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SIMPLICIAL VOLUME OF ONE-RELATOR GROUPS

joint work with Nicolaus Heuer

I ONE-RELATOR GROUPS II SCL III SIMPLICIAL VOLUME

I

one-relator group $\underbrace{\langle \overset{\text{set}}{S} \mid r \rangle}_{:= F(S) / \langle r \rangle^{\triangleleft}_{F(S)}}$ $r \in F := F(S)$

Examples:

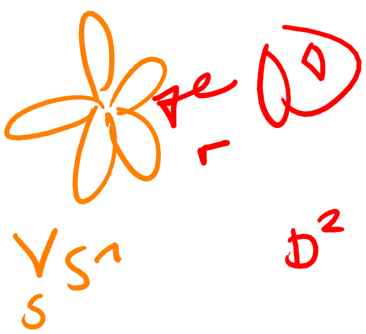
- $\langle a \mid a^2 \rangle \cong \mathbb{Z}/_2$
- $\langle a, b \mid [a,b] \rangle \cong \mathbb{Z}^2$
- ⚠ → $\langle a_1, \dots, a_g, b_1, \dots, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle \cong \pi_1(\Sigma_g)$
- Baumslag–Solitar groups

$\vdots$

Question: How are properties of $\langle S \mid r \rangle$ related to properties of r?

Well-known:

- If r is not a proper power, then $\langle S \mid r \rangle$ is torsion-free and the pres. complex is already a $K(\pi, 1)$ space. ($\leadsto \operatorname{cd} \langle S \mid r \rangle \leq 2$)



- If r is a proper power, then $\langle S \mid r \rangle$ is hyperbolic.
- If $r \in [F,F] \setminus \{e\}$, then $\langle S \mid r \rangle$ amenable $\iff \langle S \mid r \rangle \cong \mathbb{Z}^2$

- If $r \in [F,F] \setminus \{e\}$, then $H_2(\langle S \mid r \rangle; \mathbb{Z}) \cong \mathbb{Z}$ "gen by r"

Now: focus on $r \in [F,F] \setminus \{e\}$ and the invariants:
- stable commutator length $\mathrm{scl}_S\, r$ ← inv. of r
- simplicial volume $\| \langle S \mid r \rangle \|$ ← inv of $\langle S \mid r \rangle$

II SCL

Definition (scl). For $g \in F$, we set

$$d_S(g) := \inf \{ n \in \mathbb{N} \mid \exists_{x_1, \dots, x_n, y_1, \dots, y_n} \; g = [x_1,y_1] \cdots [x_n,y_n] \} \in \mathbb{N} \cup \{\infty\}.$$

(hard to compute! [Hmer])

$$\mathrm{scl}_S(g) := \inf_{n \in \mathbb{N}_{>0}} \frac{d_S(g^n)}{n} \in \cancel{\mathbb{R}_{\geq 0}}\, \mathbb{Q}_{\geq 0} \cup \{\infty\}$$

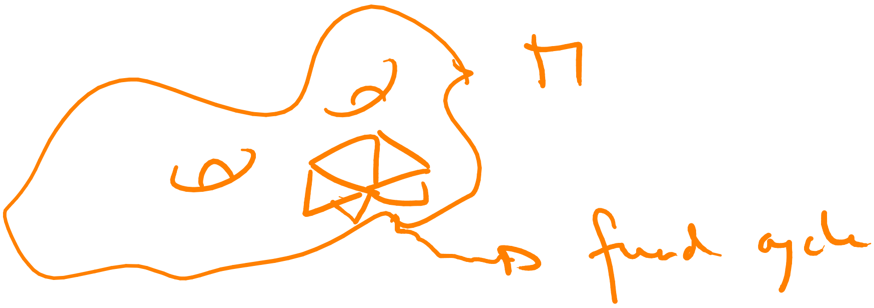
(can be computed efficiently [Calegari], i.e., there ex. a polynomial (in lg's) algorithm)

Known:
- gap: $\forall_{g \in [F,F] \setminus \{e\}} \quad \mathrm{scl}_S\, g \geq \frac{1}{2}$
- open problem: $\exists_{g \in [F,F]} \quad 1/2 < \mathrm{scl}_S\, g < \frac{7}{12}$?
- $\mathrm{scl}_{\{a_1, \dots, a_g, b_1, \dots, b_g\}} [a_1,b_1] \cdots [a_g,b_g] = g - \frac{1}{2}$.

