transitive and irreflexive.*

Proof. Transitivity. Let $\alpha, \beta, \gamma \in S$ with $\alpha <_n \beta$ and $\beta <_n \gamma$. Thus we have

$$\begin{aligned} \vdash \gamma &\to \langle n \rangle \beta, \\ &\to \langle n \rangle \langle n \rangle \alpha, \\ &\to \langle n \rangle \alpha. \end{aligned}$$

Irreflexivity. Obviously it is sufficient to prove irreflexivity for $<_0$. *Assume* $\alpha \in S$ and $\alpha <_0 \alpha$, i.e. $\vdash \alpha \to \langle 0 \rangle \alpha$. Thus, by contraposition,

$$\vdash [0]\neg\alpha \to \neg\alpha.$$

Box-necessitation yields

$$\vdash [0]([0]\neg\alpha \to \neg\alpha)$$

and on the other hand we have – by Löb's axiom –

$$\vdash [0]([0]\neg\alpha \to \neg\alpha) \to [0]\neg\alpha.$$

Putting it all together we get $\vdash \neg\alpha$. Since $\alpha \in S$, every arithmetical realisation of $\neg\alpha$ is false, thus $\nvdash \neg\alpha$, which is a contradiction. $\square$

2.1 Normal forms for S

Some of the words in S are equivalent, so we define a subset of S of normal forms: For that we use induction over the width of words in S.

Definition. Let $\alpha \in S$. Then the **width** of α, denoted by $w(\alpha)$, is the number of different letters occuring in α.

Definition. The set $\mathrm{NF} \subset S$ of **normal forms** is defined inductively by:

- Any word of width ≤ 1 belongs to NF
- Assume $\alpha \in S$ and $w(\alpha) > 1$. Then we can write (if n is the smallest letter in α)

  $$\alpha = \alpha_k n \cdots n \alpha_1,$$

  where $\alpha_1, \ldots, \alpha_k \in S_{n+1}$ (therefore their width is less then the width of α). Then $\alpha \in \mathrm{NF}$ if and only if the words $\alpha_1, \ldots, \alpha_k$ are in normal form and

  $$\forall_{j \in \{1,\ldots,k-1\}} \quad \alpha_{j+1} \not>_{n+1} \alpha_j.$$

So $\top, \langle 1\rangle\top, \langle 1\rangle\langle 1\rangle\top, \langle 1\rangle\langle 2\rangle\langle 1\rangle\langle 2\rangle\top$ are in normal form, but $\langle 4\rangle\langle 2\rangle\top$ and $\langle 1\rangle\langle 1\rangle\langle 0\rangle\langle 1\rangle\top$ are not.
Our next goal is to prove the corresponding normal form theorem for S:

Theorem (Normal form theorem). *Every word $\alpha \in S$ can be brought into a* GLP*-equivalent normal form, that is, there exists an $\alpha' \in \mathrm{NF}$ such that*

$$\alpha' \sim \alpha$$

and α' is effectively computable.

The following lemma is needed in the proof and also later very helpful.

Lemma 2. *(i) Let $m, n \in \mathbf{N}$ with $m < n$ and ϕ, ψ modal sentences in the language of* GLP *. Then*

$$\vdash \langle n\rangle\phi \wedge \langle m\rangle\psi \leftrightarrow \langle n\rangle(\phi \wedge \langle m\rangle\psi).$$

(ii) If $\alpha \in S_{m+1}$ and $\beta \in S$ then

$$\vdash \alpha \wedge m\beta \leftrightarrow \alpha m \beta.$$

Proof. (i) $\leftarrow$. Obviously we have

$$\vdash \langle n\rangle(\phi \wedge \langle m\rangle\psi) \rightarrow \langle n\rangle\phi$$

and

$$\begin{aligned} \vdash \langle n\rangle(\phi \wedge \langle m\rangle\psi) &\rightarrow \langle n\rangle\langle m\rangle\psi, \\ &\rightarrow \langle m\rangle\langle m\rangle\psi, \\ &\rightarrow \langle m\rangle\psi, \end{aligned}$$

which together gives $\vdash \langle n\rangle(\phi \wedge \langle m\rangle\psi) \to \langle n\rangle\phi \wedge \langle m\rangle\psi$.
$\to$. Since $m < n$ we get (by the axioms of GLP)

$$\vdash \langle m\rangle\psi \to [n]\langle m\rangle\psi,$$

respectively

$$\vdash \neg[n]\langle m\rangle\psi \to \neg\langle m\rangle\psi.$$

Thus

$$\begin{aligned}\vdash [n](\neg(\phi \wedge \langle m\rangle\psi)) &\to [n](\langle m\rangle\psi \to \neg\phi),\\ &\to ([n]\langle m\rangle\psi \to [n]\neg\phi),\\ &\to (\langle n\rangle\phi \to \neg[n]\langle m\rangle\psi),\\ &\to (\langle n\rangle\phi \to \neg\langle m\rangle\psi),\\ &\to \neg(\langle n\rangle\phi \wedge \langle m\rangle\psi),\end{aligned}$$

which proves the first part.
(ii) We use induction on the length of α. Clearly

$$\vdash \top \wedge m\beta \leftrightarrow \Lambda m\beta.$$

Let $\alpha \in S_{m+1}$, say $\alpha = \langle n\rangle\alpha'$ with $\alpha' \in S_{m+1}$. Then

$$\begin{aligned}\vdash \alpha \wedge m\beta &\leftrightarrow \langle n\rangle\alpha' \wedge \langle m\rangle\beta,\\ &\leftrightarrow \langle n\rangle(\alpha' \wedge \langle m\rangle\beta), && \text{(by part (i))}\\ &\leftrightarrow \langle n\rangle\alpha'\langle m\rangle\beta, && \text{(by the induction hypothesis)}\\ &\leftrightarrow \alpha\langle m\rangle\beta.\end{aligned}$$

□

The main step in the proof of the normal form theorem is the following lemma. It allows us to delete the parts which do not ft into the definition of normal forms.

