COMPUTATIONS IN BOUNDED COHOMOLOGY

(JOINT WORK WITH FRANCESCO FOURNIER-FACIO &
MARCO MORASCHINI)

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I BOUNDED COHOMOLOGY

Def. (bounded cohomology)

spaces: singular
groups: chains of bar

$$H_b^*(\cdot;\mathbb{R}) := H^*\Big(\underbrace{C_*(\cdot;\mathbb{R})}_{\|\cdot\|_1}{}^{\#}\Big) \quad \leftarrow \text{top. dual}$$

forget b $\Downarrow$ comparison map

$$H^*(\cdot;\mathbb{R})$$

⚠. in general: neither inj, nor surj.

- no MV-seq for H_b^*

Examples. • $H_b^1(\Gamma;\mathbb{R}) \cong$ bounded homs $\Gamma \to \mathbb{R}$ $\cong 0$.

$\exists$ Γ-inv. mean $\ell^\infty(\Gamma,\mathbb{R}) \to \mathbb{R}$

• If Γ is ~~amenable~~, then

$$H_b^*(\Gamma;\mathbb{R}) \overset{\text{transfer}}{\cong} H_b^*(1;\mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } *=0 \\ 0 & \text{if } *>0 \end{cases}$$

i.e.: amenable groups are boundedly acyclic (BAc).

- Mapping theorem [Gromov, Ivanov]. If $f : \Gamma \to \Lambda$ is a group epi with amenable kernel, then $H^*_b(f;\mathbb{R}) : H^*_b(\Lambda;\mathbb{R}) \to H^*_b(\Gamma;\mathbb{R})$ is an isometric iso.

  The classif. map $X \to B\pi_1(X)$ induces an isometric iso
$$H^*_b(\pi_1(X);\mathbb{R}) \cong H^*_b(X;\mathbb{R}).$$

- $H^2_b(F_2;\mathbb{R})$, $H^3_b(F_2;\mathbb{R})$: ∞-dim [Brooks, Soma]

Open problem: $H^4_b(F_2;\mathbb{R})$?

- If Γ is a hyperbolic group, then the comp. map $H^*_b(\Gamma;\mathbb{R}) \to H^*(\Gamma;\mathbb{R})$ is surj. if $* \geq 2$. [Gromov, Thurston, Mineyev]

Why?

- simplicial volume [Gromov]: For occ mfds M,
$$\|M\| = \frac{1}{\|[M]^*\|_\infty}$$
  - degree thms
  - rigidity results for Riem. vol.

• quasi-morphisms / stable commutator length:

$$\ker\left(H^2_b(\Gamma;\mathbb{R}) \xrightarrow{\text{comp map}} H^2(\Gamma;\mathbb{R})\right) \cong \underbrace{QM(\Gamma)}_{\text{quasi-morphisms } \Gamma \to \mathbb{R}} \big/ \text{trivial}$$

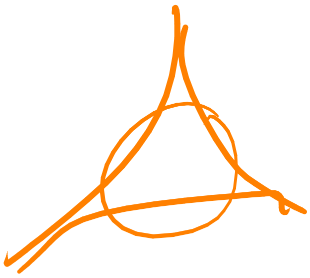
and

$$QM(\Gamma)\big/\text{trivial} \cong 0 \iff scl_\Gamma = 0.$$

[Bavard]

counting qm's: $w \in F_2$

$$\varphi_w(x) = \#w \text{ in } x - \#\bar{w} \text{ in } x.$$



Challenge: compute $H^*_b(\cdot;\mathbb{R})$!

Rmks: • $H^2_b(\cdot;\mathbb{R}) \overset{?}{\cong} 0$ is undecidable [FLM]

• There ex. many non-amenable BAc groups [Matsumoto-Morita, FLM]

• There ex. fin pres groups with "large" bounded cohomology [Burger-Monod + FLM].

II Actions

General principle:

action $\Gamma \curvearrowright X$

$\rightsquigarrow$ (co)simplicial res

cohomological control on stabilisers

$\rightsquigarrow$ comp. of (bounded) cohomology

Theorem. [Moraschini, Raptis]. If $\mathbb{R} \to V^*$ is a res by normed modules and each V^k is Γ-BAc, then there is a canonical iso

$$H_b^*(\Gamma;\mathbb{R}) \cong H^*((V^*)^\Gamma).$$

i.e., $H_b^*(\Gamma; V^k) \cong 0$ for $* > 0$.

Prop. If $\Gamma \curvearrowright X$ has fin. many orbit types and all stabilisers are BAc, then $\ell^\infty(X)$ is Γ-BAc. (Shapiro$_b$).

$\rightsquigarrow$ can compute $H_b^*(\Gamma;\mathbb{R})$ from BAc actions.

(past: successful: actions with amenable stab.)

III T

Recall: Thompson groups:

- $F < \mathrm{Homeo}^+_{PL}([0,1])$ s.t. the elts of F
  - have only fin. many breakpts, all in $\mathbb{Z}[1/2]$
  - all slopes are powers of 2.
- $T < \mathrm{Homeo}^+_{PL}(\mathbb{R}/\mathbb{Z})$ st. elts of T
  - have only fin. many breakpts, all in $\mathbb{Z}[1/2]/\mathbb{Z}$
  - all slopes are powers of 2
  - $\mathbb{Z}[1/2]/\mathbb{Z}$ is preserved

(fin. pres. inf. simple group)

Open problem: Is F amenable?

Weaker open problem: Is F BA_c?
(known: 2-BA_c)

Observe: $T \curvearrowright S^1$ with stabilizer F.

Theorem [FLM]. If F is BA_c, then

$$\underbrace{H_b^*(T;\mathbb{R})}_{\mathbb{R}\text{-alg}} \cong \mathbb{R}[\underbrace{eu_b^T}_{\in H_b^2(T;\mathbb{R})}]$$

bounded Euler class of $T \curvearrowright S^1$ ($= \frac{1}{2} \cdot$ or. class of $T \curvearrowright S^1$)

Proof. $T \curvearrowright S^1$: • highly transitive on one orbit S

• stabilizers: cartesian powers of F $\leadsto BA_c$

If F is BA_c

$$\overset{\text{(II)}}{\leadsto} H_b^*(T;\mathbb{R}) \cong H^*(\underbrace{\overbrace{\ell^\infty_{alt}(S^{*+1},\mathbb{R})^T}^{T\text{-}BA_c}}_{\cong\ \mathbb{R} \to 0 \to \mathbb{R} \to 0 \to \mathbb{R}})$$

$$\cong \mathbb{R}[eu_b^T]. \qquad \square$$

$H^*(T;\mathbb{R})$: Ghys, Sergiescu.

Theorem [Monod, Nariman].

$$H_b^*(\mathrm{Homeo}^+(S^1);\mathbb{R}) \cong \mathbb{R}[eu].$$