III SIMPLICIAL VOLUME

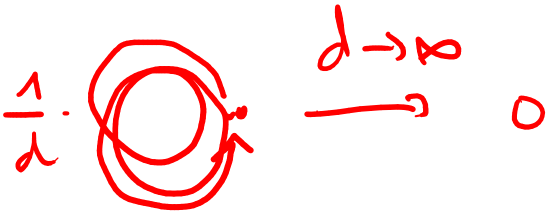
Definition. [Gromov] The *simplicial volume* of an occ n-mfd M is

$$\|M\| := \|[M]_{\mathbb{R}}\|_1 := \inf\Big\{\sum_j |a_j| \;\Big|\; \sum_j \overset{\in\mathbb{R}}{a_j}\cdot \overset{\Delta^n\to M}{\sigma_j} \in C_n(M;\mathbb{R}) \text{ fund cycle}\Big\}$$



Some properties:

- $\|S^1\| = 0$



⇝ $\|S^n\| = 0$ if $n>0$, tori, ...

- if M is hyperbolic, then $\|M\| = \dfrac{\operatorname{vol} M}{v_n} \neq 0$ [Gromov, Thurston]

⇝ $\|\Sigma_g\| = 4\cdot(g-1)$ if $g \geq \cancel{2}\,1$

if M is aspherical and $\pi_1(M)$ is hyp, then $\|M\| > 0$ [Mineyev]

- if $\pi_1(M)$ is amenable (and $\dim > 0$), then $\|M\| = 0$. [Gromov, Ivanov]

in general: hard to compute!

Definition [Heuer, L] Let $r \in [F,F]\setminus\{e\}$. Then

$$\|\langle S|r\rangle\| := \|\underbrace{[\langle S|r\rangle]_{\mathbb{R}}}_{\text{gen. of } H_2(\langle S|r\rangle;\, \mathbb{R}) \cong \mathbb{R}}\|_1 \in \mathbb{R}_{\geq 0}$$

$\bigvee_S S^1$

Example:

$$\|\langle \overbrace{a_1,\dots,a_g,b_1,\dots,b_g}^{S} \mid \overbrace{[a_1,b_1]\cdots[a_g,b_g]}^{r}\rangle\| \qquad \mathrm{scl}_S: g-\tfrac{1}{2}$$

$$= \|\Sigma_g\| = 4\cdot(g-1)$$

$$= 4\cdot\left(\mathrm{scl}_S\, r - \tfrac{1}{2}\right).$$

Question.

- When do we have

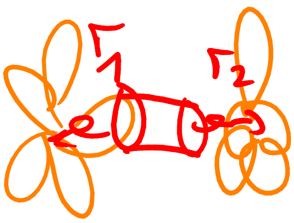
$$\boxed{*} \quad \|\langle S|r\rangle\| = 4\cdot(\underbrace{\mathrm{scl}_S\, r - \tfrac{1}{2}}_{\text{gap } \geq 0}) \qquad ?$$

- (How) Can $\|\langle S|r\rangle\|$ be computed ?

Some answers [Heuer, L]

- if r is a proper power, then $\langle S|r\rangle$ is hyperbolic and so $\|\langle S|r\rangle\| > 0$
- $\boxed{*}$ holds if r is decomposable:

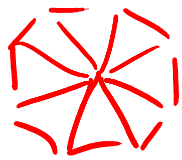
$$\underbrace{r = r_1\cdot r_2}_{\text{diff gens and in } [F,F]} \qquad \text{or} \qquad \underbrace{r = r_1\cdot t\cdot r_2\, t^{-1}}_{r_1, r_2 \text{ do not contain } t}$$



$\rightsquigarrow$ mod 1, we can realise all of $\mathbb{Q}_{\geq 0}$ as $\|\langle S|r\rangle\|$.

- always: $\|\langle S \mid r\rangle\| \cancel{\neq} \; < \; 4 \cdot \mathrm{scl}_S\, r$

$\bigvee_S S^1$



$4g$ gon

$r = \prod_g [\cdot,\cdot]$

- asymptotically $\boxed{*}$ holds:

use surfaces + combinatorics

$$\lim_{N \to \infty} \frac{\|\langle S \mid r^N \rangle\|}{N} = 4 \cdot \mathrm{scl}_S\, r .$$

- $\|\langle S \mid r\rangle\|$ and $4 \cdot \mathrm{scl}_S\, r$ have similar distributions

  (use scl-results of Calegari–Walker)

- There is a function lallop $: [F,F] \to \mathbb{Q}_{\geq 0}$

  (similar to scallop by Calegari for scl)

  with
  - $\mathrm{lallop}(r) \leq \|\langle S \mid r\rangle\|$
  - $\mathrm{lallop}(r) \leq 4 \cdot (\mathrm{scl}_S\, r - \frac{1}{2})$
  - admits an algo in polynomial time

    linear programming
  - in "many" examples: $\mathrm{lallop}(r) = \|\langle S \mid r\rangle\|$

Example that does not satisfy $\boxed{*}$:

$$r = a^4 b a^{-1} b^{-1} a^{-1} b a b^{-1} a^{-2} b a^{-1} b^{-1}$$

$\mathrm{scl}_{\{a,b\}}\, r = \frac{5}{8}$ but $\|\langle S \mid r\rangle\| = 2$.