Lemma 3. *Let $\alpha = \alpha_k \ldots \alpha_1$ with $\alpha_1, \ldots, \alpha_k \in S_{n+1}$. If $\alpha_k >_{n+1} \alpha_{k-1}$ then*

$$\alpha \sim \alpha_k n \alpha_{k-2} n \cdots n\alpha_1.$$

Proof. Let

$$\beta := \alpha_{k-2} n \cdots n\alpha_1.$$

Since $\alpha_k, \alpha_{k-1} \in S_{n+1}$, two applications of lemma 2 yield

$$\begin{aligned}\vdash \alpha = \alpha_k n\alpha_{k-1} n\beta &\leftrightarrow \alpha_k \wedge n\alpha_{k-1} n\beta, &(1)\\ &\leftrightarrow \alpha_k \wedge \langle n\rangle(\alpha_{k-1} \wedge n\beta). &(2)\end{aligned}$$

By the first part of lemma 2 and the axioms of GLP we have

$$\begin{aligned}\vdash \langle n+1\rangle\alpha_{k-1} \wedge n\beta &\to \langle n+1\rangle(\alpha_{k-1} \wedge n\beta), &(3)\\ &\to \langle n\rangle(\alpha_{k-1} \wedge n\beta). &(4)\end{aligned}$$

Since $\alpha_k >_{n+1} \alpha_{k-1}$, it follows

$$\begin{aligned}
\vdash \alpha_k n\beta &\leftrightarrow \alpha_k \wedge n\beta, && \text{(by lemma 2)}\\
&\to \alpha_k \wedge \langle n+1\rangle\alpha_{k-1} \wedge n\beta, &&\\
&\to \alpha_k \wedge \langle n\rangle(\alpha_{k-1} \wedge n\beta), && \text{(by 4)}\\
&\to \alpha && \text{(by 2).}
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\vdash \alpha &\leftrightarrow \alpha_k \wedge \langle n\rangle(\alpha_{k-1} \wedge n\beta), && \text{(by 2)}\\
&\to \alpha_k \wedge \langle n\rangle(n\beta), &&\\
&\to \alpha_k n\beta && \text{(by lemma 2).}
\end{aligned}$$ □

Proof. (of the normal form theorem). We are proceeding by induction on $w(\alpha)$. Per definitionem of NF the case $w(\alpha) \leq 1$ is trivial. Thus we consider the case $w(\alpha) > 1$. Let $\alpha \in S$ and let n be the smallest letter in α. Then we can write

$$\alpha = \alpha_k n \cdots n\alpha_1,$$

with $\alpha_1, \ldots, \alpha_k \in S_{n+1}$. By the induction hypothesis we get normal forms $\alpha'_1, \ldots, \alpha'_k \in S_{n+1}$ with

$$\alpha_1 \sim \alpha'_1, \ldots, \alpha_k \sim \alpha'_k.$$

Then it follows that

$$\begin{aligned}
\vdash \alpha &\leftrightarrow \alpha_k \wedge (n\alpha_{k-1}n \cdots \alpha_1), && \text{(by lemma 2)}\\
&\vdots &&\\
&\leftrightarrow \alpha_k \wedge (n(\alpha_{k-1} \wedge n(\cdots \wedge n(\alpha_1)\cdots))), && \text{(repeated application of lemma 2)}\\
&\leftrightarrow \alpha'_k \wedge (n(\alpha'_{k-1} \wedge n(\cdots \wedge n(\alpha'_1)\cdots))), &&\\
&\vdots &&\\
&\leftrightarrow \alpha'_k n \cdots n\alpha'_1. &&
\end{aligned}$$

If α is not already in normal form, we find a largest integer $j \in \{1, \ldots, k-1\}$ with $\alpha'_j <_{n+1} \alpha'_{j+1}$. Then lemma 3 yields

$$\alpha'_j n\alpha'_{j-1} n \cdots n\alpha'_1 \sim \alpha'_j n\alpha'_{j-2} n \cdots n\alpha'_1,$$

and therefore

$$\alpha \sim \alpha'_k n \cdots n\alpha'_j n\alpha'_{j-2} n \cdots n\alpha'_1.$$

Thus we "deleted" the evil element α'_{j+1}. So, repeating this step we will arrive at a GLP-equivalent for α.
This proof shows, that the element α' can be effectively computed if we are able to compare normal forms with respect to $<_{n+1}$ effectively. This will be proved below by lemma 4. □

Remark. The length and width of the so constructed word α' do not exceed the length of α. □

Lemma 4. *Any two normal forms $\alpha, \beta \in S_n$ are $<_n$-comparable, that is*

$$\alpha <_n \beta \text{ or } \beta <_n \alpha \text{ or } \alpha = \beta.$$

And it can effectively decided which of these statements holds.

Proof. We reason by induction on the joint width of α and β and wlog assume $n = 0$. If $w(\alpha\beta) \leq 1$ then $\alpha = \langle 0\rangle^k\top$ and $\beta = \langle 0\rangle^m\top$ with $k, m \in \mathbf{N}$ and wlog $k < m$. Then

$$\begin{aligned} \vdash \beta &\to \langle 0\rangle^m\top, \\ &\to \langle 0\rangle\langle 0\rangle^k\top, && \text{(since } k < m) \\ &\to \langle 0\rangle\alpha, \end{aligned}$$

i.e. $\alpha <_n \beta$. Thus α and β are $<_0$-comparable.
So we consider the case that $w(\alpha\beta) > 1$. As before we write

$$\alpha = \alpha_k 0 \dots 0 \alpha_1 \text{ and } \beta = \beta_m 0 \dots 0 \beta_1,$$

where $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k$ do not contain 0. By the induction hypothesis we have (since α and β are in normal form)

$$\alpha_1 \geq_1 \alpha_2 \geq_1 \dots \geq_1 \alpha_k \text{ and } \beta_1 \geq_1 \beta_2 \geq 1 \dots \geq_1 \beta_m$$

and

$$\alpha_1 <_1 \beta_1 \text{ or } \beta_1 <_1 \alpha_1 \text{ or } \alpha_1 = \beta_1.$$

Now we are going to compare α and β backwards:
Claim. If $\alpha_1 <_1 \beta_1$ then $\alpha <_0 \beta$ and vice versa.

