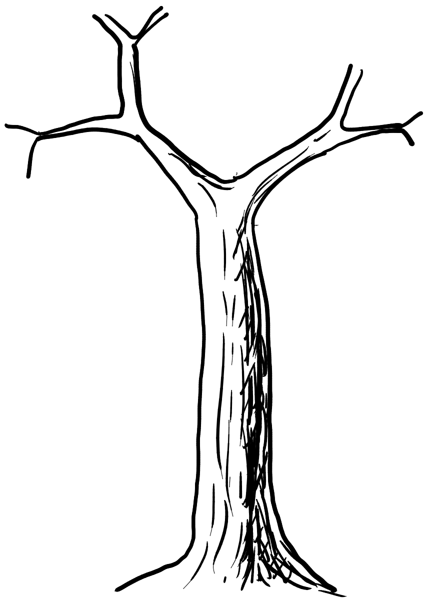
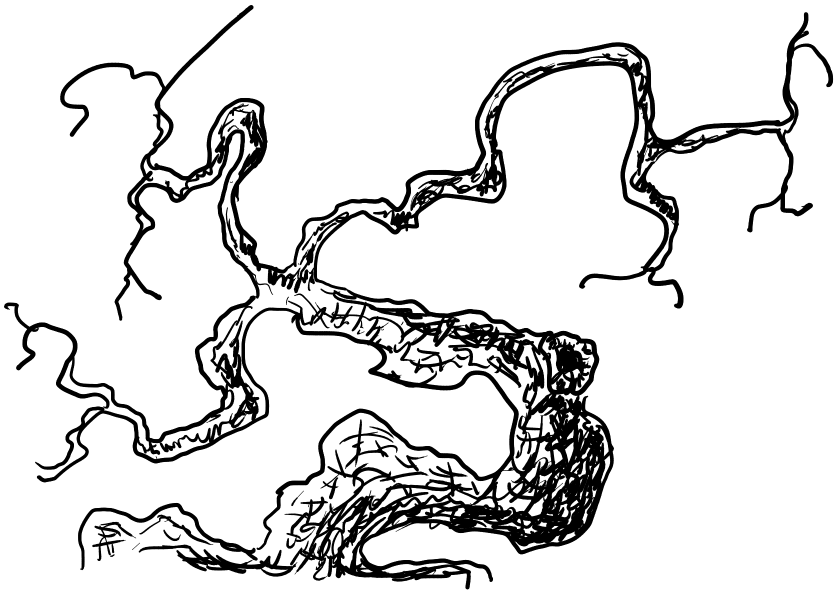
# COHOMOLOGICAL FILLINGS
# & GNARLEDNESS OF CHAIN COMPLEXES

I GNARLEDNESS OF SIMPLICIAL COMPLEXES

II COHOMOLOGICAL LIFTING

III GOTO I





04/2026

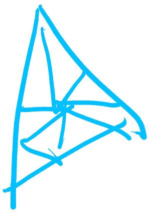
I

Context: quantitative null-cobordism

Theorem [Chambers, Dotterrer, Manin, Weinberger]

Let $X$ be a fin. simplicial complex, $k \in \mathbb{N}$.

Then: there ex. $K \in \mathbb{R}_{>0}$ s.t.:



For "every" subdivision $X'$ of $X$ and for every $\varphi \in \operatorname{im} \pi \subset H^k(X'; \mathbb{Q}/\mathbb{Z})$,

$\pi \colon H^k(X';\mathbb{Q}) \to H^k(X';\mathbb{Q}/\mathbb{Z})$ reduction hom

there ex. $\tilde{\varphi} \in H^k(X';\mathbb{Q})$ with:

$$\pi(\tilde{\varphi}) = \varphi \quad \text{and} \quad \|\tilde{\varphi}\|_\infty^\Delta \le K.$$

indep of $X'$ (!)

Sketch of proof:

1. def. gnarledness $G_k$ of fin s.c. via $k$-spanning trees
2. prove that $k$-spanning trees lead to a lifting estimate with "$K$" $= G_k(X') + k + 1$.
3. show that $G_k(X') \le G_k(X)$ via explicit constructions of $k$-spanning trees. □

# II

**Idea:** Turn the desired property into a def.
$\leadsto$ universal solution

**Setup:** $C_*$ : marked free $\mathbb{Z}$-chain complex
$k \in \mathbb{N}$
$$\pi : \underbrace{H^k(C_*;\mathbb{Q})}_{:= H^k(\mathrm{Hom}_{\mathbb{Z}}(C_*,\mathbb{Q}))} \longrightarrow H^k(C_*;\mathbb{Q}/\mathbb{Z}) \quad \text{reduction hom}$$

**Def (lifting volume).**

- For $\varphi \in \operatorname{im} \pi \subset H^k(C_*;\mathbb{Q}/\mathbb{Z})$ set
$$L_k(\varphi) := \inf_{\substack{\tilde{\varphi} \in H^k(C_*;\mathbb{Q}) \\ \pi(\tilde{\varphi}) = \varphi}} \|\tilde{\varphi}\|_\infty \in [0,\infty]$$
- $$L_k(C_*) := \sup_{\varphi \in \operatorname{im}\pi} L_k(\varphi) \in [0,\infty].$$

**Plan:**
1. Basic examples
2. "functoriality" / monotonicity
3. chain ext. estimate
4. finiteness

1. **Example(s):** If $\partial_k = 0 = \partial_{k+1}$, then
$$L_k(C_*) \le \frac{1}{2}.$$

(b) $C_*$: $\quad 0 \to \overset{2}{\mathbb{Z}} \to \overset{1}{\mathbb{Z}^2} \xrightarrow[0]{} \overset{0}{\mathbb{Z}} \to 0$

$1 \mapsto (d,1)$ with $d \in \mathbb{N}_{\geq 1}$.

