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Stable Integral Simplicial Volume

I Residually Finite View }
II Dynamical View } on Simplicial Volume

I Definition (stable integral simplicial volume). If M is an occ mfd, then

oriented, closed, conn.

gradient inv. of $\|\cdot\|_{\mathbb{Z}}$

$$\|M\|_{\mathbb{Z}}^{\infty} := \inf\left\{ \frac{\|N\|_{\mathbb{Z}}}{|\deg p|} \;\middle|\; N \xrightarrow{p} M \text{ finite cov.} \right\} \in \mathbb{R}_{\geq 0}.$$

$$\leadsto \quad \|M\| \leq \|M\|_{\mathbb{Z}}^{\infty} \leq \|M\|_{\mathbb{Z}}$$

Questions. (1) For which occ aspherical mfds M (with residually finite $\pi_1(M)$) do we have

integral approx. problem for simplicial volume

$$\|M\|_{\mathbb{Z}}^{\infty} = \|M\| \quad ?!$$

(2) Why should we care about (1)?

Observation: • If $\pi_1(M) \cong 1$, then $\|M\|_{\mathbb{Z}}^{\infty} = \|M\|_{\mathbb{Z}} \geq 1$ $\neq 0 = \|M\|$

$\leadsto$ focus on the aspherical case

• For all $k \in \mathbb{N}$: $b_k(M) \overset{PD}{\leq} \|M\|_{\mathbb{Z}}$ [Gromov]

$$\leadsto \quad \underbrace{|\chi(M)|}_{\text{multiplicative}} \leq (\dim M + 1) \cdot \|M\|_{\mathbb{Z}}^{\infty}.$$

Thus: $\|M\|_{\mathbb{Z}}^{\infty} = 0 \implies \chi(M) = 0.$

Question. [Gromov]. Let M be an occ aspherical mfd. Then

$$\|M\| = 0 \overset{?!}{\implies} \chi(M) = 0.$$

Elementary examples:

- $\|S^1\|_{\mathbb{Z}}^{\infty} = 0 = \|S^1\|$.

 

 S^1

 $\downarrow$

 S^1

 (similarly: all tori, mfd $\times S^1$)

- For all $g \in \mathbb{N}_{\geq 1}$: $\|\Sigma_g\|_{\mathbb{Z}}^{\infty} = \|\Sigma_g\|$ [Gromov]

 $\geq$: ✓

 $\leq$: $\|\Sigma_h\|_{\mathbb{Z}} \leq 4 \cdot h - 2$

 

 4h-gon

 \+ relation between genus & covering degree

 $\leadsto \|\Sigma_g\|_{\mathbb{Z}}^{\infty} \leq 4g - 4 = \|\Sigma_g\|$.

Problem: How to compute $\|M\|_{\mathbb{Z}}^{\infty}$ in more cases ?

Inspiration: other gradient invariants (of groups):

[Lück] [Gaboriau]

Betti numbers $\quad b_k(\Gamma) \leadsto \lim_{\substack{\Lambda \triangleleft \Gamma \\ \text{f.i.}}} \frac{b_k(\Lambda)}{[\Gamma : \Lambda]} = b_k^{(2)}(\Gamma) = b_k^{(2)}\big(\mathcal{R}_{\Gamma \curvearrowright (X,\mu)}\big)$

profin. completion of Γ

rank $\quad \operatorname{rank} d(\Gamma) \leadsto \inf_{\substack{\Lambda < \Gamma \\ \text{f.i.}}} \frac{d(\Lambda)\cancel{-1}}{[\Gamma : \Lambda]} = \operatorname{rg}(\Gamma) = \operatorname{cost}(\Gamma \curvearrowright \widehat{\Gamma}) - 1$

rank gradient

[Abért, Nikolov]

$\leadsto$ can use tools from ergodic theory !

(II)

Definition. [Gromov, Schmidt] (integral foliated simplicial volume)

Let M be an occ mfd and let $\alpha : \pi_1(M) \curvearrowright (X,\mu)$ be a pmp action (on a standard Borel) prob. space. Then we define

twisted coeffs via α

$$\|M\|^{\alpha} := \|M\|_{L^\infty(X,\mu;\mathbb{Z})}$$

flexibility — rigidity

defined via $L^\infty(X,\mu;\mathbb{Z}) \otimes_{\mathbb{Z}\pi_1(M)} C_*(\widetilde{M};\mathbb{Z})$

and

$$\|M\|_{\mathcal{F},\mathbb{Z}} := \inf\{\|M\|^{\alpha} \mid \alpha : \pi_1(M) \curvearrowright (X,\mu)\} \in \mathbb{R}_{\geq 0}$$

$$\leadsto \quad \|M\| \underset{\text{integration}}{\leq} \|M\|_{\mathcal{F},\mathbb{Z}} \leq \|M\|^{\infty}_{\mathbb{Z}} \leq \|M\|_{\mathbb{Z}}$$

Theorem [L-Pagliantini, Frigerio-L-Pagliantini-Sauer] If M is an occ mfd with res. fin. $\pi_1(M)$, then

$$\|M\|^{\infty}_{\mathbb{Z}} = \|M\|^{\pi_1(M) \curvearrowright \widehat{\pi_1(M)}}$$

with Haar measure

Theorem. [L-Pagliantini, Frigerio-L-Pagliantini-Sauer, Fauser-L-Moraschini-Quintanilha] Let M be an occ aspherical 3-mfd. Then

$$\|M\|^{\infty}_{\mathbb{Z}} = \|M\| = \frac{\operatorname{hypvol}(M)}{v_3}.$$

⚠ <u>Not</u> true for occ hyperbolic mfds in dim ≥ 4
[Francaviglia, Frigerio, Martelli]

Sketch of proof for the hyp 3-mfd case:

① geometric input: There ex. a sq $(M_n)_{n\in\mathbb{N}}$ of occ hyp 3-mfds with

$$\lim_{n\to\infty} \frac{\|M_n\|_{\mathbb{Z}}}{\|M_n\|} = 1.$$

② proportionality principle: Let M, N be occ hyp n-mfds. Then

$$\frac{\|M\|_{\mathcal{F},\mathbb{Z}}}{\operatorname{vol}(M)} \leq \frac{\|N\|^{\infty}_{\mathbb{Z}}}{\operatorname{vol}(N)}$$

Idea of proof: $\pi_1(M) \curvearrowright \mathbb{H}^n \curvearrowleft \pi_1(N)$ ME-coupling

$\leadsto$ ME-cocycle

$$\leadsto \frac{\|M\|_{\mathcal{F},\mathbb{Z}}}{\operatorname{vol} M} \leq \frac{\|M\|^{\pi_1(M)\curvearrowright \operatorname{Isom}^+(\mathbb{H}^n)/\pi_1(N)}}{\operatorname{vol} M} \leq \frac{\|N\|^{\infty}_{\mathbb{Z}}}{\operatorname{vol} N}$$

"smearing"

③ more ergodic theory: If $\dim M = 3$, then

$$\|M\|_{\mathcal{F},\mathbb{Z}} \underset{\pi_1(M) \text{ has EMD}^*}{=} \|M\|^{\pi_1(M)\curvearrowright \widehat{\pi_1(M)}} = \|M\|^{\infty}_{\mathbb{Z}}$$

□