HOMOLOGY AND SEMINORMS

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ABSTRACT. These are the notes for a talk given in the "Kleines Seminar" at the WWU Münster. After motivating the simplicial volume via triangulations, some of the following points were presented:

- Define singular homology and the fundamental class of oriented closed connected manifolds.
- Define the ℓ^1-seminorm and the simplicial volume.
- State properties of the simplicial volume, including the proportionality principle.
- Introduce measure homology as tool for proving the proportionality principle.
- Sketch proof that measure homology and singular homology are algebraically isomorphic.
- Sketch proof that measure homology and singular homology are isometrically isomorphic.
- Sketch proof of the proportionality principle.

1. SINGULAR HOMOLOGY REFRESHED

Before introducing functorial seminorms and the simplicial volume, we recall some basic definitions concerning singular homology: The main idea of the singular theory is to examine all continuous maps from the standard simplices to a topological space to detect the "holes" of this space, i.e., to measure which "cycles" are no "boundaries".

Definition (1.1). Let X be a topological space, let $k \in \mathbf{N}$ and let R be a commutative ring with unit.

- The set
$$\Delta^k := \left\{ (x_0, \dots, x_k) \in \mathbf{R}^{k+1}_{\geq 0} \,\middle|\, \sum_{j=0}^{k} x_j = 1 \right\}$$
is called the **k-dimensional standard simplex**. For $j \in \{0, \dots, k\}$ we let
$$\begin{aligned} \partial_j \colon \Delta^{k-1} &\longrightarrow \Delta^k \\ (x_0, \dots, x_{k-1}) &\longmapsto (x_0, \dots, x_{j-1}, 0, x_j, \dots, x_{k-1}) \end{aligned}$$
be the inclusion of the **j-th face of** Δ^k.

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- The set $S_k(X)$ of **singular k-simplices** is the set of all continuous functions $\Delta^k \longrightarrow X$. If $\sigma \in S_k(X)$ and $j \in \{0, \dots, k\}$, then $\sigma \circ \partial_j$ is called the **j-th face of** σ.
- The set $C_k(X, R)$ of **singular k-chains of X with coefficients in R** is the free R-module generated by $S_k(X)$.
- The **boundary operator** $\partial\colon C_{k+1}(X, R) \longrightarrow C_k(X, R)$ is the R-module homomorphism given on the generators $\sigma \in S_{k+1}(X)$ by

$$\partial(\sigma) := \sum_{j=0}^{k+1} (-1)^j \cdot \sigma \circ \partial_j.$$

The elements of $\operatorname{im} \partial$ are called – as geometry suggests – **boundaries**, and the elements of $\ker \partial$ are called **cycles**. $\diamond$

A lengthy calculation shows that in the situation of the above definition the equality $\partial \circ \partial = 0$ holds (i.e., the singular chains form a chain complex). Hence the following definition makes sense:

Definition (1.2). Let X be a topological space, $k \in \mathbf{N}$, and let R be a commutative ring with unit. The R-module

$$H_k(X, R) := \frac{\ker\big(\partial\colon C_k(X, R) \to C_{k-1}(X, R)\big)}{\operatorname{im}\big(\partial\colon C_{k+1}(X, R) \to C_k(X, R)\big)}$$

is called **k-th singular homology of X with coefficients in R**. $\diamond$

Of course, singular homology is a functor:

Definition (1.3). Let $f\colon X \longrightarrow Y$ be a continuous map and let $k \in \mathbf{N}$. Then we write

$$C_k(f, R)\colon C_k(X, R) \longrightarrow C_k(Y, R)$$

for the homomorphism induced by f on the singular chain complex which is uniquely determined by

$$\begin{aligned} S_k(X) &\longrightarrow S_k(Y) \\ \sigma &\longmapsto f \circ \sigma. \end{aligned}$$

Then $C_*(f, R)$ is a chain map. The homomorphism on singular homology induced by $C_*(f, R)$ is denoted by $H_*(f, R)$. $\diamond$

Moreover, there are **relative singular homology groups** for pairs of topological spaces (defined as the homology of the quotient chain complex).

What has singular homology to do with triangulations?

