Representation theory of Lie algebras

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Abstract. In these notes, we give a brief overview of the (finite dimensional) representation theory of finite dimensional semisimple Lie algebras. We first study the example of $\mathfrak{sl}_2(\mathbb{C})$ and then provide the general (additive) theory, along with an analysis of the representations of $\mathfrak{sl}_3(\mathbb{C})$. In the last section, we have a look at the multiplicative structure of the representation ring, discussing examples for the Lie algebras $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{sl}_3(\mathbb{C})$. The main source for these notes is the book *Representation Theory, A First Course* by William Fulton and Joe Harris.

The name of the game

In this section, we present the fundamental terminology and notation used in the sequel. In particular, we introduce representations of Lie algebras (Subsection 1.2) and some basic constructions (Subsection 1.4).

1.1 Lie algebras

The theory of Lie algebras is the linear algebraic counterpart of the (rather geometric) theory of Lie groups.

Definition (1.1). A **Lie algebra** is a finite dimensional complex vector space $\mathfrak{g}$, equipped with a skew-symmetric bilinear map

$$[\,\cdot\,,\,\cdot\,]\colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g},$$

the so-called **Lie bracket**, satisfying the **Jacobi identity**, i.e., for all $x, y \in \mathfrak{g}$,

$$\big[x, [y,z]\big] + \big[y, [z,x]\big] + \big[z, [x,y]\big] = 0.$$

A **homomorphism of Lie algebras** is a homomorphism $\varphi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ of complex vector spaces that is compatible with the Lie brackets, i.e., for all $x, y \in \mathfrak{g}$ we have

$$\varphi\big([x,y]_{\mathfrak{g}}\big) = \big[\varphi(x), \varphi(y)\big]_{\mathfrak{h}}. \qquad \diamond$$

Example (1.2). Let V be a finite dimensional complex vector space. Then the trivial Lie bracket $[\,\cdot\,,\,\cdot\,] = 0$ turns V into a Lie algebra. Lie algebras of this type are called **Abelian**. ■

Example (1.3). The main source of Lie algebras are matrix algebras:

- Let V be a finite dimensional complex vector space. Then the set of endomorphsisms of V is a Lie algebra, when endowed with the Lie bracket

  $$\begin{aligned} \operatorname{End} V \times \operatorname{End} V &\longrightarrow \operatorname{End} V \\ (A, B) &\longrightarrow A \circ B - B \circ A. \end{aligned}$$

  We denote this Lie algebra by $\mathfrak{gl}(V)$. Moreover, we use the notation

  $$\mathfrak{gl}_n(\mathbf{C}) := \mathfrak{gl}(\mathbf{C}^n),$$

  and view the elements of $\mathfrak{gl}_n(\mathbf{C})$ as matrices rather than endomorphisms.

- Let $n \in \mathbf{N}$. Then the traceless matrices form a subalgebra of $\mathfrak{gl}_n(\mathbf{C})$, denoted by $\mathfrak{sl}_n(\mathbf{C})$. ■

Example (1.4). If G is a Lie group, then the tangent space T_eG at the unit element can be endowed with a Lie algebra structure (using the Lie derivative of vector fields or the derivative of the adjoint representation of G [2; Section 8.1]). ■

Definition (1.5). A Lie algebra is **simple**, if it contains no non-trivial ideals. Non-trivial Lie algebras that can be decomposed as a direct product of simple Lie algebras are called **semisimple**. $\diamond$

Simple Lie algebras can be classified by means of Dynkin diagrams, a purely combinatorial tool [2; Chapter 21].

Example (1.6). For example, the Lie algebras $\mathfrak{sl}_n(\mathbf{C})$ are all semisimple (they are even simple Lie algebras) [2; Chapter 21]. ■

1.2 Representations of Lie algebras

Like any algebraic object, Lie algebras can be represented on vector spaces. Clearly, these actions should be compatible with the Lie algebra structure, leading to the following definition.

