Simplicial volume and L^2-Betti numbers

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Abstract. Gromov conjectured the following deep connection between two at first glance quite different topological invariants: *If the simplicial volume of an aspherical oriented closed connected manifold is zero, then all its L^2-Betti numbers vanish.* When the simplicial volume is replaced by the integral foliated simplicial volume, the corresponding statement is true. Hence it is interesting to study the relation between the simplicial volume and the integral foliated simplicial volume of aspherical manifolds. In these notes, we focus on the multiplicativity of these invariants under finite coverings. This provides us with a proof that the simplicial volume and the integral foliated simplicial volume coincide in the world of oriented closed connected surfaces.

1 The name of the game

The following conjecture of Gromov [4; p. 232] suggests a deep connection between the simplicial volume and L^2-Betti numbers of aspherical manifolds:

Conjecture (Gromov). *If M is an aspherical oriented closed connected manifold with $\|M\| = 0$, then all L^2-Betti numbers of M vanish. (In particular, $\chi(M) = 0$).*

The conjecture is mainly interesting, because it yields a link between two at first sight quite different invariants, but it would also make it possible to carry some results concerning the simplicial volume over to the world of L^2-invariants.

1.1 Simplicial volume

The simplicial volume is a topological invariant measuring the complexity of a manifold introduced by Gromov (in his proof of Mostow rigidity). Moreover, the simplicial volume of Riemannian manifolds is in some sense a topological approximation of the volume.

Let M be an occ n-manifold. Then the n-th singular homology group

occ: *oriented closed connected*

$$H_n(M, \mathbf{Z}) = \frac{\ker\bigl(\partial \colon C_n(M, \mathbf{Z}) \to C_{n-1}(M, \mathbf{Z})\bigr)}{\ker\bigl(\partial \colon C_{n+1}(M, \mathbf{Z}) \to C_n(M, \mathbf{Z})\bigr)}$$

is generated by the so-called **fundamental class** $[M]$ of M. All cycles in $C_n(M, \mathbf{Z})$ representing $[M]$ are called **fundamental cycles** of M.

The image in $H_n(M, \mathbf{R}) \cong \mathbf{R}$ of $[M]$ is also a generator and denoted by $[M]_{\mathbf{R}}$. Furthermore, all cycles in $C_n(M, \mathbf{R})$ representing the class $[M]_{\mathbf{R}}$ are called fundamental cycles of M.

Definition (1.1). The **simplicial volume** of an occ n-manifold M is defined as

$$\|M\| := \Bigl\{ \sum_{j \in J} |a_j| \Bigm| \sum_{j \in J} a_j \cdot \sigma_j \in C_n(M, \mathbf{R}) \text{ is a fundamental cycle} \Bigr\}. \qquad \diamond$$

For example, if M can be triangulated, then the simplicial volume of M is at most the minimal number of simplices needed in a triangulation of M.

1.2 L^2-Betti numbers

The L^2-Betti numbers are an extension of ordinary Betti numbers to "large" topological spaces, taking advantage of a group action.

Recall that if X is a topological space with fundamental group G, then the homology of $\widetilde{X}$ with local coefficients in the right $\mathbf{Z}G$-module R is the homology of the twisted chain complex

$$C_*^G(\widetilde{X}, R) := R \otimes_{\mathbf{Z}G} C_*(\widetilde{X}, \mathbf{Z}),$$

equipped with the boundary operator $\mathrm{id} \otimes \partial$.

For example, homology of $\widetilde{X}$ with local coefficients in the trivial G-module $\mathbf{R}$ is just homology of X with real coefficients. On the other hand, homology of $\widetilde{X}$ with local coefficients in $\mathbf{R}G$ coincides with the homology of $\widetilde{X}$ with real coefficients.

Of course, the Betti numbers of $\widetilde{X}$ are in general infinite. However, if we replace the homology of $\widetilde{X}$ with homology with local coefficients in a special $\mathbf{Z}G$-module, then we obtain finite dimensions that also encode some information about the action of the fundamental group on the universal covering space:

Definition (1.2). Let X be a topological space with fundamental group G, and let $k \in \mathbf{N}$. Then the **k-th L^2-Betti number** of X is given by

$$b_k^{(2)}(X) := b_k^{(2)}(\widetilde{X}, \mathcal{N}G) := \dim_{\mathcal{N}G} H_k^G(\widetilde{X}, \mathcal{N}G).$$

Here $H_k^G(\widetilde{X}, \mathcal{N}G)$ denotes the k-th homology of $\widetilde{X}$ with local coefficients in the group von Neumann algebra $\mathcal{N}G$, and $\dim_{\mathcal{N}G}$ is the dimension function corresponding to $\mathcal{N}G$ defined by Lück [7]. $\diamond$

This definition can easily be generalised to more general G-spaces than universal covering spaces [7].