Example (1.4). The singular homology of oriented closed connected manifolds looks very special. For example, if M is an n-dimensional oriented closed connected manifold, then there is a distinguished homology class $[M]_R \in H_n(M, R) \cong R$, called **fundamental class**. E.g., if M is triangulated by simplices $\sigma_1, \dots, \sigma_k \in S_n(M)$, then $\sum_{j=1}^{k} \varepsilon_j \cdot \sigma_j$ is a cocycle representing the fundamental class (where $\varepsilon_1, \dots, \varepsilon_k$ are appropriate signs). ■

Singular homology has the following properties (so-called **Eilenberg-Steenrod axioms**):

- Functoriality
- Homotopy invariance
- Long exact pair sequence
- Excision
- Dimension axiom
- Additivity

Singular homology is even uniquely characterised by this property on the category of CW-complexes. For concrete computations it is often useful to look at cellular homology instead of singular homology because of its combinatorial character.

2. FUNCTORIAL SEMINORMS

How can we measure the "complexity" of the fundamental class?

Gromov introduced another piece of structure on homology theories:

Definition (2.1). Suppose for each topological space X a seminorm $\|\cdot\|$ on $H_*(X, \mathbf{R})$ is given. Such a seminorm is called **functorial** if for all continuous maps $f\colon X \to Y$ and all $\alpha \in H_*(X, \mathbf{R})$ the inequality

$$\big\|H_*(f, \mathbf{R})(\alpha)\big\| \leq \|\alpha\|$$

holds. ◇

Conjecture. *Any functorial seminorm vanishes on simply connected spaces.*

One of the most interesting functorial seminorms is the ℓ^1-seminorm.

Definition (2.2). Let $p \in [1, \infty]$, let X be a topological space and let $k \in \mathbf{N}$. The **ℓ^p-norm** on $C_k(X, \mathbf{R})$ is defined by

$$\|c\|_p := \sqrt[p]{\sum_{\sigma \in S_k(X)} |a_\sigma|^p}$$

for all $c = \sum_{\sigma \in S_k(X)} a_\sigma \cdot \sigma \in C_k(X, \mathbf{R})$. This induces a seminorm on singular homology via

$$\|\alpha\|_p := \inf\big\{\|c\|_p \;\big|\; c \in C_k(X, \mathbf{R}), \partial c = 0, [c] = \alpha\big\}$$

for all $\alpha \in H_k(X, \mathbf{R})$. ◇

Remark (2.3). Functoriality of the ℓ^p-seminorm on singular homology is non-trivial. Moreover, $\|\cdot\|_p = 0$ for all $p \in \mathbf{R}_{>1}$. ◇

3. SIMPLICIAL VOLUME

The simplicial volume is a homotopy invariant of oriented closed connected manifolds measuring the complexity of the real fundamental class.

Definition (3.1). Let M be an oriented closed connected manifold. Then the **simplicial volume** of M is given by

$$\|M\| := \|[M]_{\mathbf{R}}\|_1 .$$ ◇

Remark (3.2).
- The simplicial volume can also be calculated using singular homology with rational coefficients. Furthermore, the simplicial volume can be read off the singular chain complex with integral coefficients (however, this requires more care).
- The simplicial volume of oriented closed connected smooth manifolds can be calculated via smooth singular simplices.
- The simplicial volume of oriented closed connected triangulated manifolds can be computed via simplicial singular simplices. ◇

Theorem (3.3). *The simplicial volume has the following properties:*
- *Functoriality*
- *Multiplicativity*
- *Additivity*
- *Amenable fundamental groups*

Example (3.4). In particular, $\|S^n\| = 0$ for all $n \in \mathbf{N}$, and $\left\|S^1 \times S^1\right\| = 0$. On the other hand it is well-known that the simplicial volume of hyperbolic manifolds (or more generally, of negatively curved manifolds) is non-zero. ■

Despite of its topological nature, the simplicial volume is related to Riemannian geometry in various ways:

Theorem (3.5). *If M is an n-dimensional oriented closed connected smooth manifold, then*
$$\|M\| \leq (n-1)^n \cdot n! \cdot \operatorname{minvol} M.$$

Theorem (3.6) (Proportionality principle of simplicial volume)**.** *Let M and N be oriented closed connected Riemannian manifolds with isometric Riemannian universal covering. Then*
$$\frac{\|M\|}{\operatorname{vol} M} = \frac{\|N\|}{\operatorname{vol} N}.$$

4. Measure homology

How can we get more room (i.e., more chains) in the singular chain complex?