Definition (1.7). Let $\mathfrak{g}$ be a Lie algebra.

- A **representation of the Lie algebra** $\mathfrak{g}$ is a (finite dimensional) complex vector space V together with a homomorphism $\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ of Lie algebras.
- A **subrepresentation** of a representation $\varrho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ consists of a subspace W satisfying
$$\forall_{x\in\mathfrak{g}} \quad \bigl(\varrho(x)\bigr)(W) \subset W.$$
- A representation of the Lie algebra $\mathfrak{g}$ is called **irreducible**, if it contains no proper subrepresentations.
- A **homomorphism of representations** $\varrho_V\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ and $\varrho_W\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(W)$ of the Lie algebra $\mathfrak{g}$ is a linear map $\varphi\colon V \longrightarrow W$ such that
$$\varphi \circ \bigl(\varrho_V(x)\bigr) = \bigl(\varrho_W(x)\bigr) \circ \varphi$$
holds for all $x \in \mathfrak{g}$. ◇

$$\begin{tikzcd} V \arrow[r, "\varrho_V(x)"] \arrow[d, "\varphi"'] & V \arrow[d, "\varphi"] \\ W \arrow[r, "\varrho_W(x)"'] & W \end{tikzcd}$$

Clearly, a representation $\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is the same as linear map $\varphi\colon \mathfrak{g} \otimes V \longrightarrow V$ satisfying
$$\varphi\bigl([x,y] \otimes v\bigr) = \varphi\bigl(x \otimes \varphi(y \otimes v)\bigr) - \varphi\bigl(y \otimes \varphi(x \otimes v)\bigr)$$
for all $x, y \in \mathfrak{g}$ and all $v \in V$. We will sometimes als make use of this description and write $x \cdot v := \varphi(x \otimes v)$ for $x \in \mathfrak{g}$ and $v \in V$.

Example (1.8). Let $\mathfrak{g}$ be a Lie algebra. Then
$$\begin{aligned} \mathrm{ad}\colon \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{g}) \\ x &\longmapsto \bigl(y \mapsto [x,y]\bigr) \end{aligned}$$
defines a representation of $\mathfrak{g}$ on itself, the **adjoint representation of** $\mathfrak{g}$. ■

Example (1.9). Let V be a finite dimensional complex vector space and let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a subalgebra. Then the **standard representation of** $\mathfrak{g}$ is given by
$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathfrak{gl}(V) \\ x &\longmapsto x. \end{aligned}$$
■

With this terminology, the goal of this talk can be summarised as follows: *We want to classify (up to isomorphism) all finite dimensional representations of finite dimensional semisimple Lie algebras.*

Lie algebras	*Lie groups*
Lie algebra homomorphism	Lie group homomorphism
ideal	normal subgroup
Cartan subalgebra	maximal torus
Lie algebra representation	Lie group representation

Figure 1: Translation between Lie algebras and Lie groups

1.3 Travelling between the Lie group and the Lie algebra universe

Using the usual translation mechanisms (i.e., differentiating and the exponential map) between Lie groups and Lie algebras (cf. Figure 1), we obtain Lie algebra representations out of Lie group representations (and vice versa):

$$\begin{tikzcd} \mathfrak{g} = T_eG \arrow[r, "T_e\varphi"] \arrow[d, "\exp"'] & \mathrm{End}(V) = \mathfrak{gl}(V) \arrow[d, "\exp"] \\ G \arrow[r, "\varphi"'] & \mathrm{Aut}(V) \end{tikzcd}$$

Let G be a Lie group and let $\mathfrak{g} := T_eG$ be the corresponding Lie algebra. If $\varphi\colon G \longrightarrow \mathrm{Aut}(V)$ is a representation of G, then the derivative $T_e\varphi\colon \mathfrak{g} \longrightarrow \mathrm{End}(V)$ is a representation of the Lie algebra $\mathfrak{g}$.