Proof. (of the claim). Defining

$$\forall_{j \in \{1,\dots,k\}} \quad \bar{\alpha}_j := \alpha_j 0 \dots 0 \alpha_1$$

we prove – by induction on j – that $\bar{\alpha}_j <_1 \beta_1$ for all $j \leq k$: Since $\beta_1 >_1 \alpha_1 = \bar{\alpha}_1$ we only need to look at the case $j > 1$. Since $\beta_1 >_1 \alpha_1 \geq_1 \dots \geq_1 \alpha_{j+1}$ we have $\beta_1 >_1 \alpha_{j+1}$, and by the induction hypothesis we have $\beta_1 >_1 \bar{\alpha}_j$. Hence

$$\begin{aligned} \vdash \beta &\to \langle 1\rangle\alpha_{j+1} \wedge \langle 1\rangle\bar{\alpha}_j, \\ &\to \langle 1\rangle\alpha_{j+1} \wedge \langle 0\rangle\bar{\alpha}_j, \\ &\to \langle 1\rangle(\alpha_{j+1} \wedge 0\bar{\alpha}_j), && \text{(by lemma 2)} \\ &\to \langle 1\rangle(\alpha_{j+1} 0\bar{\alpha}_j), && \text{(by lemma 2)} \\ &\to \langle 1\rangle\bar{\alpha}_{j+1}. \end{aligned}$$

Thus $\bar{\alpha}_j <_j \beta_1$ for all $j \leq k$. In particular

$$\alpha = \bar{\alpha}_k <_1 \beta_1 \leq_1 \beta,$$

which proves the claim (since $<_1$ is stronger than $<_0$). □

By the claim we are done if $\alpha_1 <_1 \beta_1$ or $\beta_1 <_1 \alpha_1$. If $\alpha_1 = \beta_1$ we have to compare α_2 and β_2 using the induction hypothesis again. Continuing in this way

- we will find a least integer j with $\alpha_j \neq \beta_j$ or
- one of the two words α, β is a proper end segment of the other.

In the second case α and β are obviously comparable by $<_0$. Thus we consider the first case: wlog we assume $\alpha_j <_1 \beta_j$. It follows that

$$\begin{aligned}
\vdash \bar{\beta}_j &\to \beta_j \wedge 0\bar{\beta}_{j-1}, && \text{(by lemma 2)}\\
&\to \beta_j \wedge 0\bar{\alpha}_{j-1}, && \text{(for } \bar{\alpha}_{j-1} = \bar{\beta}_{j-1}\text{)}\\
&\to \langle 1\rangle\alpha_j \wedge 0\bar{\alpha}_{j-1}, &&\\
&\to \langle 1\rangle(\alpha_j \wedge 0\bar{\alpha}_{j-1}), && \text{(by lemma 2)}\\
&\to \langle 1\rangle\alpha_j 0\bar{\alpha}_j. &&
\end{aligned}$$

As in the proof of the above claim we get for all $\ell \in \{j+1, \ldots, k\}$

$$\bar{\alpha}_\ell <_1 \bar{\beta}_j \leq_0 \beta$$

and therefore $\alpha <_0 \beta$.
This proof in particular shows that the comparison can be effectively (by recursion) computed. □

Corollary. *The normal form of a given word is graphically unique.*

Proof. Let $\alpha \in S$ and $\alpha', \alpha'' \in S$ normal forms with

$$\alpha \sim \alpha' \text{ and } \alpha \sim \alpha'',$$

hence $\alpha' \sim \alpha''$. Since normal forms are comparable we have wlog

$$\alpha' <_0 \alpha'' \text{ or } \alpha' = \alpha''.$$

Assume $\alpha' <_0 \alpha''$. Since $<_0$ respects $\sim$ we have $\alpha' <_0 \alpha'$, in contradiction to the irreflexivity of $<_0$. Thus $\alpha' = \alpha''$. □

Corollary. *Any two words $\alpha, \beta \in S$ satisfy the trichtomy*

$$\alpha <_n \beta \textit{ or } \beta <_n \alpha \textit{ or } \alpha \sim \beta.$$

Proof. This follows directly by considering normal forms and the fact that $<_n$ respects GLP-equivalence. □

Since the proof of lemma 4 provides an effective comparison algorithm for words in normal form we can also effectively check (by recursion) wether a given word is in normal form or not. Furthermore the last corollary yields:

Corollary. *For any $\alpha, \beta \in S_n$ either $\vdash n\alpha \to n\beta$ or $\vdash n\beta \to n\alpha$ and this can be effectively decided.*

Proof. We simply have to build the respective normal forms and to compare them. Since both is effectively computable we are done. □

2.2 From notations to ordinals

Now we are going to establish the correspondence between notations and ordinals:

Definition. For every $n \in \mathbf{N}$ we define

$$o_n : S_n \longrightarrow \varepsilon_0$$

by recursion on width and a subsidiary recursion on $\min(\alpha) - n$ where for $\alpha \in S$ we denote by $\min(\alpha)$ the minimal letter in α: Let $\alpha \in S_n$.

- If $k \in \mathbf{N}$ and $\alpha = \langle n \rangle^k \top$ then
$$o_n(\alpha) := k.$$
- If $\alpha = \alpha_k n \cdots n \alpha_1$ with $\alpha_1, \ldots, \alpha_k \in S_{n+1}$ and $\alpha_1 \cdots \alpha_k \neq \Lambda$, then
$$o_n(\alpha) := \omega^{o_{n+1}(\alpha_1)} + \cdots + \omega^{o_{n+1}(\alpha_k)}.$$

In the latter case all the $o_{n+1}(\alpha_j)$ are already defined, since

- If $k > 1$ then $w(\alpha_1) < w(\alpha), \ldots, w(\alpha_k) < w(\alpha)$.
- If $k = 1$ then $\alpha_1 = \alpha \neq \Lambda$, hence
$$\min(\alpha_1) - (n+1) < \min(\alpha) - n.$$

Definition. The function $o \colon S \longrightarrow \varepsilon_0$ is defined via

$$\forall_{\alpha \in S} \quad o(\alpha) := o_0(\alpha).$$

Then we have for example

$$\begin{aligned} o(101) &= o_0(101) = \omega^{o_1(1)} + \omega^{o_1(1)} = \omega^1 + \omega^1, \\ o_1(212) &= \omega^{o_2(2)} + \omega^{o_2(2)} = \omega^1 + \omega^1, \\ o(2101) &= o_0(2101) = \omega^{o_1(1)} + \omega^{o_1(21)} = \omega^1 + \omega^{\omega^{o_2(\Lambda)} + \omega^{o_2(2)}} \\ &= \omega + \omega^{\omega^0 + \omega^1} = \omega + \omega^{1+\omega} = \omega + \omega^\omega = \omega^\omega. \end{aligned}$$