One of the most important aspects of L^2-Betti numbers is that they can be used to compute the Euler characteristic [7]: If X is a topological space with finite homology, then

$$\chi(X) = \sum_{k \in \mathbf{N}} b_k^{(2)}(X).$$

Moreover, if one believes the Singer conjecture, then Gromov's conjecture is equivalent to the following conjecture:

Conjecture. *If the simplicial volume of an aspherical occ manifold vanishes, then its Euler characteristic must be zero.*

1.3 Status of Gromov's conjecture

To understand why Gromov's conjecture is plausible, we have a look at the common properties of simplicial volume and L^2-Betti numbers of (aspherical) manifolds:

Some reasons to believe Gromov's conjecture.

- The simplicial volume and L^2-Betti numbers are multiplicative under finite coverings. We investigate this property in more detail in Section 3.
- More general, they both satisfy a *proportionality principle*:

  If M and N are occ Riemannian manifolds with isometric universal Riemannian coverings, then [11, 3, 10, 7]

$$\frac{\|M\|}{\operatorname{vol} M} = \frac{\|N\|}{\operatorname{vol} N},$$
$$\forall_{k \in \mathbf{N}} \quad \frac{b_k^{(2)}(M)}{\operatorname{vol} M} = \frac{b_k^{(2)}(N)}{\operatorname{vol} N}.$$

  In particular, for all flat occ manifolds, both the simplicial volume and the L^2-Betti numbers must vanish.
- The simplicial volume and L^2-Betti numbers of smooth manifolds are bounded (by some multiple of) the minimal volume [7, 3, 9].
- The simplicial volume of occ manifolds with amenable fundamental group vanishes [3]. Moreover, if M is an aspherical CW-complex with amenable fundamental group, then all L^2-Betti numbers of M are zero [7].
- If M is a smooth occ manifold with a non-trivial S^1-action, then $\|M\| = 0$ [3, 12]. On the other hand, if M is an aspherical occ smooth manifold with non-trivial S^1-action, then also all L^2-Betti numbers of M vanish [7; Corollary 1.43].

Some variations of Gromov's conjecture.

- Of course, the asphericity condition cannot be easily dropped: For example $\|S^2\| = 0$, but $b_0^{(2)}(S^2) = 1 \neq 0$.
- The converse of the conjecture does not hold. That it is wrong in odd dimensions is predicted by the Singer conjecture. However, it is also possible to construct counterexamples in even dimensions: Let M be a hyperbolic occ manifold fibering over the circle S^1 (such manifolds exist for example in dimension 3). Then the product $M \times M$ has non-zero simplicial volume and vanishing L^2-Betti numbers.

A connected subset $Y \subset X$ is amenable, *if the image of the homomorphism $\pi_1(Y) \to \pi_1(X)$ induced by the inclusion is an amenable group. A subset $Y \subset X$ is* amenable, *if all its connected components are amenable.*

Some consequences of Gromov's conjecture.

- If Gromov's conjecture is true, then the following result of Ivanov [6] could be carried over to L^2-Betti numbers of aspherical manifolds:

  If M is an occ n-manifold admitting an amenable covering such that any point of M is contained in at most n sets of this covering, then $\|M\| = 0$.

  Sauer is currently working on a direct proof of the corresponding result for L^2-Betti numbers [9].
- If M is an occ manifold admitting a selfmap of non-trivial degree, then obviously $\|M\| = 0$. If Gromov's conjecture is true, then also the L^2-Betti numbers of aspherical occ manifolds with selfmaps of non-trivial degree would be zero. At the time of writing this is only known in special cases [7; Theorem 14.40].

2 A possible(?) strategy

The key idea to approach Gromov's conjecture is to introduce an invariant similar to the simplicial volume for which the corresponding statement is more accessible – the integral foliated simplicial volume. It then remains to investigate the relation between the simplicial volume and the integral foliated simplicial volume of aspherical manifolds.