If G is simply connected, this process also can be reversed [2; Second Principle on p. 119]: If $\varrho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a representation (that is, a homomorphism of Lie algebras), then there is a Lie group homomorphism (that is, a representation of G) of type $\varphi\colon G \longrightarrow \mathrm{Aut}(V)$ such that $\varrho = T_e\varphi$.

1.4 Combining representations

The usual suspects of linear algebraic constructions give also rise to composite representations of Lie algebras. For example, there is an obvious way for defining the direct sum of representations:

Definition (1.10). Let $\varrho_V\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ and $\varrho_W\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(W)$ be two representations of the Lie algebra $\mathfrak{g}$. Then the **direct sum** of these representations is defined by

$$\begin{aligned} \varrho_V \oplus \varrho_W\colon \mathfrak{g} &\longrightarrow \mathfrak{gl}(V \oplus W) \\ x &\longmapsto \begin{pmatrix} \varrho_V(x) & 0 \\ 0 & \varrho_W(x) \end{pmatrix}. \end{aligned} \qquad \diamond$$

This definition is compatible with the direct sum of group representations (in the sense of Subsection 1.3).

Remark (1.11). Let G be a Lie group with corresponding Lie algebra $\mathfrak{g} := T_eG$. If $\varrho_V\colon G \times V \longrightarrow V$ and $\varrho_W\colon G \times W \longrightarrow W$ are two representations of G, then the direct sum of the differentiated representations, which are representations of $\mathfrak{g}$, is the same as the derivative of the direct sum of the group representations ϱ_V and ϱ_W. $\diamond$

How can we define the tensor product of two Lie algebra representations? Let $\mathfrak{g} \otimes V \longrightarrow V$ and $\mathfrak{g} \otimes W \longrightarrow W$ be two representations of the Lie algebra $\mathfrak{g}$. The naïve definition

$$\begin{aligned} \mathfrak{g} \times (V \otimes W) &\longrightarrow V \otimes W \\ (x, (v \otimes w)) &\longmapsto x \cdot v \otimes x \cdot w \end{aligned}$$

is not linear in $\mathfrak{g}$ and hence does not give rise to a representation of $\mathfrak{g}$ on $V \otimes W$. However, by differentiating the tensor product of Lie group representations, we are led to the following definition:

Definition (1.12). Let $\varrho_V \colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ and $\varrho_W \colon \mathfrak{g} \longrightarrow \mathfrak{gl}(W)$ be two representations of the Lie algebra $\mathfrak{g}$. Then the **tensor product** of these representations is defined by

$$\begin{aligned} \varrho_V \otimes \varrho_W \colon \mathfrak{g} &\longrightarrow \mathfrak{gl}(V \otimes W) \\ x &\longmapsto \Bigl(v \otimes w \mapsto \bigl(\varrho_V(x)\bigr)(v) \otimes w + v \otimes \bigl(\varrho_W(x)\bigr)(w)\Bigr). \end{aligned}$$ ◇

Similarly, differentiating dual representations of Lie groups gives us the corresponding representations for Lie algebras:

Definition (1.13). Let $\varrho \colon \mathfrak{g} \longrightarrow \mathfrak{gl} V$ be a representation of the Lie algebra $\mathfrak{g}$. Then the **dual representation of** ϱ is given by

$$\begin{aligned} \varrho^* \colon \mathfrak{g} &\longrightarrow \mathfrak{gl}(V^*) \\ x &\longmapsto -\varrho(x)^T. \end{aligned}$$ ◇

The tensor product of representations of Lie algebras is bilinear (up to isomorphism) with respect to the direct sum of representations of Lie algebras. Hence, we can reorganise the world of representations of Lie algebras into a single algebraic object:

Definition (1.14). Let $\mathfrak{g}$ be a Lie algebra. Then the $\mathbf{Z}$-module generated by the set of all isomorphism classes of $\mathfrak{g}$-representations, divided by the equivalence relation generated by

$$[\varrho] \oplus [\varrho'] \sim [\varrho \oplus \varrho'],$$

and equipped with the direct sum and tensor product of $\mathfrak{g}$-representations is called the **representation ring of** $\mathfrak{g}$. We denote the representation ring by $R(\mathfrak{g})$. ◇

Since we are only dealing with finite dimensional representations, it is easy to see that the collection of all isomorphism classes of representations of a given Lie algbra indeed form a set (and hence the above definition makes sense).