In order to prove the proportionality principle, Thurston introduced a new homology theory, called (smooth) measure homology. (For simplicity, we consider only the version of measure homology defined by Zastrow and Hansen.)

Measure homology is a curious generalisation of singular homology: Let X be topological space and let $k \in \mathbf{N}$. The idea of measure homology is to think of a singular chain $\sum_{\sigma \in S_k(X)} a_\sigma \cdot \sigma \in C_k(X, \mathbf{R})$ as a signed measure on $S_k(X)$ having the mass a_σ on the set $\{\sigma\}$. The measure chain complex consists of all signed measures on $S_k(X)$ satisfying some finiteness condition.

As a first step, we recall some basic notions from measure theory.

Definition (4.1). Let $(X, \mathcal{A})$ be a measurable space.

- A map $\mu\colon \mathcal{A} \longrightarrow \mathbf{R} \cup \{\infty, -\infty\}$ is called a **signed measure** if $\mu(\emptyset) = 0$, not both ∞ and $-\infty$ are contained in the image of μ, and μ is σ-additive.
- A **null set** of a signed measure μ on $(X, \mathcal{A})$ is a measurable set $A \in \mathcal{A}$ with $\mu(B) = 0$ for all $B \in \mathcal{A}$ with $B \subset A$.
- A **determination set** of a signed measure μ on $(X, \mathcal{A})$ is a subset D of X such that each measurable set contained in the complement of D is a μ-null set.
- The **total variation** of a signed measure μ on $(X, \mathcal{A})$ is given by
$$\|\mu\| := \sup_{A \in \mathcal{A}} \mu(A) - \inf_{A \in \mathcal{A}} \mu(A).$$
- If $f\colon (X, \mathcal{A}) \longrightarrow (Y, \mathcal{B})$ is a measurable map and μ is a signed measure on $(X, \mathcal{A})$, then
$$\forall_{B \in \mathcal{B}} \quad \mu^f(B) := \mu\bigl(f^{-1}(B)\bigr)$$
defines a signed measure μ^f on $(Y, \mathcal{B})$. ◇

Definition (4.2). Let X be a topological space and let $k \in \mathbf{N}$.

- The **k-th measure chain group**, denoted by $\mathcal{C}_k(X)$, is the $\mathbf{R}$-vector space of signed measures on $\operatorname{map}(\Delta^k, X)$ possessing a compact determination set and finite total variation, where the mapping space $\operatorname{map}(\Delta^k, X)$ is equipped with the compact-open topology. The elements of $\mathcal{C}_k(X)$ are called **measure k-chains**.
- For each $j \in \{0, \dots, k+1\}$ the inclusion $\partial_j\colon \Delta^k \longrightarrow \Delta^{k+1}$ of the j-th face induces a continuous map $\operatorname{map}(\Delta^{k+1}, X) \longrightarrow \operatorname{map}(\Delta^k, X)$ and hence a homomorphism (which we will also denote by ∂_j)
$$\begin{aligned} \partial_j\colon \mathcal{C}_{k+1}(X) &\longrightarrow \mathcal{C}_k(X) \\ \mu &\longmapsto \mu^{(\sigma \mapsto \sigma \circ \partial_j)}. \end{aligned}$$
The **boundary operator** of measure chains is then defined by
$$\partial := \sum_{j=0}^{k+1} (-1)^j \cdot \partial_j \colon \mathcal{C}_{k+1}(X) \longrightarrow \mathcal{C}_k(X).$$
- The $\mathbf{R}$-vector space $\mathcal{H}_k(X) := H_k\bigl(\mathcal{C}_*(X), \partial\bigr)$ is called the **k-th measure homology group** of X.
- The total variation $\|\cdot\|$ turns $\mathcal{C}_k(X)$ into a normed vector space and thus induces a seminorm on $\mathcal{H}_k(X)$ as follows: For all $\mu \in \mathcal{C}_k(X)$ we define
$$\|\mu\|_{\mathrm{m}} := \inf\bigl\{\|\nu\| \bigm| \nu \in \mathcal{C}_k(X), \partial(\nu) = 0, [\nu] = \mu\bigr\}. \qquad ◇$$

Of course, measure homology can be turned into a functor. Analogously to singular homology, relative measure homology groups can be defined.

