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COMPUTING BOUNDED COHOMOLOGY ?

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I BOUNDED COHOMOLOGY   II WHY?   III VANISHING?

I Def. (bounded cohomology of groups/spaces).

$$H_b^*(\cdot\,;\mathbb{R}) := H^*\big(C_*(\cdot\,;\mathbb{R})^{\#}\big)$$

top. dual

- singular chain cpx
- bar cpx

$\Downarrow$ "forgetting bdd"

$H^*(\cdot\,;\mathbb{R})$

⚠ in general: · neither injective, nor surjective
· no MV-seq for $H_b^*(\cdot\,;\mathbb{R})$

Examples:

- for all grps Γ:

$$H_b^1(\Gamma;\mathbb{R}) \cong \text{bounded homs } \Gamma \to \mathbb{R} \cong 0$$

- if Γ is amenable, then

∃ Γ-inv. mean $\ell^\infty(\Gamma,\mathbb{R}) \to \mathbb{R}$

$$H_b^*(\Gamma;\mathbb{R}) \underset{\text{transfer}}{\cong} H_b^*(1;\mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } * = 0 \\ 0 & \text{if } * > 0 \end{cases}$$

- Mapping thm [Gromov, Ivanov]

  If $f:\Gamma \to \Lambda$ is an epi with amenable kernel, then $H_b^*(f;\mathbb{R}): H_b^*(\Lambda;\mathbb{R}) \to H_b^*(\Gamma;\mathbb{R})$ is an isometric iso.

  The classifying map $X \to B\pi_1(X)$ induces an isometric iso

$$H_b^*(\pi_1(X);\mathbb{R}) \longrightarrow H_b^*(X;\mathbb{R}).$$

- $H_b^2(F_2;\mathbb{R})$, $H_b^3(F_2;\mathbb{R})$ are inf-dim (!)

  [Brooks, Soma]

free gp of rk 2

Open problem: $H_b^4(F_2;\mathbb{R}) \neq 0$??

- If Γ is a hyp. gp, then the comp. map

$$H_b^*(\Gamma;\mathbb{R}) \longrightarrow H^*(\Gamma;\mathbb{R})$$

  is surj. if $*\geq 2$.

  [Thurston, Gromov, Mineyev].

II WHY?

- simplicial volume [Gromov]

$$\|M\| = \frac{1}{\|[M]^*\|_\infty}$$

dual fd class

⇝ · rigidity results for Riem. mfd

· degree thms

- Quasi-morphisms / stable commutator length [Brooks, Bavard]

If Γ is a group, then

$$\ker \left(H_b^2(\Gamma;\mathbb{R}) \xrightarrow{\text{comp.}} H^2(\Gamma;\mathbb{R}) \right) \cong \underbrace{QM(\Gamma)}_{\text{span of quasi-morphisms } \Gamma \to \mathbb{R}} / \text{trivial qm's}$$

and

$$QM(\Gamma)/\text{trivial} \cong 0 \iff scl_\Gamma = 0.$$

- volume of rep's in $SL_n(\mathbb{R})$ and rep. rigidity [Burger, Iozzi, Wienhard, Bucher, ...]
- estimates for the amenable "LS" category [Gromov, Ivanov, ...]

III VANISHING?

Theorem [FF, L, M]. There ex. an (explicit) functor $\mu : \text{Group} \to \text{Group}$ such that: for all groups Γ:

- Γ embeds into $\mu(\Gamma)$
- $H^*_b(\mu(\Gamma);\mathbb{R}) \cong H^*_b(1;\mathbb{R})$
- If Γ is fin gen, then $\mu(\Gamma)$ is fin gen.

Proof. based on [Matsumoto–Morita, Baumslag–Dyer–Heller]

$\Gamma \rightsquigarrow \underbrace{\text{standard mitosis of } \Gamma \ \circlearrowleft \text{ iterate}}_{\text{suitable ascending HNN-ext.}}$ $\square$

$\downarrow$ ascending HNN ext, universal fin pres. group [Higman]

Corollary: There ex. a fin pres. group Γ s.t.

- $H^*_b(\Gamma;\mathbb{R}) \cong H^*_b(1;\mathbb{R})$
- every fin pres. gp embeds into Γ.

in particular: Γ is not amenable!

Theorem [F., L., M.] Let $d \in \mathbb{N}_{\geq 2}$. Then the following decision problem is undecidable:
Given a fin. pres $\langle S \mid R \rangle$,
decide whether $H^d_b(\langle S \mid R\rangle; \mathbb{R}) \cong 0$ or not.

in approp. sense
$\leadsto$ $H^d_b(\text{fin. simpl. cpx}; \mathbb{R})$ is <u>not</u>
then algorithmically computable!

Proof. for $d = 2$ (similar to [Gordon])

- There ex. a fin pres $\langle S \mid R\rangle$ of a gp with unsolvable word problem and an algo

$$\begin{aligned} \text{words over } S &\longrightarrow \text{fin pres.} \\ w &\longmapsto \langle S_w \mid R_w \rangle =: \Lambda_w \end{aligned}$$

s.t. [Novikov-Boone-Britton, Adian-Rabin]
w rep. the trivial elt of $\langle S \mid R\rangle$ $\iff$ $\Lambda_w \cong 1$.

- We then consider

$$\begin{aligned} \text{words over } S &\longrightarrow \text{fin pres} \\ w &\longmapsto \langle S_w, t \mid R_w \rangle \cong \Lambda_w * \mathbb{Z} \end{aligned}$$

Then:

- w reps 1 in $\langle S \mid R\rangle$ $\leadsto$ $H^2_b(\underbrace{\Lambda_w * \mathbb{Z}}_{\cong \mathbb{Z} \text{ amenable}}; \mathbb{R}) \cong 0$
- w rep $\neq 1$ in $\langle S \mid R\rangle$ $\leadsto$ $H^2_b(\underbrace{\Lambda_w * \mathbb{Z}}_{\text{non-amenable free product}}; \mathbb{R}) \not\cong 0$ QM □