Theorem. *Let $n \in \mathbf{N}$.*

1. *For all $\alpha, \beta \in S_n$*
$$\alpha <_n \beta \implies o_n(\alpha) < o_n(\beta).$$
2. *For all $\alpha \in S_n$*
$$\forall_{z < o_n(\alpha)} \quad \exists_{\beta <_n \alpha} \quad o_n(\beta) = z.$$
3. *The function $o_n \colon S_n \longrightarrow \varepsilon_0$ is surjective.*

*The first two items state that o_n is a p-***morphism** *from the structure $(S_n, <_n)$ to ε_0.*

Proof. 1. The algorithm of bringing a given word $\alpha \in S_n$ into normal form preserves the ordinal $o_n(\alpha)$: The algorithm merely consists of elimination steps of the form

$$\begin{aligned} \alpha &= \alpha_k n \alpha_{k-1} n \cdots n \alpha_1 \\ &\sim \alpha_k n \alpha_{k-2} n \cdots n \alpha_1, \end{aligned}$$

where $\alpha_1, \ldots, \alpha_k \in S_{n+1}$ and $\alpha_k >_{n+1} \alpha_{k+1}$ (in particular $\alpha_k \neq \Lambda$ or $\alpha_{k-1} \neq \Lambda$). Proceeding by induction on the width and $\min(\alpha) - n$ we get $o_{n+1}(\alpha_{k-1}) < o_{n+1}(\alpha_k)$ and therefore

$$\begin{aligned} o_n(\alpha) &= \omega^{o_{n+1}(\alpha_1)} + \ldots \omega^{o_{n+1}(\alpha_k - 1)} + \omega^{o_{n+1}(\alpha_k)} \\ &= \omega^{o_{n+1}(\alpha_1)} + \ldots \omega^{o_{n+1}(\alpha_k - 2)} + \omega^{o_{n+1}(\alpha_k)} \\ &= o_n(\alpha_k n \alpha_{k-2} n \cdots n \alpha_1). \end{aligned}$$

Thus o_n stays invariant under our normalising algorithm. So without loss of generality we may assume $\alpha, \beta \in \mathrm{NF} \cap S_n$.
Let $\gamma \in \mathrm{NF} \cap S_n$. Using induction on $w(\gamma)$ we show that $o_n(\gamma)$ is represented in Cantor normal form: Without loss of generality we may assume $\gamma = \gamma_k n \cdots n \gamma_1$ with $\gamma_1, \ldots, \gamma_k \in \mathrm{NF} \cap S_{n+1}$ and $\gamma_k \cdots \gamma_1 \neq \Lambda$.

- If $k = 1$ then $o_n(\gamma) = \omega^{o_{n+1}(\gamma_1)}$, where γ_1 is in normal form. Thus it follows (by a subsidiary induction on $\min(\gamma) - n$) that $o_{n+1}(\gamma_1)$ is represented in Cantor normal form. Therefore also $o_n(\gamma)$ is represented in Cantor normal form.

- If $k > 1$ then
$$o_n(\gamma) = \omega^{o_{n+1}(\gamma_1)} + \cdots + \omega^{o_{n+1}(\gamma_k)}$$
where – by the induction hypothesis – $o_{n+1}(\gamma_1), \ldots, o_{n+1}(\gamma_k)$ are represented in Cantor normal form. Since γ is in normal form and normal forms are comparable, we have
$$\forall_{j \in \{1,\ldots,k-1\}} \quad \gamma_j \geq_{n+1} \gamma_{j+1}.$$
The proof of lemma 4 shows that comparing normal forms in S corresponds to comparing Cantor normal forms. Therefore
$$\forall_{j \in \{1,\ldots,k-1\}} \quad o_{n+1}(\gamma_j) \geq o_{n+1}(\gamma_{j+1}),$$
i.e. $o_n(\gamma)$ is represented in Cantor normal form.

Therefore $o_n(\alpha)$ and $o_n(\beta)$ are represented in Cantor normal form. Since comparing normal forms is the same as comparing Cantor normal forms we obtain $o_n(\alpha) < o_n(\beta)$.
3. Let z be an ordinal in Cantor normal form. By the definition of o_n and induction on the shape of Cantor normal forms we can construct a word $\beta \in S_n$ with

$$o_n(\beta) = z.$$

2. Now let $\alpha \in S_n$ and z an ordinal less than $o_n(\alpha)$. Part 3. yields a word $\beta \in S_n$ with $o_n(\beta) = z$. We now consider the normal forms $\alpha', \beta' \in S_n$ of α and β, respectively. Then

$$o_n(\beta') = o_n(\beta) = z < o_n(\alpha) = o_n(\alpha').$$

Since comparing Cantor normal forms corresponds to comparing normal forms by $<_n$, it follows that

$$\beta \sim \beta' <_n \alpha' \sim \alpha,$$

which proves 2. $\square$

3 Normal forms for letterless sentences of GLP

In this section we are going to prove a normal form theorem for the letterless sentences of GLP . This will also lead to an algorithm to decide whether a given letterless sentence is provable in GLP or not.

Definition. A modal sentence (in the language of GLP) is called **letterless** if it contains no propositional variables.

For example $\top$ is letterless, but the equivalent sentence $p \to p$ is not.

Definition. Every ordinal $\alpha > 0$ can be written in **Cantor normal form**, that is

$$\begin{aligned} \alpha &= \omega^{\lambda_1} + \cdots + \omega^{\lambda_n} \\ &=: \langle \lambda_1, \ldots, \lambda_n \rangle_\omega, \end{aligned}$$

where $\lambda_1, \ldots, \lambda_n$ are ordinals with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We write (for every ordinal $\beta > 0$)

$$\begin{aligned} \alpha^- &:= \omega^{\lambda_1} + \cdots + \omega^{\lambda_{n-1}}, \\ d(\alpha) &:= \lambda_n, \\ \alpha^{+\beta} &:= \min\{\gamma > \alpha | d(\gamma) \geq \beta\}, \\ \alpha^{-\beta} &:= (\alpha^{+\beta})^-, \end{aligned}$$

and (if $\alpha_0, \ldots, \alpha_m$ are also ordinals)

$$\begin{aligned} B(\alpha) &:= \alpha \\ B(\alpha, \beta) &:= \alpha^{+\beta} \\ B(\alpha_0, \ldots, \alpha_m) &:= B(\alpha_0, B(\alpha_1, \ldots, \alpha_m)). \end{aligned}$$

For every modal formula A the formula $\uparrow A$ is the result of raising each $[n]$ in A to $[n+1]$.