Thurston originally defined measure homology only for smooth manifolds. Instead of the set of singular simplices with the compact open topology, he considered the space of smooth singular simplices endowed with the $\mathcal{C}^1$-topology. This distinction is important when considering the integration of measure chains over smooth forms (as is needed in Thurston's proof of the proportionality principle).

4.1. **The algebraic isomorphism.** Establishing the Eilenberg-Steenrod axioms for measure homology, Zastrow and Hansen independently showed that singular homology with real coefficients and measure homology coincide on the category of CW-complexes. (However, there are spaces for which singular homology and measure homology are not isomorphic).

Theorem (4.3). *Let X be a CW-complex. Then the inclusion $i_X \colon C_* (X, \mathbf{R}) \longrightarrow \mathcal{C}_* (X)$ induces a natural isomorphism*

$$H_* (X, \mathbf{R}) \cong \mathcal{H}_* (X)$$

of real vector spaces.

In particular (manifolds are homotopy equivalent to CW-complexes), if M is an oriented closed connected manifold, there is also a fundamental class in measure homology, which we denote by $[M]_{\mathrm{m}}$.

4.2. **The isometric isomorphism.** Despite of their straightforward definition, the seminorms on singular homology and measure homology are not very well understood. More or less the only systematic tool available is bounded cohomology and the duality principle:

Theorem (4.4) (Duality Principle)**.** *Let X be a topological space, $k \in \mathbf{N}$ and $\alpha \in H_k (X)$.*

(1) *Then $\|\alpha\|_1 = 0$ if and only if*

$$\forall_{\varphi \in \widehat{H}^k(X)} \quad \langle \varphi, \alpha \rangle = 0.$$

(2) *If $\|\alpha\|_1 > 0$, then*

$$\|\alpha\|_1 = \sup\Big\{ \frac{1}{\|\varphi\|_\infty} \;\Big|\; \varphi \in \widehat{H}^k (X), \langle \varphi, \alpha \rangle = 1 \Big\}.$$

When proving that singular homology and measure homology are isometrically isomorphic, we construct analogs of bounded cohomology and the duality principle for measure homology. Then the results of Ivanov and Monod help to prove that singular homology and measure homology are isometrically isomorphic.

More precisely, we construct a dual $\widehat{\mathcal{C}}^*(X)$ of the measure chain complex (together with a corresponding duality principle) fitting into a commutative diagram

$$\begin{tikzcd}
 & \mathsf{I}^*(X) \arrow[dl] \arrow[dd] \\
\widehat{\mathcal{C}}^*(X) \arrow[dr] & \\
 & \widehat{C}^*(X),
\end{tikzcd}$$

where $\widehat{C}^*(X)$ is the cochain complex defining bounded cohomology. The crux of this diagram is that the vertical arrow induces an isometric isomorphism on the level of cohomology and that the arrows on the left do not increase the seminorm. Then the duality principles allow us to deduce that the algebraic isomorphism $H_*(X) \cong \mathcal{H}_*(X)$ must be isometric:

Theorem (4.5). *Let X be a topological space. Then the canonical inclusion*

$$i_X \colon C_*(X, \mathbf{R}) \longrightarrow \mathcal{C}_*(X)$$

induces a natural isometric isomorphism

$$H_*(X, \mathbf{R}) \cong \mathcal{H}_*(X)$$

with respect to the ℓ^1-seminorm on singular homology and the seminorm on measure homology induced by the total variation.

Corollary (4.6). *For all oriented closed connected manifolds M, we can compute the simplicial volume via measure homology, namely*

$$\|M\| = \big\|H_*(i_M)\big([M]_{\mathbf{R}}\big)\big\|_{\mathrm{m}}.$$

5. Proof of the proportionality principle

Let M and N be oriented closed connected Riemannian manifolds of dimension n with isometric Riemannian universal covering U. We write $p_M \colon U \longrightarrow M$ and $p_N \colon U \longrightarrow N$ for the Riemannian universal covering maps.