2.1 The integral simplicial volume and (L^2-)Betti numbers

If in the definition of the simplicial volume of an occ manifold M, we only allow fundamental cycles with integral coefficients, we obtain the **integral simplicial volume of** M, denoted by $\|M\|_\mathbb{Z}$.

Of course, we always have

$$\|M\| \leq \|M\|_\mathbb{Z}$$

and $\|M\|_\mathbb{Z} \geq 1$. In particular, in general, $\|M\| \neq \|M\|_\mathbb{Z}$.

On the other hand, using Poincaré duality, one easily shows the following estimate for the Betti numbers [7]:

Theorem (2.1). *Let M be an occ n-manifold. Then*

$$\sum_{p \in \mathbf{N}} \dim H_p(M, \mathbf{R}) \leq (n+1) \cdot \|M\|_{\mathbf{Z}}.$$

Clearly, the corresponding statement for the simplicial volume would be false.

Proof. The proof is essentially the same as in Lück's book – we merely count simplices a little more carefully:

Let $\sum_{j=0}^{k} a_j \cdot \sigma_j \in C_n(M, \mathbf{Z})$ be a fundamental cycle of M. For $p \in \{0, \dots, n\}$, we consider the set

$$S_p := \{ {}_p\lfloor \sigma_j \mid j \in \{1, \dots, k\} \}$$

of all p-back faces of the simplices $\sigma_1, \dots, \sigma_k$. Poincaré duality yields

$$b_p(M) \leq |S_p|.$$

Proof. Let $\varphi = [f] \in H_p(M, \mathbf{R})$ with $f|_{S_p} = 0$. By definition of the cap product, we obtain

$$\varphi \cap [M] = \left[\sum_{j=1}^{k} a_j \cdot f({}_p\lfloor \sigma_j) \cdot \sigma_j \rfloor_{n-p} \right] = [0].$$

Hence Poincaré duality shows $\varphi = 0 \in H_p(M, \mathbf{R})$. Therefore, the p-th Betti number is bounded by the cardinality of S_p. □

On the other hand, we clearly have $|S_p| \leq k$, which gives the result. □

Using equivariant Poincaré duality, the same argument can be applied in the setting of L^2-Betti numbers [8]:

Theorem (2.2). *Let M be an occ n-manifold. Then*

$$\sum_{k \in \mathbf{N}} b_k^{(2)}(M) \leq (n+1) \cdot \|M\|_{\mathbf{Z}}.$$

However, since the integral simplicial volume is too rigid, we have to use a larger but not too large coefficient ring instead of $\mathbf{Z}$. In the next subsection, we introduce a candidate for such a ring, using measure theory and the group action of the fundamental group on the universal covering.

2.2 The integral foliated simplicial volume and its relation to L^2-Betti numbers

Based on ideas of Gromov [5; Section 5.38], Schmidt [8] defined the integral foliated simplicial volume as follows:

A standard G-space is a standard probability space with a measure preserving G-action by Borel automorphisms

Definition (2.3). Let M be an occ n-manifold with fundamental group G. Then the **integral foliated simplicial volume** of M is defined by

$$|M| := \inf\{ |M|^X \mid [X] \text{ is an isomorphism class of standard } G\text{-spaces} \},$$

where for any standard G-space (X, μ)

$$|M|^X := \inf\biggl\{ \sum_{j \in J} \int_X |f_j| \, d\mu \biggm| \sum_{j \in J} f_j \otimes \sigma_j \in L^\infty(X, \mathbf{Z}) \otimes_{\mathbf{Z}G} C_*(\widetilde{M}, \mathbf{Z}) \text{ is a fundamental cycle of } M \biggr\}. \qquad \diamond$$

The measure spaces involved provide us with some flexibility and allow us to "divide" some cycles, as we are used to in the world of real or rational coefficients. Hence the integral foliated simplicial volume is not as rigid as the integral simplicial volume.