Definition. For each ordinal α (below ε_0) we define the formula $D(\alpha)$ by induction on α:

$$\begin{aligned} D(0) &:= \bot \\ D(\alpha) &:= \Box D(\alpha^-) \vee \uparrow D(d(\alpha)). \end{aligned}$$

So, for example (where $\sim$ denotes equivalence in GLP)

$$\begin{aligned} D(1) &= \Box\bot \vee \uparrow D(0) \\ &= \Box\bot \vee \bot \\ &\sim \Box\bot \\ D(2) &\sim \Box\Box\bot \\ D(\omega) &\sim [1]\bot \\ D(\omega+1) &\sim \Box[1]\bot. \end{aligned}$$

Definition. For all ordinals $\alpha < \varepsilon_0$ we write (and $H(\infty) := \top =: \boxed{1}^{\infty}\bot$)

$$H(\alpha) := \Box D(\alpha) \sim D(\alpha+1).$$

Definition. We say that a letterless sentence A is in **normal form** if A is a (possibly empty) conjunction of formulas of the following form

$$\begin{aligned} &(H(\alpha_0) \to H(\beta_0)) \vee (\uparrow H(\alpha_1) \to \uparrow H(\beta_1)) \vee \cdots \vee (\uparrow^n H(\alpha_n) \to \uparrow^n H(\beta_n)) \\ &\vee \uparrow^{n+1} \neg H(\alpha_{n+1}) \vee \cdots \vee \uparrow^m \neg H(\alpha_m) \end{aligned}$$

where $m \in \mathbf{N}, n \in \mathbf{Z}_{\geq -1}$ (if $n = -1$, we consider the formula $\neg H(\alpha_0) \vee \uparrow \neg H(\alpha_1) \vee \cdots \vee \uparrow^m \neg H(\alpha_m)$), $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in \varepsilon_0 \dot\cup \{\infty\}$ and

$$\begin{aligned} H(\infty) &:= \top \\ d(0) := d(\infty) &:= \infty \\ \forall_{j\in\{1,\ldots,n\}} \quad & \alpha_j > \beta_j \\ \forall_{j\in\{1,\ldots,n-1\}} \quad & d(\alpha_j), d(\beta_j) \geq B(\beta_{j+1}, \ldots, \beta_{n-1}, \beta_n + 1). \end{aligned}$$

For instance, the formula $\Box[1]\bot \to [1]\bot$ has the following normal form

$$\begin{aligned} \vdash (\Box[1]\bot \to [1]\bot) &\leftrightarrow (\Box[1]\bot \to \Box\bot) \vee [1]\bot, \\ &\leftrightarrow ((H(\omega) \to H(0)) \vee (\uparrow H(\infty) \to \uparrow H(0))). \end{aligned}$$

We are going to show how to find (effectively) from any given letterless sentence A of GLP a truth-functional combination of sentences $D(\alpha)$ that is GLP-equivalent to A. Since GLP is arithmetically sound (cf. [Ignatiev], [Boolos]) and nothing false is "n-provable" (the interpretation of $[n]$ in PA), it is evident from the definition of $D(\alpha)$, that every $D(\alpha)$ is false in the standard model (under every realisation). Therefore we can determine the truth-value of A. That is, for a given letterless sentence we can effectively decide, whether it is provable in GLP or not.

Theorem (Normal form theorem for GLP**).** *(a) Each letterless sentence is* GLP*-equivalent to a boolean combination of some formulas of the form* $\uparrow^n H(\alpha)$.
(b) Each letterless sentence has a GLP*-equivalent normal form. (Actually a stronger equivalence holds – cf. [Ignatiev, Section 10]).*
(c) Each letterless sentence is GLP*-equivalent to a boolean combination of some formulas of the form* $D(\alpha)$.
(d) If A *has a normal form which is not the empty conjunction, then* GLP $\nvdash A$.

The proof consists of two steps:

Lemma 5. *(i) If A is a boolean combination of formulas of the form $\uparrow^n H(\alpha)$, then A has a* GLP*-equivalent in normal form.*
(ii) If A has a normal form, then for any $n \in \mathbf{N}$ the formula $[n]A$ is GLP*-equivalent to a boolean combination of formulas of the form $\uparrow^n H(\alpha)$.*
(ii') More precisely: If A has a normal form, then

$$\Box A \sim H(\beta)$$

for some ordinal β.

Remark. As in the normal form theorem, here too, stronger equivalences can be proved (as described in [Ignatiev, Section 10]).

Proof. (of the normal form theorem). (a) This easily follows with the above lemma by induction over the construction of modal sentences.
(b) This can be deduced from (a) and the first part one of the last lemma.
(c) Since for all ordinals $\alpha < \varepsilon_0$

$$\begin{aligned} D(\omega^{\alpha+1}) &= \Box D(0) \vee \uparrow D(\alpha+1) \\ &\sim \Box\bot \vee \uparrow \Box D(\alpha) \\ &\sim \Box D(\alpha) \\ &= \uparrow H(\alpha), \end{aligned}$$

we get

$$\forall_{n\in\mathbf{N}} \quad \uparrow^n H(\alpha) \sim D(\Omega^n(\alpha+1)),$$

where $\Omega(\alpha) := \omega^\alpha$. Thus part (c) is implied by (a).
(d) By lemma 5(ii1/2) we have

$$\text{GLP} \vdash \Box A \leftrightarrow H(\beta).$$

Since $H(\alpha) = D(\alpha+1)$ holds in no realisation, we get GLP $\nvdash H(\alpha)$. Thus GLP $\nvdash \Box A$, and by $\Box$-necessitation GLP $\nvdash A$. $\Box$

We prove lemma 5 only for GLB instead of GLP . The proof for GLP is analogous (but a little more complicated) and can be found in [Ignatiev].

Definition. The bimodal calculus GLB is defined as GLP, but contains only two box operators, namely $\Box$ and $\boxed{1}$.

