http://www.mathematik.uni-r.de/loeh/talk.pdf

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THE SPECTRUM OF SIMPLICIAL VOLUME OF NON-COMPACT MANIFOLDS

(JOINT WORK WITH NICOLAUS HEUER)

I SIMPLICIAL VOLUME OF NON-COMPACT MANIFOLDS

II SV^{lf}

III SV^{lf}_{tame}

I Definition. [Gromov] Let M be an oriented ~~closed~~ w/ boundary connected n-mfd. Then the simplicial volume of M is defined as

locally finite

$$\|M\|^{lf} := \inf\{|c|_1 \mid c \in C_n^{lf}(M;\mathbb{R}) \text{ is a fund. cycle of } M\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}$$

for geometry: better to use $\|M\|^{lf}_{Lip}$

Questions: What can be said about

$$SV(n) := \{\|M\| \mid M \text{ occ } n\text{-mfd}\} \subset \mathbb{R}_{\geq 0}$$ (?)

[Gromov/Thurston, $SV(1) = \{0\}$, $SV(2) = 4 \cdot \mathbb{N}$,
$SV(3) = \mathbb{N}\left[\frac{\text{hyperbolic}}{v_3}\right]$, $\underbrace{SV(n)}_{\geq 4} \subset \mathbb{R}_{\geq 0}$ dense [Heuer, L]]

- What can be said about

$$SV^{lf}(n) := \{\|M\|^{lf} \mid M \text{ or, conn, } n\text{-mfd}, \partial M = \emptyset\}$$ (?)

$[SV^{lf}(1) = \{\overset{S^1}{0}, \overset{\mathbb{R}}{\infty}\}]$

II **Theorem.** [Heuer, L] Let $n \in \mathbb{N}_{\geq 4}$. Then
$$SV^{lf}(n) = \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Remarks: • $SV^{lf}(2) = 2 \cdot \mathbb{N} \cup \{\infty\}$

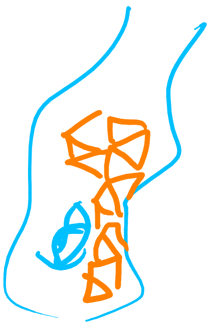
• $SV^{lf}(3) = ?$ Recent conj: [Bargagnati, Frigerio]
$$SV^{lf}(3) = SV(3) \cup \{\infty\}$$

Proof of Thm: Let $\alpha \in \mathbb{R}_{\geq 0} \cup \{\infty\}$.

$SV(n)$ dense in $\mathbb{R}_{\geq 0}$ $\Longrightarrow$ there ex. a seq. $(\alpha_k)_{k \in \mathbb{N}}$ in $SV(n)$ with $\sum_{k=0}^{\infty} \alpha_k = \alpha$.

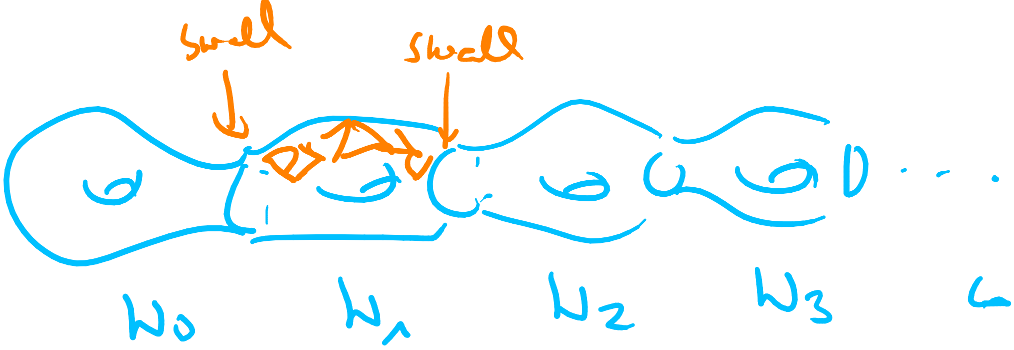
Recap: • $c = \sum_\sigma a_\sigma \cdot \sigma$ ($\sigma \colon \Delta^n \to M$) is a lf-chain on M if
$$\forall_{K \subset M \text{ compact}} \quad \infty > \left|\{\sigma \mid a_\sigma \neq 0 \text{ and } \operatorname{im} \sigma \cap K \neq \emptyset\}\right|.$$

• a lf-chain c on M is a fund. cycle of M if for all $x \in M$
$$\operatorname{res}[c] \in H_n(M, M \setminus \{x\}, \mathbb{R})$$
is the gen. compatible with the orientation.