1.5 Reduction to irreducibility

When dealing with representations of semisimple Lie algebras, we can restrict ourselves to irreducible representations:

Theorem (1.15) (Complete reducibility). *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then every (finite dimensional) representation of $\mathfrak{g}$ is a direct sum of irreducible representations of $\mathfrak{g}$.*

In particular, the representation ring of $\mathfrak{g}$ is generated by the irreducible $\mathfrak{g}$-representations.

Since we are dealing with finite dimensional objects, to prove the theorem it clearly suffices to show the following:

Proposition (1.16) (Existence of complementary submodules). *Let V be a representation of the semisimple Lie algebra $\mathfrak{g}$ and let $W \subset V$ be a $\mathfrak{g}$-subrepresentation. Then there is a $\mathfrak{g}$-subrepresentation W' of V such that V is the direct sum of the representations of $\mathfrak{g}$ on W and W'.*

Proof. For example, this can be reduced by Weyl's "unitary trick" [2; Section 9.3] to the corresponding statement about representations of compact Lie groups, where it can easily be shown be means of averaged inner products [1; Theorem 3.20]. But there are purely algebraic proofs of this fact [2; Proposition C.15]. $\square$

Hence we only need to analyse irreducible representations and can thus reformulate our goal as follows: *We want to classify (up to isomorphism) all irreducible finite dimensional representations of finite dimensional semisimple Lie algebras.*

The atoms of representation theory

To get a first impression how a successful classification of the irreducible representations might look like, we sketch an analysis of the representations of the simplest semisimple Lie algebra, namely $\mathsf{sl}_2(\mathbf{C})$. Moreover, the representations of $\mathsf{sl}_2(\mathbf{C})$ turn out to be basic components of the representations of all semisimple Lie algebras.

To this end, we choose the following (vector space) basis of $\mathsf{sl}_2(\mathbf{C}) = \{A \in \mathbf{C}^{2\times 2} \mid \operatorname{Tr} A = 0\}$:

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These elements clearly satisfy the following relations:

$$[H, X] = 2 \cdot X, \qquad [H, Y] = -2 \cdot Y, \qquad [X, Y] = H.$$

2.1 Decomposition into eigenspaces

To find out how the (diagonalised) element H acts on a a representation of $\mathsf{sl}_2(\mathbf{C})$, we make use of the Jordan decomposition on Lie algebras [2; Corollary C.18]:

An element is called semisimple, if its action is diagonalisable

Theorem (2.1) (Jordan decomposition). *Let $\varrho\colon \mathsf{g} \longrightarrow \mathsf{gl}(V)$ be a representation of a semisimple Lie algebra g, and let $x \in \mathsf{g}$. If x_s and $x_n \in \mathsf{gl}(\mathsf{g})$ are the semisimple and the nilpotent part of $\operatorname{ad}(x)$ respectively, then $\varrho(x_s)$ and $\varrho(x_n) \in \mathsf{gl}(V)$ are the semisimple and the nilpotent part of $\varrho(x)$ respectively.*

Corollary (2.2) (Diagonalisability survives). *Let $\varrho\colon \mathsf{g} \longrightarrow \mathsf{gl}(V)$ be a representation of a semisimple Lie algebra g, and let $x \in \mathsf{g}$. If $\operatorname{ad}(x) \in \mathsf{gl}(\mathsf{g})$ is diagonalisable, then also $\varrho(x) \in \mathsf{gl}(V)$ is diagonalisable.* $\square$