In the case of GLB we need only to consider ordinals below ω^ω (as we will see in the proofs). Thus for GLB the definition of normal forms boils down to:

Definition. A letterless GLB-formula is in **normal form** if it is a (possibly empty) conjunction of formulas of the following form (where $\alpha \in \omega^\omega \dot\cup \{\infty\}, \beta \in \omega^\omega, i \in \mathbf{N} \cup \{\infty\}, j \in \mathbf{N}$)

(1) $\neg H(\alpha) \vee \neg [1]^i \bot$

(2) $\neg H(\alpha) \vee H(\beta) \vee \neg [1]^i \bot$ with $\alpha > \beta$

(3) $\neg H(\alpha) \vee H(\beta) \vee \neg [1]^i \bot \vee [1]^j \bot$ with $\alpha > \beta, i > j > 0, d(\alpha), d(\beta) \geq j$.

Since we are in the rest of the section working in GLB, $\vdash$ denotes GLB $\vdash$, and ordinals are ordinals below ω^ω. The next lemma is fundamental to what follows:

Lemma 6. *Let α, β be ordinals and $\alpha < \beta$. Then*

$$\vdash \Box D(\alpha) \to D(\beta).$$

With some extra work it should be provable from the comparability-results of section 2. But we will give a direct proof:

Proof. Since $D(\alpha + 1) = \Box D(\alpha)$, we may assume that for all γ with $\alpha < \gamma < \beta$ already $\vdash \Box D(\alpha) \to D(\gamma)$.
If $\alpha < \beta^-$, then (since $\beta^- < \beta$)

$$\vdash \Box D(\alpha) \to D(\beta^-),$$

whence

$$\begin{aligned} \vdash \Box D(\alpha) &\to \Box\Box D(\alpha), \\ &\to \Box D(\beta^-), \\ &\to D(\beta). && \text{by definition of } D(\beta) \end{aligned}$$

Thus we may assume $\beta^- \leq \alpha < \beta$, i.e. there are natural numbers $i_1, \dots, i_{m+1}, j_1, \dots, j_n$ (where either m or n may be 0), such that

$$\begin{aligned} \beta &= \langle i_1, \dots, i_m, i_{m+1} \rangle_\omega, \\ \alpha &= \langle i_1, \dots, i_m, j_1, \dots, j_n \rangle_\omega, \\ \beta^- &= \langle i_1, \dots, i_m \rangle_\omega, \end{aligned}$$

and $i_{m+1} > j_1 \geq \cdots \geq j_n$.

- If $n = 0$ then $\alpha = \beta^-$ and so

$$\begin{aligned} \vdash \Box D(\alpha) &\to \Box D(\beta^-), \\ &\to D(\beta). \end{aligned}$$

For every $k \in \mathbf{N}$ and every modal sentence B we write

$$\Delta_k := \Box(B \vee \boxed{1}^k \bot).$$

- If $n > 0$, then $D(\alpha) = \Box D(\alpha^-) \vee \boxed{1}^{j_n}\bot$ and

$$\begin{aligned}\Box D(\alpha) &= \Delta_{j_n}\Box D(\alpha^-)\\ &\vdots\\ &= \Delta_{j_n}\cdots\Delta_{j_2}\Delta_{j_1}\Box D(\beta^-).\end{aligned}$$

Lemma 7. *If $j < i$ then for every modal sentence A*

$$\vdash \Box(\Box A \vee \boxed{1}^j\bot) \to \Box A \vee \boxed{1}^i\bot,$$

i.e. $\vdash \Delta_j\Box A \to \Box A \vee \boxed{1}^i\bot$.

Proof. By the axioms of GLB we have

$$\begin{aligned}\vdash \Box(\Box A \vee \boxed{1}^j\bot) &\to \boxed{1}(\Box A \vee \boxed{1}^j\bot),\\ \vdash (\boxed{1}(\Box A \vee \boxed{1}^j\bot) \wedge \boxed{1}\neg\Box A) &\to \boxed{1}((\Box A \vee \boxed{1}^j\bot) \wedge \neg\Box A),\\ &\to \boxed{1}\boxed{1}^j\bot,\\ &\to \boxed{1}^i\bot, \qquad\qquad \text{(since } j < i)\\ \vdash \neg\Box A &\to \boxed{1}\neg\Box A,\end{aligned}$$

which – by propositional logic – prove this sublemma. □

Repeated application of the last lemma therefore yields

$$\begin{aligned}\vdash \Delta_{j_1}\Box D(\beta^-) &\to \Box D(\beta^-) \vee \boxed{1}^{i_{m+1}}\bot,\\ \vdash \Delta_{j_2}\Delta_{j_1}\Box D(\beta^-) &\to \Delta_{j_1}\Box D(\beta^-) \vee \boxed{1}^{i_{m+1}}\bot\\ &\vdots\\ \vdash \Box D(\alpha) &\to \Box D(\beta^-) \vee \boxed{1}^{i_{m+1}}\bot,\\ &\to D(\beta).\end{aligned}$$

□

Definition. We call a modal sentence **nice** if it is a truth-functional combination of sentences $H(\alpha)$ and $\boxed{1}^j\bot$.

We now come to the proof of lemma 5:

Proof. (of lemma 5) (i) Let A be a nice modal sentence. We now show how to find an equivalent of A in normal form:
Since A is nice, we can rewrite A as a conjunction of disjunctions of sentences of the form $H(\alpha), \neg H(\alpha), \boxed{1}^j\bot$ and $\neg\boxed{1}^j\bot$. Let B be a conjunct of A. Now it will suffice to determine

- whether B is equivalent to $\top$ (for if so, it may be deleted from the conjunction A)
- and if not, to find an equivalent of B in form (1),(2) or (3).

Recalling that $\neg H(\infty)$ and $\neg\boxed{1}^{\infty}\bot$ are equivalent to $\bot$,we may suppose that for some $\alpha \in \omega^\omega \dot\cup \{\infty\}$ and some $i \in \mathbf{N} \cup \{\infty\}$ the formulas $\neg H(\alpha)$ and $\neg\boxed{1}^{i}\bot$ are disjuncts of B. Using the equivalence (for all $j \in \mathbf{N}$)

$$\begin{aligned} \vdash \boxed{1}^{j}\bot &\leftrightarrow \boxed{1}^{j}\bot \vee \Box\bot \\ &\leftrightarrow \boxed{1}^{j}\bot \vee H(0), \end{aligned}$$

we may suppose that if some sentence $\boxed{1}^{j}\bot$ is a disjunct of B, so is some sentence $H(\beta)$ (where $\beta < \omega^\omega$).
According to lemma 6, either $\vdash H(\alpha) \to H(\beta)$ or $\vdash H(\beta) \to H(\alpha)$. Moreover, either $\vdash \boxed{1}^{i}\bot \to \boxed{1}^{j}\bot$ or $\vdash \boxed{1}^{j}\bot \to \boxed{1}^{i}\bot$. Thus by deleting disjuncts $\neg H(\alpha)$ of B that imply other disjuncts $\neg H(\alpha')$, and similarly for $H(\alpha), \neg\boxed{1}^{i}\bot, \boxed{1}^{i}\bot$, we may assume that B has one of the three following forms

$$\begin{aligned} &\neg H(\alpha) \vee \neg\boxed{1}^{i}\bot, \\ &\neg H(\alpha) \vee H(\beta) \vee \neg\Box^{i}\bot, \\ &\neg H(\alpha) \vee H(\beta) \vee \neg\Box^{i}\bot \vee \boxed{1}^{j}\bot. \end{aligned}$$