For $k \in \mathbb{N}$: choose an occ n-mfd M_k with $\|M_k\| = \alpha_k$.

$\leadsto$ Set $M := M_0 \# M_1 \# M_2 \# \dots$ "linear connected sum"



compact mfds with boundary

Computation:

- $\boxed{\|M\|^{lf} \leq} \sum_{k=0}^{\infty} \|W_k, \partial W_k\|$ — equivalence thm [Gromov] + UBC [Matsumoto, Morita]

$= \sum_{k=0}^{\infty} \underbrace{\|M_k\|}_{=\alpha_k}$

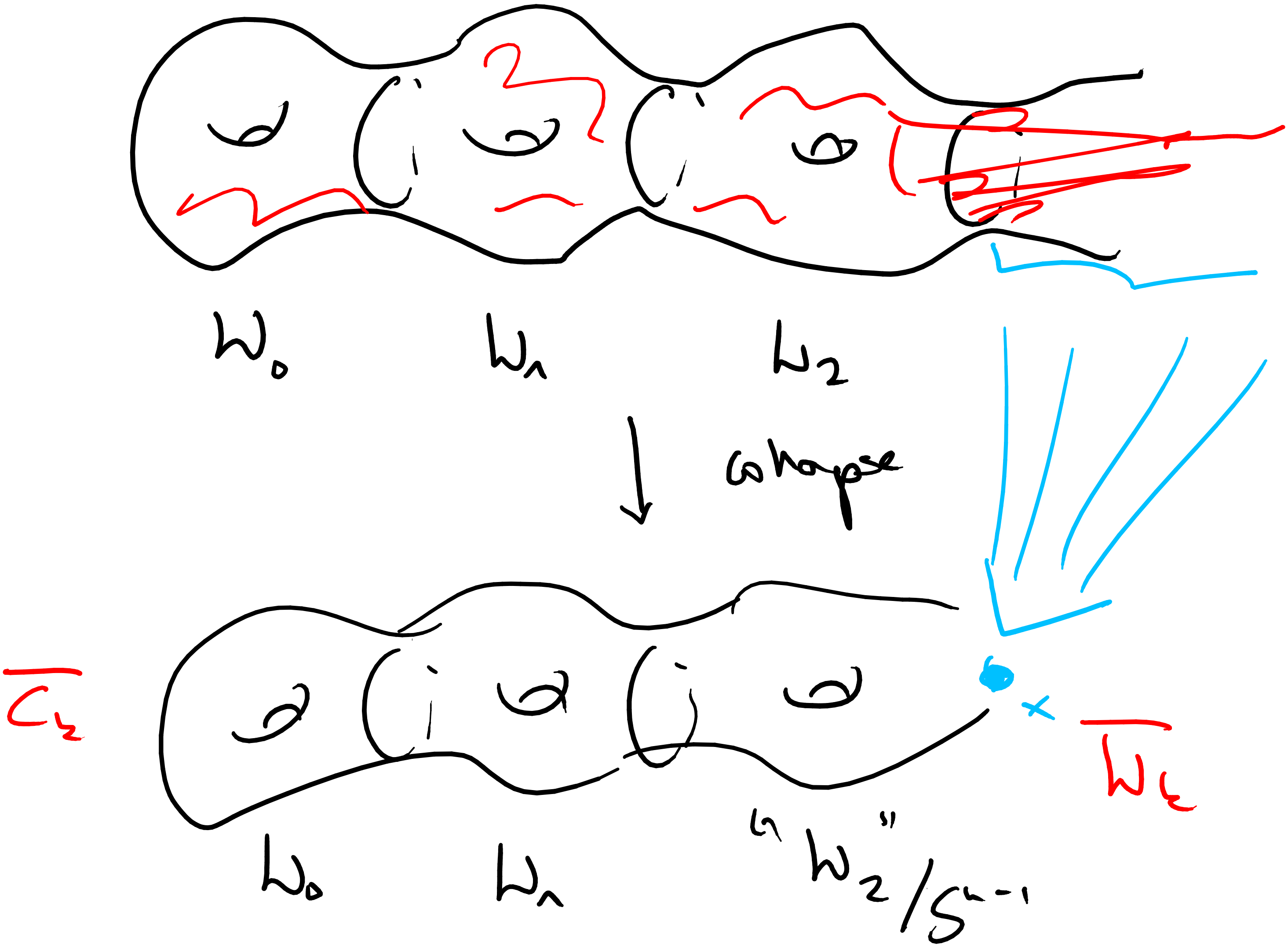
$= \boxed{\alpha}$

- $\|M\|^{lf} \geq \alpha$: Wlog $\|M\|^{lf} < \infty$.

Let $\varepsilon \in \mathbb{R}_{>0}$.

$\leadsto$ ex. lf-fund cycle c of M with $|c|_1 \leq \|M\|^{lf} + \varepsilon$.

For $k \in \mathbb{N}$: $c_k \big|_{W_0 \cup_{S^{n-1}} W_1 \cup \dots \cup_{S^{n-1}} W_k}$ (finite!)



$\overline{C_k}$ is "rel. fund. cycle" of $(\overline{W_k}, \{x\})$.

$\rightsquigarrow |c|_1 \geq |\overline{C_k}|_1 \geq \|\overline{W_k}, \{x\}\| = \|\overline{W_k}\|$ ← equivalence thm

$$= \dots = \sum_{j=0}^{k} \|M_j\| = \sum_{j=0}^{k} \alpha_j$$

$\overset{k\to\infty}{\rightsquigarrow}$ $|c|_1 \geq \alpha$

$\wedge$

$\|M\|^{lf} + \varepsilon$

$\overset{\varepsilon\to 0}{\rightsquigarrow}$ $\|M\|^{lf} \geq \alpha$. ∎

III

