

STABLE INTEGRAL SIMPLICIAL VOLUME

I RESIDUALLY FINITE VIEW } ON SIMPLICIAL VOLUME
II DYNAMICAL VIEW

oriented, closed, conn.

I Definition (stable integral simplicial volume). If M is an n -oc wfd, then
gradient nv. of $\|\cdot\|_{\mathbb{Z}}$
$$\|M\|_{\mathbb{Z}}^{\infty} := \inf \left\{ \frac{\|N\|_{\mathbb{Z}}}{|\deg p|} \mid N \xrightarrow{p} M \text{ finite cv.} \right\} \in \mathbb{R}_{\geq 0}$$

$$\implies \|M\| \leq \|M\|_{\mathbb{Z}}^{\infty} \leq \|M\|_{\mathbb{Z}}$$

Questions. (1) For which n -oc wfds M (with residually finite $\pi_1(M)$) do we have
integral approx. problem for simplicial volume
$$\|M\|_{\mathbb{Z}}^{\infty} = \|M\| \quad (?!)$$

(2) Why should we care about (1) (?) $\neq \|M\|$

Observation: If $\pi_1(M) \cong 1$, then $\|M\|_{\mathbb{Z}}^{\infty} = \|M\|_{\mathbb{Z}} \geq 1$

\implies focus on the aspherical case

For all $k \in \mathbb{N}$: $b_k(M) \stackrel{+P}{\leq} \|M\|_{\mathbb{Z}}$ [Gromov]

$$\implies |\chi(M)| \leq (\dim \pi + 1) \cdot \|M\|_{\mathbb{Z}}^{\infty}$$

$\underbrace{\chi(M)}_{\text{multiplicative}}$

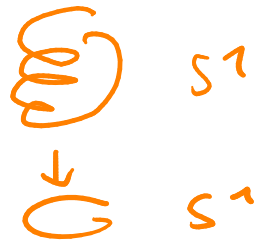
$$\text{Thus: } \|M\|_{\mathbb{Z}}^{\infty} = 0 \implies \chi(M) = 0$$

Question [Gromov]. Let M be an n -oc aspherical wfd. Then

$$\|M\| = 0 \implies \chi(M) = 0 \quad (?!)$$

Elementary examples:

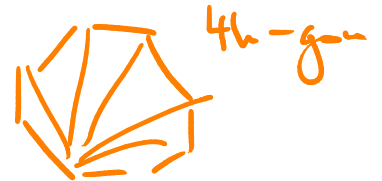
• $\|S^1\|_{\mathbb{Z}}^{\infty} = 0 = \|S^1\|$.



(similarly: all tori, wfd $\times S^1$)

• For all $g \in \mathbb{N}_{\geq 1}$: $\|\Sigma_g\|_{\mathbb{Z}}^{\infty} = \|\Sigma_g\|$ [Gromov]
 \geq : \checkmark

\leq : $\|\Sigma_h\|_{\mathbb{Z}} \leq 4 \cdot h - 2$



+ relation between genus covering degree

$\leadsto \|\Sigma_g\|_{\mathbb{Z}}^{\infty} \leq 4g - 4 = \|\Sigma_g\|$.

Problem: How to compute $\|M\|_{\mathbb{Z}}^{\infty}$ in more cases (?)

Inspiration: other gradient invariants (of groups):
 [Lück] [Gaboriau]

Betti numbers

$b_k(\Gamma) \leadsto \lim_{\substack{\Lambda < \Gamma \\ \text{fin.}}} \frac{b_k(\Lambda)}{[\Gamma:\Lambda]} \stackrel{(2)}{=} b_k^{(2)}(\Gamma) \stackrel{(2)}{=} b_k^{(2)}(\mathcal{K}_{\Gamma, \mathbb{Z}}(X, \Gamma))$

profin. completion of Γ

rank

$\text{rank } d(\Gamma) \leadsto \inf_{\substack{\Lambda < \Gamma \\ \text{fin.}}} \frac{d(\Lambda)}{[\Gamma:\Lambda]} = \text{rg}(\Gamma) = \text{cost}(\Gamma \curvearrowright \hat{\Gamma}) - 1$

rank gradient

[Abért, Nikody]

\leadsto can use tools from ergodic theory!

II

Definition. [Gromov, Schmidt] (integral foliated simplicial volume)
 let M be an o.c.c. wfd and let $\alpha: \pi_1(M) \curvearrowright (X, \mu)$
 be a pmp action (on a standard Borel) prob. space. Then we define

$\|M\|^\alpha := \|M\|$

flexibility \swarrow $L^\infty(X, \mu; \mathbb{Z})$ twisted coeffs via α
 rigidity \searrow
 defined via $L^\infty(X, \mu; \mathbb{Z}) \otimes_{\mathbb{Z}\pi_1(M)} C_*(\tilde{M}; \mathbb{Z})$

and

$\|M\|_{\mathbb{F}, \mathbb{Z}} := \inf \{ \|M\|^\alpha \mid \alpha: \pi_1(M) \curvearrowright (X, \mu) \}$
 $\in \mathbb{R}_{\geq 0}$

$\implies \|M\| \leq \|M\|_{\mathbb{F}, \mathbb{Z}} \leq \|M\|_{\mathbb{Z}}^\infty \leq \|M\|_{\mathbb{Z}}$
 integrality

Theorem [L-Paglicci, Frigerio-L-Paglicci-Sauer] If M
 is an o.c.c. wfd with res. fin. $\pi_1(M)$, then

$\|M\|_{\mathbb{Z}}^\infty = \|M\|_{\pi_1(M) \curvearrowright \widehat{\pi_1(M)}}$
 with Haar measure

Theorem. [L-Pagani, Frigio - L-Pagani-Sans, Favre - Maraschini - Quirion]
 let M be an occ aspherical 3-mfld. Then

$$\|M\|_{\mathbb{Z}}^{\infty} = \|M\| = \frac{\text{hypvol}(M)}{\sqrt{3}}$$

⚠ Not true for occ hyperbolic mflds in $\dim \geq 4$
 [Franco vigi, Frigenio, Martelli]

Sketch of proof for the hyp 3-mfld case:

① geometric input: There ex. a seq. ^{(M_n)_n} of occ hyp 3-mflds with
 $\lim_{n \rightarrow \infty} \frac{\|M_n\|_{\mathbb{Z}}}{\|M_n\|} = 1$.

② proportionality principle: let M, N be occ hypⁿ-mflds.
 Then

$$\frac{\|M\|_{\mathbb{F}, \mathbb{Z}}}{\text{vol}(M)} \leq \frac{\|N\|_{\mathbb{Z}}^{\infty}}{\text{vol}(N)}$$

Idea of proof: $\pi_1(M) \curvearrowright H^n \curvearrowright \pi_1(N)$ ME-coupling
 \leadsto ME-cycle

$$\leadsto \frac{\|M\|_{\mathbb{F}, \mathbb{Z}}}{\text{vol} M} \leq \frac{\|M\|_{\pi_1(M) \curvearrowright \text{ball}^n(H^n)}}{\text{vol} M} \stackrel{\pi_1(N)}{\leq} \frac{\|N\|_{\mathbb{Z}}^{\infty}}{\text{vol} N}$$

"Swearing"

③ more ergodic theory: If $\dim M = 3$, then

$$\|M\|_{\mathbb{F}, \mathbb{Z}} = \underbrace{\|M\|_{\pi_1(M) \curvearrowright \widehat{\pi_1(M)}}}_{\pi_1(M) \text{ has EMS}^*} = \|M\|_{\mathbb{Z}}^{\infty}$$

□