- If B is of the first form, we are done.
- If B is of the second form but $\alpha \le \beta$, then by lemma 6 $\vdash H(\alpha) \to H(\beta)$, and therefore B is equivalent to $\top$.
- If B is of the third form and $i \le j$, then $\vdash \boxed{1}^{i}\bot \to \boxed{1}^{j}\bot$, i.e. B is equivalent to $\top$.

  Thus let $i > j$. If $\alpha \neq \infty$ let $\alpha' := \alpha^{-j}$, otherwise let $\alpha' := \infty$. Furthermore let $\beta' := \beta^{-j}$. Then by lemma 8(a)

  $$\vdash \neg\boxed{1}^{j}\bot \to (H(\alpha) \leftrightarrow H(\alpha')) \wedge (H(\beta) \leftrightarrow H(\beta')).$$

  Lemma 8. *(a) For all $j \in \mathbf{N}$ and all α*

  $$\vdash \neg\boxed{1}^{j}\bot \to (\Box D(\alpha) \leftrightarrow D(\alpha^{-j})).$$

  (b) If $d(\alpha) \ge j$, then

  $$\vdash D(\alpha^{+j}) \leftrightarrow (\Box D(\alpha) \vee \boxed{1}^{j}\bot).$$

  (A proof can be found in [Boolos, Chapter 16].)

  Thus B is equivalent to

  $$\neg H(\alpha') \vee H(\beta') \vee \neg\boxed{1}^{i}\bot \vee \boxed{1}^{j}\bot.$$

  If $\alpha' \le \beta'$, then $\vdash H(\alpha') \to H(\beta')$, i.e. $B \sim \bot$. Thus $\alpha' > \beta'$. If $\alpha' = 0$ or ∞ then $d(\alpha') = \infty \ge j$ and otherwise $d(\alpha') = d(\alpha^{-j}) \ge j$. Similarly $d(\beta') \ge j$.

So A has an equivalent in normal form.
(ii) We now have to show how to find nice equivalents of $\Box A$ and $\boxed{1}A$ from a sentence A in normal form:
Since $\Box$ and $\boxed{1}$ distribute over $\wedge$, we may assume that A consists of a single conjunct having one of the forms (1),(2),(3).
First, we prove part (ii'). (1) If $A = \neg H(\alpha) \vee \neg\boxed{1}^i\bot$, then $\Box A \sim \Box\bot = H(0)$.
(2) Suppose $A = \neg H(\alpha) \vee H(\beta) \vee \neg\boxed{1}^i\bot$, where $\alpha > \beta$. Then $\Box A \sim H(\beta+1)$:
On the one hand

$$\vdash \Box H(\beta) \to \Box A.$$

On the other hand, by lemma 6,

$$\vdash \neg H(\alpha) \to \neg H(\beta+1).$$

Thus

$$\begin{aligned} \vdash \Box A &\to \Box(\neg H(\beta+1) \vee H(\beta) \vee \neg\boxed{1}^i\bot), \\ &\to \Box(\neg H(\beta+1) \vee H(\beta) \vee \neg\boxed{1}\bot). \end{aligned}$$

For any modal sentence C

$$\begin{aligned} \vdash \neg(\neg\Box\Box C \vee \Box C) &\to (\Box\Box C \wedge \neg\Box C), \\ &\to (\boxed{1}\Box C \wedge \boxed{1}\neg\Box C), \\ &\to \boxed{1}(\Box C \wedge \neg\Box C), \\ &\to \boxed{1}\bot, \end{aligned}$$

thus $\vdash \neg\boxed{1}\bot \to (\neg\Box\Box C \vee \Box C)$. Since $H(\beta) = \Box D(\beta)$, it follows that

$$\vdash \neg\boxed{1}\bot \to (\neg H(\beta+1) \vee H(\beta)),$$

and therefore

$$\begin{aligned} \vdash \Box A &\to \Box(\neg H(\beta+1) \vee H(\beta)), \\ &\to \Box H(\beta) && \text{(by Löb's axiom).} \end{aligned}$$

Thus $\vdash \Box A \leftrightarrow H(\beta+1)$.
(3) Finally, we suppose that $A = \neg H(\alpha) \vee H(\beta) \vee \neg\boxed{1}^i\bot \vee \boxed{1}^j\bot$, where $\alpha > \beta, i > j$ and $d(\alpha), d(\beta) \geq j$. Then $\Box S \sim H(\beta^{+j})$:
Since $\alpha \geq \beta^{+j}$, we have

$$\vdash A \to \neg H(\beta^{+j}) \vee H(\beta) \vee \neg\boxed{1}^i\bot \vee \boxed{1}^j\bot.$$

Since $i > j$,

$$\begin{aligned} \vdash \neg\boxed{1}^i\bot &\to \neg\boxed{1}^{j+1}\bot, \\ &\to (H(\beta^{+j}) \to H((\beta^{+j})^{-(j+1)})), && \text{(by lemma 8(b))} \\ &\to (H(\beta^{+j}) \to H(\beta)) && \text{(by lemma 6).} \end{aligned}$$

Thus $\vdash A \to (\neg H(\beta^{+j}) \vee H(\beta) \vee [1]^j\bot)$. Since $d(\beta) \geq j$, by lemma 8(b),

$$\vdash D(\beta^{+j}) \leftrightarrow H(\beta) \vee D(\beta^{+j}).$$

Therfore

$$\begin{aligned} \vdash A &\to \neg H(\beta^{+j}) \vee D(\beta^{+j}), \\ &\to \neg\Box D(\beta^{+j}) \vee D(\beta^{+j}), \end{aligned}$$

and so (by Löb's axiom)

$$\begin{aligned} \Box A &\to \Box D(\beta^{+j}), \\ &\to H(\beta^{+j}). \end{aligned}$$