To prove the proportionality principle, we proceed in several steps:

- The isometry group $\mathrm{Isom}^+_\infty(U)$ of orientation preserving isometries of U is a locally compact second countable Hausdorff group (with respect to the compact open topology and the $\mathcal{C}^1$-topology).
- Hence there exists a Haar measure h on $\mathrm{Isom}^+_\infty(U)$. Moreover, it can be shown that $\pi_1(M)$ has finite covolume in $\mathrm{Isom}^+_\infty(U)$ and that $\mathrm{Isom}^+_\infty(U)$ is unimodular. Therefore, there is a right invariant Borel probability measure h_M on the quotient $\pi_1(M) \setminus \mathrm{Isom}^+_\infty(U)$.

- The map
$$F_\sigma \colon \pi_1(M) \setminus \mathrm{Isom}^+_\infty(U) \longrightarrow \mathrm{map}_\infty\left(\Delta^k, M\right)$$
$$\big(\pi_1(M)\big) \cdot f \longmapsto p_M \circ f \circ \sigma$$
induces a chain map
$$\mathrm{smear}_M \colon C^{\mathrm{s}}_* (U, \mathbf{R}) \longrightarrow \mathcal{C}_* (M)$$
by
$$\sigma \longmapsto h_M^{F_\sigma}.$$
- Since h_M is right-invariant, the map
$$\mathrm{smear}_{N,M} \colon C^{\mathrm{s}}_* (N, \mathbf{R}) \longrightarrow \mathcal{C}_* (M)$$
$$\sigma \longmapsto \mathrm{smear}_M(\widetilde{\sigma})$$
is a well-defined chain map.
- The smearing map $\mathrm{smear}_{N,M}$ does not increase the norm because h_M is a probability measure. Furthermore, integration shows that for all cycles $z \in C^{\mathrm{s}}_n (N, \mathbf{R})$ with $[z] = [N]_{\mathrm{s}}$ the equality
$$\big[\mathrm{smear}_{N,M}(z)\big] = \frac{\mathrm{vol}(N)}{\mathrm{vol}(M)} \cdot [M]_{\mathrm{m}} \in \mathcal{H}_n (M)$$
holds (i.e., smearing induces multiplication by $\mathrm{vol}(N)/\,\mathrm{vol}(M)$ on top homology).

Putting all pieces together yields Thurston's proof of the proportionality principle of simplicial volume:

Proof. Suppose $z \in C^{\mathrm{s}}_n (N, \mathbf{R})$ is a cycle that represents the (smooth) fundamental class $[N]_{\mathrm{s}}$. Then $\mathrm{smear}_{N,M}$ induces the factor $\mathrm{vol}(N)/\,\mathrm{vol}(M)$ on top homology, i.e.,
$$\big[\mathrm{smear}_{N,M}(z)\big] = \frac{\mathrm{vol}(N)}{\mathrm{vol}(M)} \cdot [M]_{\mathrm{m}}\,.$$
Since we can compute the simplicial volume via measure homology and since smearing does not increase the norm, we obtain
$$\begin{aligned} \|M\| &= \|[M]_{\mathrm{m}}\|_{\mathrm{m}} \\ &\le \frac{\mathrm{vol}(M)}{\mathrm{vol}(N)} \cdot \|\mathrm{smear}_{N,M}(z)\| \\ &\le \frac{\mathrm{vol}(M)}{\mathrm{vol}(N)} \cdot \|z\|^{\mathrm{s}}_1\,. \end{aligned}$$
Now taking the infimum over all (smooth) representatives of $[N]_{\mathrm{s}}$ gives the estimate
$$\|M\| \le \frac{\mathrm{vol}(M)}{\mathrm{vol}(N)} \cdot \|[N]_{\mathrm{s}}\|^{\mathrm{s}}_1 = \frac{\mathrm{vol}(M)}{\mathrm{vol}(N)} \cdot \|N\|$$

because $\|[N]_s\|_1^s = \|N\|$. Swapping the rôles of the manifolds M and N shows the reverse inequality

$$\|N\| \leq \frac{\mathrm{vol}(N)}{\mathrm{vol}(M)} \cdot \|M\|$$

and hence proves the proportionality principle. □

6. Applications of the proportionality principle

The proportionality principle can be applied to show that the simplicial volume of oriented closed connected flat manifolds vanishes. Moreover, the proportionality principle yields certain mapping lemmas and a degree theorem.

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