Theorem (2.4). *Let M be an occ manifold of dimension n with fundamental group G. Then*

$$\sum_{k \in \mathbf{N}} b_k^{(2)}(M) \leq (n+1) \cdot |M|^X$$

holds for all standard G-spaces. In particular

$$\sum_{k \in \mathbf{N}} b_k^{(2)}(M) \leq (n+1) \cdot |M| .$$

Proof. The proof given by Schmidt [8] relies on the fact that the L^2-Betti numbers of M can also be computed using the von Neumann algebra of the standard equivalence relation induced by the G-action on X (instead of the group von Neumann algebra $\mathcal{N}G$) [8, 2], and then takes advantage of Poincaré duality as in the proof of the estimate

$$\sum_{k \in \mathbf{N}} b_k(M) \leq \|M\|_{\mathbf{Z}}$$

for ordinary Betti numbers and the integral simplicial volume (Theorem 2.1). $\square$

2.3 Known properties of the integral foliated simplicial volume

- Let M be an occ manifold with fundamental group G and let X be a standard G-space. Then the inclusion $C_*(M, \mathbf{Z}) \hookrightarrow L^\infty(X, \mathbf{Z}) \otimes_{\mathbf{Z}G} C_*(\widetilde{M}, \mathbf{Z})$ shows $|M|^X \leq \|M\|_{\mathbf{Z}}$. In particular,

  $$|M| \leq \|M\|_{\mathbf{Z}} .$$

  On the other hand, integration of the coefficients yields $\|M\| \leq |M|^X$ and thus

  $$\|M\| \leq |M| .$$

  In general, both of these inequalities are strict.

- The integral foliated simplicial volume is compatible with products in the following sense: For each $n \in \mathbf{N}$ there is a constant $c(n) \in \mathbf{R}_{>0}$ such that for all occ manifolds M and N

  $$|M \times N| \leq c(\dim M + \dim N) \cdot |M| \cdot |N| .$$

  This can be seen by inspecting the cross product on homology [8].

- Whenever X is an ergodic Z-space, then

  $$\lvert S^1 \rvert^X = 0.$$

  In particular, the integral foliated simplicial volume of tori is always zero [8].

- If M is a simply connected occ manifold, then [8]

  $$\lvert M \rvert = \|M\|_{\mathbf{Z}}.$$

- As we will see in the following part, the integral foliated simplicial volume is multiplicative with respect to finite coverings (Theorem (3.2)). This implies that the integral foliated simplicial volume and the simplicial volume of hyperbolic occ surfaces coincide (Corollary (3.5)).

3 The (intregal foliated) simplicial volume of finite coverings

In this section, we give a proof of multiplicativity of the (integral foliated) simplicial volume under finite coverings and present an application to hyperbolic occ surfaces.

Using covering theory, it is not hard to show that the simplicial volume is multiplicative under finite coverings:

Proposition (3.1). *If $p\colon N \longrightarrow M$ is a finite covering of occ manifolds, then*

$$\|M\| = \frac{1}{|\deg p|} \cdot \|N\|.$$

Proof. For brevity, we write $d := 1/\deg p$. Why does "$\leq$" hold? Let $c := \sum_{j=0}^{m} a_j \cdot \sigma_j \in C_*(N, \mathbf{R})$ be a fundamental cycle of N. Then

$$\begin{aligned}\left[\frac{1}{d}\sum_{j=0}^{m} a_j p \circ \sigma_j\right] &= \frac{1}{d} \cdot H_n(p, \mathbf{R})([N]_{\mathbf{R}}) \\ &= \frac{1}{d} \cdot \deg p \cdot [M]_{\mathbf{R}},\end{aligned}$$

which shows that

$$\|M\| \leq \frac{1}{|\deg p|} \cdot \|c\|_1.$$

Taking the infimum over all fundamental cycles of N proves

$$\|M\| \leq \frac{1}{|\deg p|} \cdot \|N\|.$$

Conversely, let $c := \sum_{j=0}^{m} a_j \cdot \sigma_j \in C_*(M, \mathbf{R})$ be a fundamental cycle of M. Then

$$\sum_{j=0}^{m} a_j \cdot \sum_{\tau \in C_n(p,\mathbf{R})^{-1}(\sigma_j)} \tau$$

is a fundamental cycle of N. Thus

$$\|c\|_1 \geq \frac{1}{|\deg p|} \cdot \|N\|,$$

and taking the infimum over all fundamental cycles of M yields

$$\|M\| \geq \frac{1}{|\deg p|} \cdot \|N\|. \qquad \square$$

For the integral foliated simplicial volume the proof of the corresponding statement is a bit more complicated: we have to implement the division by $|\deg p|$ into the measure theory of the parameter spaces. To get a geometric intuition about the parametrised fundamental cycles in $L^\infty(X, \mathbf{Z}) \otimes_{\mathbf{Z}G} C_n(\widetilde{M}, \mathbf{Z})$ it is convenient (yet not entirely correct) to think of them as a G-equivariant "triangulation" of $X \times \widetilde{M}$, or as a "triangulation" of a G-fundamental domain of $X \times \widetilde{M}$.