Conversely, by lemma 8(b), $\vdash \Box D(\beta^{+j}) \to \Box(H(\beta) \vee [1]^j\bot)$, which yields

$$\vdash H(\beta^{+j} \to \Box A.)$$

Thus (ii') is proved.
Now we are looking at the easier case of $[1]A$:

Lemma 9. *Let F be a truth-functional combination of sentences $\Box C$. Then for all modal sentences G*

$$\vdash [1](F \vee G) \leftrightarrow (F \vee [1]G).$$

Proof. In the propositional calculus F is equivalent to some conjunction of disjunctions of sentences $\Box C$ and $\neg\Box C$, and so is $\neg F$. Since

$$\begin{aligned} &\vdash \Box C \to [1]\Box C, \\ &\vdash \neg\Box C \to [1]\neg\Box C, \\ &\vdash [1]F_1 \vee [1]F_2 \to [1](F_1 \vee F_2), \\ &\vdash [1]F_1 \wedge [1]F_2 \to [1](F_1 \wedge F_2), \end{aligned}$$

it follows inductively that

$$\vdash F \to [1]F \text{ and } \vdash \neg F \to [1]\neg F.$$

Since

$$\begin{aligned} \vdash F &\to [1]F, \\ &\to [1](F \vee G) \end{aligned}$$

and $\vdash [1]G \to [1](F \vee G)$, we get

$$\vdash (F \vee [1]G) \to [1](F \vee G).$$

Conversly, $\vdash \neg F \to [1]\neg F$, whence

$$\vdash (\neg F \wedge [1](F \vee G)) \to [1]G,$$

and therefore $\vdash [1](F \vee G) \to (F \vee [1]G)$. $\Box$

The sentences of the form $H(\alpha)$ are of course truth-functional combninations of some sentences $\Box C$.

- If A is of form (1), then by the above lemma

$$\begin{aligned}\vdash \boxed{1}A &\leftrightarrow \neg H(\alpha) \vee \boxed{1}\neg\boxed{1}^i\bot,\\ &\leftrightarrow \neg H(\alpha) \vee \boxed{1}\bot.\end{aligned}$$

- If A is of form (2), then

$$\begin{aligned}\vdash \boxed{1}A &\leftrightarrow \neg H(\alpha) \vee H(\beta) \vee \boxed{1}\neg\boxed{1}^i\bot,\\ &\leftrightarrow \neg H(\alpha) \vee H(\beta) \vee \boxed{1}\bot.\end{aligned}$$

- If A is of form (3), then

$$\begin{aligned}\vdash \boxed{1}A &\leftrightarrow \neg H(\alpha) \vee H(\beta) \vee \boxed{1}(\neg\boxed{1}^i\bot \vee \boxed{1}^j\bot),\\ &\leftrightarrow \neg H(\alpha) \vee H(\beta) \vee \boxed{1}^{j+1}\bot.\end{aligned}$$

Thus $\boxed{1}A$ has a nice GLB-equivalent. $\square$

3.1 Relations with the normal forms of S

By the normal form theorem, we can write every letterless sentence as a boolean combination of sentences $D(\alpha)$. Now we are going to show, that these $D(\alpha)$ are dual to elements in S.

Lemma 10. *For each ordinal $\alpha < \varepsilon_0$, the formula $D(\alpha)$ is* GLP*-equivalent to the negation of some element in S.*

Proof. We have

$$\forall_{j\in\mathbf{N}} \quad D(j) \sim [0]^j\bot = \neg\langle 0\rangle^j\top,$$

and – by induction – for all ordinals α, that $\Box D(\alpha^-) \sim \neg\beta$ and $D(d(\alpha)) \sim \neg\gamma$ with $\beta, \gamma \in S$. Thus $\uparrow\gamma \in S$ and

$$\uparrow D(d(\alpha)) \sim \uparrow\neg\gamma = \neg\uparrow\gamma.$$

Hence

$$\begin{aligned}D(\alpha) &= \Box D(\alpha^-) \vee \uparrow D(d(\alpha))\\ &\sim \neg\beta \vee \neg\uparrow\gamma\\ &\sim \neg(\beta \wedge \uparrow\gamma),\end{aligned}$$

and we are done by the following lemma. $\square$

Lemma 11. *For all $\alpha, \beta \in S_n$ there is an (effectively constructible) $\gamma \in S_n$ with*

$$\gamma \sim (\alpha \wedge \beta).$$

Proof. We reason by induction on the joint width of α and β. The case $w(\alpha\beta) \leq 1$ is trivial. Thus let $w(\alpha\beta) > 1$. Without loss of generality assume $\alpha \in S_n \setminus S_{n+1}$. So we can write

$$\alpha = \alpha_1 n \alpha'$$

with $\alpha_1 \in S_{n+1}$.

- The letter n does not occur in β, i.e. $\beta \in S_{n+1}$. By the induction hypothesis we get a $\gamma_1 \in S_{n+1}$ with

  $$\gamma_1 \sim \alpha_1 \wedge \beta.$$

  Therefore

  $$\begin{aligned} \vdash \alpha \wedge \beta &\leftrightarrow \alpha_1 \wedge \beta \wedge n\alpha', \\ &\leftrightarrow \gamma_1 \wedge n\alpha', \\ &\leftrightarrow \gamma_1 n\alpha', && \text{(by lemma 2)} \end{aligned}$$

  which has the required form.

- We have $\beta \in S_n \setminus S_{n+1}$, i.e. we can write $\beta = \beta_1 n \beta'$ with $\beta_1 \in S_{n+1}$. Since α' and β' are $<_n$ comparable, it follows, that

  $$\vdash n\alpha' \to n\beta' \text{ or } \vdash n\beta' \to n\alpha'.$$

  Assume, that $n\alpha'$ is stronger. Thus, by lemma 2,

  $$\begin{aligned} \vdash \alpha \wedge \beta &\leftrightarrow (\alpha_1 \wedge n\alpha') \wedge (\beta_1 \wedge n\beta') \\ &\leftrightarrow \alpha_1 \wedge \beta_1 \wedge n\alpha', \end{aligned}$$

  and we can proceed as in the first case to obtain γ. □

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