Then: $L_1(C_*) = \frac{d}{2}$. $\leadsto$ $L_k$ not $\simeq$-inv!

(2) Prop (monotonicity). Let $f_*: C_* \otimes_{\mathbb{Z}} \mathbb{Q} \to D_* \otimes_{\mathbb{Z}} \mathbb{Q}$ be a chain map that induces a $\mathbb{Q}$-iso $H_*(f_*): H_*(C_*;\mathbb{Q}) \to H_*(D_*;\mathbb{Q})$ and $H_k(f_*)(H_k(C_*)_{\mathbb{Q}}) \subset H_k(D_*)_{\mathbb{Q}}$.

Then
$$L_k(C_*) \leq \|f_k\| \cdot L_k(D_*).$$

Proof. straightforward calc. $\square$

(3) Cor (chain ext. est.). Let $s: C_k \otimes_{\mathbb{Z}} \mathbb{Q} \to H_k(C_*;\mathbb{Q})$ be a chain extension,

$\widetilde{\ }$ $\mathbb{Q}$-lin and $\forall$ cycles $c \in C_k \otimes_{\mathbb{Z}} \mathbb{Q}$ $\quad s(c) = [c]$.

let $B \subset H_k(C_*;\mathbb{Q})$ be a cointegral $\mathbb{Q}$-basis. $\quad H_k(C_*)_{\mathbb{Q}} \subset \mathbb{Z}B$

Then:
$$L_k(C_*) \leq \frac{1}{2} \cdot \|s\|^B.$$

Proof: let $D_*$ be concentrated in deg $k$ with $D_k := \bigoplus_B \mathbb{Z}$.

Then: apply (2) to $s$ and then (1) (a). $\square$

(4) Cor (finiteness). If $\mathrm{rk}\, C_k < \infty$, then

$$L_k(C_*) < \infty .$$

Proof. We choose a chain extension and a $\mathbb{Z}$-integral basis (both ex.!) and apply (3). $\square$

In particular:

If $X$ is a finite simplicial complex, then:

$$L_k(C_*^\Delta(X)) < \infty .$$

III

Theorem. Let $X$ be a finite simpl. cx, and let $X'$ be a simpl. subdiv. of $X$, $k \in \mathbb{N}$. Then: For every $\varphi \in \operatorname{im} \pi \subset H^k(X'; \mathbb{Q}/\mathbb{Z})$ there ex. $\tilde{\varphi} \in H^k(X'; \mathbb{Q})$ with

$$\pi(\tilde{\varphi}) = \varphi \quad \text{and} \quad \|\tilde{\varphi}\|_\infty^\Delta \leq \underbrace{L_k(C_*^\Delta(X))}_{\substack{< \infty \\ (4) \\ \text{indep of } X'}}$$

Proof. Suffices to show:

$$L_k(C_*^\Delta(X')) \leq L_k(C_*^\Delta(X)) .$$

Because $X'$ is a subdiv. of $X$, there ex. a simplicial map $f : X' \to X$ s.t. $H_k(f;\mathbb{Q}) : H_k(X';\mathbb{Q}) \to H_k(X;\mathbb{Q})$ is a $\mathbb{Q}$-iso.

We apply monotonicity to $C_*^\Delta(f;\mathbb{Q})$:

$$L_k(C_*^\Delta(X')) \leq \underbrace{\|C_k^\Delta(f;\mathbb{Q})\|}_{\substack{\leq 1 \\ \text{because } f \text{ is simplicial.}}} \cdot L_k(C_*^\Delta(X)).$$

$\square$

Def. (quantitative of a chain cx). We set

$$G_k(C_*) := \inf_{\substack{s : C_k \otimes \mathbb{Q} \to H_k(C_*;\mathbb{Q}) \\ \text{chain ext.}}} \ \inf_{\substack{B \subset H_k(C_*;\mathbb{Q}) \\ \text{coint.} \\ \mathbb{Q}\text{-basis}}} \|s\|^B.$$

Rmk.
- $G_k(C_*^\Delta(X)) \leq G_k(X)$.
- $G_k(C_*)$ satisfies (2)
- $L_k(C_*) \underset{(3)}{\leq} \frac{1}{2} \cdot G_k(C_*) \leq \dim_{\mathbb{Q}} H_k(C_*;\mathbb{Q}) L_k(C_*)$.