Theorem (3.2). *If $p\colon N \longrightarrow M$ is a finite covering of occ manifolds, then*

$$|M| = \frac{1}{|\deg p|} \cdot |N|.$$

Proof. We write G and H for the fundamental groups of M and N respectively. The theorem is clearly a consequence of the following two estimates:

1. For all standard H-spaces X, we have

   $$|M|^{G \times_H X} = \frac{1}{|\deg p|} \cdot |N|^X.$$

   We obtain the standard G-space structure on $G \times_H X$ by using a bijection

   $$G \times_H X \longrightarrow G/H \times X.$$

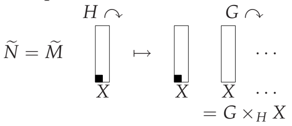
2. For all standard G-spaces X,

   $$|M|^X \cdot |\deg p| \geq |N|^{\operatorname{res}_H^G X}.$$

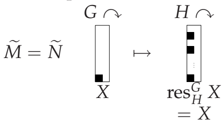
How can we prove these estimates? For the first part, we show that the isomorphism

$$\begin{aligned} L^\infty(G \times_H X, \mathbf{Z}) \otimes_{\mathbf{Z}G} C_*(\widetilde{M}, \mathbf{Z}) &\cong L^\infty(X, \mathbf{Z}) \otimes_{\mathbf{Z}H} \mathbf{Z}G \otimes_{\mathbf{Z}G} C_*(\widetilde{M}, \mathbf{Z}) \\ &\cong L^\infty(X, \mathbf{Z}) \otimes_{\mathbf{Z}H} C_*(\widetilde{N}, \mathbf{Z}) \end{aligned}$$

First part:



Second part:



of chain complexes maps parametrised fundamental cycles to parametrised fundamental cycles and is compatible with the respective norms (up to a factor of $|\deg p|$).

For the second part, we consider the chain map

$$L^\infty(X, \mathbf{Z}) \otimes_{\mathbf{Z}G} C_*(\widetilde{M}, \mathbf{Z}) \longrightarrow L^\infty(\operatorname{res}^G_H X, \mathbf{Z}) \otimes_{\mathbf{Z}H} C_*(\widetilde{N}, \mathbf{Z})$$

given by summing up all $\widetilde{p}$-lifts of a singular simplex of $\widetilde{M}$ to $\widetilde{N}$, where $\widetilde{p} \colon \widetilde{N} \longrightarrow \widetilde{M}$ is some lift of p. □

Corollary (3.3). *If the occ manifold M admits a self-covering p with $|\deg p| \geq 2$, then*

$$\mathbf{|} M \mathbf{|} = 0 = \|M\|. \qquad \square$$

This can, for example, applied to tori.

3.1 The (integral foliated) simplical volume of surfaces

If F is an occ surface, then either F is a torus, or F is hyperbolic. Since tori admit self-coverings of non-trivial degree, we obtain

$$\mathbf{|} F \mathbf{|} = 0 = \|F\|$$

if F is a torus.

Using the multiplicativity under finite coverings, we can also compute the (integral foliated) simplicial volume of hyperbolic occ surfaces [1, 5]:

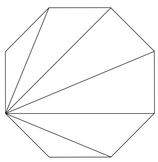
Theorem (3.4). *Let F be a hyperbolic occ surface of genus $g(F)$. Then*

$$\|F\| = 4 \cdot g(F) - 4.$$

Proof. Using Proposition (3.1), we show

$$\|F\| \leq 4 \cdot g(F) - 4.$$

Standard triangulation:



For any $d \in \mathbf{N}$ there is a d-sheeted covering $M_d \longrightarrow F$. Clearly, M_d is also a hyperbolic occ surface with genus $d \cdot g(F) - d + 1$. Therefore, the standard triangulation of the regular $4 \cdot g(M_d)$-gon shows

$$\|M_d\|_{\mathbf{Z}} \leq 4 \cdot \bigl(d \cdot g(F) - d + 1\bigr) - 2.$$

Hence, using the multiplicativity of the simplicial volume under finite coverings (Proposition (3.1)), we obtain

$$\begin{aligned} \|F\| &\leq \inf_{d \to \infty} \frac{1}{d} \cdot \|M_d\| \\ &\leq \inf_{d \to \infty} \frac{1}{d} \cdot 4 \cdot \bigl(d \cdot g(F) - d + 1\bigr) - 2 \\ &= 4 \cdot g(F) - 4, \end{aligned}$$

as desired.