Theorem [Heuer, L] Let $n \in \mathbb{N}$. Then

$$\underbrace{SV^{lf}_{\mathrm{tame}}}_{:= \{ \|M\|^{lf} \mid M \text{ or., conn, } n\text{-mfd with } \partial M = \emptyset \text{ and } M \text{ is tame} \}}(n) \subset \mathbb{R}_{\geq 0} \cup \{\infty\} \text{ is countable.}$$

tame: $\exists_{\text{cpt } n\text{-mfd } W \text{ with bdy}} \quad M \underset{\mathrm{Top}}{\cong} W^{\circ} = W \setminus \partial W$

Proof.
- Observation: there ex. only countably many proper homotopy types of tame n-mfds
- $\|\cdot\|^{lf}$ is a proper homotopy inv.

$\square$

Def. $\|M\|^{lf}_{\mathrm{Lip}} := \{ |c|_1 \mid c = \sum_{j=0}^{\infty} a_j \cdot \sigma_j \in C^{lf}_n(M;\mathbb{R})$ fund. cycle and $\sup_{j \in \mathbb{N}} \mathrm{Lip}(\sigma_j) < \infty \}$

• $\|\mathbb{R}\|^{ef} \geq \|[0,1], \{0,1\}\| \geq \frac{1}{2} \underbrace{\|\{0,1\}\|}_{=2} > 0$

$$\|W^\circ\| < \infty \overset{[\text{Gromov}]}{\Longrightarrow} \|\partial W\| = 0.$$

$\rightsquigarrow \|\mathbb{R}\|^{ef} = \infty$. (or: straightening)

• $\mathbb{R}^n \circlearrowleft$ proper self-map of deg 2

$\to \|\mathbb{R}^n\|^{ef} = 0$ if $n \geq 2$.

• hyperbolic n dim 3 has order type ω^ω

[Thurston]