Conversely, let $c := \sum_{j=0}^{m} a_j \cdot \sigma_j \in C_2(F, \mathbf{R})$ be a *straight* fundamental cycle of F, i.e., the simplices are projections under the universal covering map π_F of geodesic simplices in the hyperbolic plane $\mathbf{H}^2$. Writing vol_F for the hyperbolic volume form on F, we obtain

$$\begin{aligned}
\pi \cdot \bigl(4 \cdot g(F) - 4\bigr) &= 2\pi \cdot \chi(F) \\
&= \operatorname{vol} F \\
&= \int_c \operatorname{vol}_F \\
&= \sum_{j=0}^{m} a_j \cdot \int_F \sigma_j^* \operatorname{vol}_F \\
&\leq \sum_{j=0}^{m} |a_j| \cdot \int_F \sigma_j^* \pi_F^* \operatorname{vol}_{\mathbf{H}^2} .
\end{aligned}$$

Since the σ_j are straight, the last integral is bounded by π. Hence

$$4 \cdot g(F) - 4 \leq \|c\|_1 .$$

It is not hard to see that the infimum of the ℓ^1-norms of all straight fundamental cycles gives rise to the simplicial volume. Hence

$$4 \cdot g(F) - 4 \leq \|F\| . \qquad \square$$

Theorem (3.5). *For all hyperbolic occ surfaces F, we obtain*

$$\|F\| = |F| .$$

Proof. Let F be a hyperbolic occ surface of genus $g(F)$. Then (Theorem (3.4))

$$|F| \geq \|F\| = 4 \cdot g(F) - 4,$$

and thus it suffices to show the reverse inequality $|F| \leq 4 \cdot g(F) - 4$. To this end, we use the same idea as in the calculation of $\|F\|$:

For any $d \in \mathbf{N}$ there is a d-sheeted covering $M_d \longrightarrow F$. Clearly, M_d is also a hyperbolic occ surface with genus $d \cdot g(F) - d + 1$. Therefore, the standard triangulation of the $4 \cdot g(M_d)$-gon shows

$$|M_d|_{\mathbf{Z}} \leq 4 \cdot \bigl(d \cdot g(F) - d + 1\bigr) - 2.$$

Hence, using the multiplicativity of the integral foliated simplicial volume under finite coverings (Theorem (3.2)), we obtain

$$\begin{aligned}
|F| &\leq \lim_{d \to \infty} \frac{1}{d} \cdot |M_d| \\
&\leq \lim_{d \to \infty} \frac{1}{d} \cdot 4 \cdot \bigl(d \cdot g(F) - d + 1\bigr) - 2 \\
&= 4 \cdot g(F) - 4,
\end{aligned}$$

as desired. $\square$

4 Future work

- It would be nice to compute the integral foliated simplicial volume for more examples, such as S^1-principal bundles and hyperbolic manifolds.
- The main tool for computing the simplicial volume is bounded cohomology. It would be interesting to come up with a similar machinery for the integral foliated simplicial volume of aspherical occ manifolds. This would possibly lead to a proportionality principle for the integral foliated simplicial volume and to a vanishing result for aspherical occ manifolds with amenable fundamental group.
- It is not clear yet, how the change of the parameter space "X" affects the value of the corresponding norm. For example, it could be possible that the integral foliated simplicial volume is attained by some X.
- At least for aspherical occ manifolds with residually finite fundamental group it should be possible to compute the integral foliated simplicial volume in terms of the integral simplicial volume of the finite coverings.
- Moreover, it would be interesting to show that the integral foliated simplicial volume of aspherical occ manifolds with a selfmap of non-trivial degree is zero.
- It is clear that the obvious sum formula for the integral foliated simplicial volume of connected sums cannot hold (because the integral foliated simplicial volume is non-zero if the Euler characterstic is non-zero). It would be interesting to calculate the error term.

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