Remark (2.3). Let V be a finite dimensioanl complex vector space and let g be a subalgebra of $\mathsf{gl}(V)$. If $x \in \mathsf{g}$ is diagonalisable (viewed as an element of $\operatorname{End}(V)$), then also $\operatorname{ad}(x)$ is diagonalisable. $\diamond$

In the following, we consider an irreducible representation of $\mathfrak{sl}_2(\mathbf{C})$ on a finite dimensional vector space V. By the above theorem, the action of H on V is diagonalisable. Hence we can decompose V into the eigenspaces V_α of H, i.e.,

$$V = \bigoplus_{\alpha \in A} V_\alpha,$$

where $A \subset \mathbf{C}$ is the (finite) set of eigenvalues of H acting on V.

2.2 The interaction between the eigenspaces

Obviously, it is crucial for the further analysis to know how X and Y act on this decomposition. Let $\alpha \in A$ and $v \in V_\alpha$. Then

$$\begin{aligned} H\bigl(X(v)\bigr) &= [H,X](v) + X\bigl(H(v)\bigr) \\ &= 2 \cdot X(v) + \alpha \cdot X(v) \\ &= (\alpha + 2) \cdot X(v) \end{aligned}$$

and thus $X(v) \in V_{\alpha+2}$. Similarly, we see that $Y(v) \in V_{\alpha-2}$.

In particular, the direct sum

$$\bigoplus_{\substack{\beta \in A \\ \beta \equiv \alpha \mod 2}} V_\beta$$

is a subrepresentation of V. Since V is irreducible, we must have $A = \bigl\{\alpha + 2 \cdot j \bigm| j \in \{0, \dots, n\}\bigr\}$ for some eigenvalue α and some $n \in \mathbf{N}$, i.e.,

$$V = \bigoplus_{j \in \{0,\dots,n\}} V_{\alpha + 2 \cdot j}.$$

We can depict this situation as follows:

$$\begin{tikzcd} 0 & V_\alpha \arrow[l, "Y"] \arrow[r, shift left, "X"] \arrow[loop below, "H"] & V_{\alpha+2} \arrow[l, shift left, "Y"] \arrow[r, shift left, "X"] \arrow[loop below, "H"] & \cdots \arrow[l, shift left, "Y"] \arrow[r, shift left, "X"] & V_{\alpha+2n-2} \arrow[l, shift left, "Y"] \arrow[r, shift left, "X"] \arrow[loop below, "H"] & V_{\alpha+2n} \arrow[l, shift left, "Y"] \arrow[r, "X"] \arrow[loop below, "H"] & 0 \end{tikzcd}$$

2.3 The complete picture

It remains to investigate how large the eigenspaces $V_{\alpha+2j}$ are, how the set of eigenvalues may look like in detail, and how many irreducible representations for a given set of eigenvalues exist. The key observation is the following:

Proposition (2.4). *We use the notation established in the previous subsections. Let $v \in V_{\alpha+2n} \setminus \{0\}$. Then*

$$V = \bigoplus_{j \in \{0,\dots,n\}} \mathbf{C} \cdot Y^j(v),$$

and

$$X\big(Y^j(v)\big) = j \cdot (\alpha + 2n - j + 1) \cdot Y^{j-1}(v)$$

for all $j \in \{1, \dots, n\}$.

Proof. Clearly, the above direct sum is contained in V. Since V is irreducible, it suffices to show that the subspace

$$W := \bigoplus_{j \in \{0,\dots,n\}} \mathbf{C} \cdot Y^j(v)$$

is closed under the action of H and X. Since each $Y^j(v)$ lies in the eigenspace $V_{\alpha+2j}$ of H, we clearly obtain $H(W) \subset W$. For the assertion on X, we show inductively that

$$X\big(Y^j(v)\big) = j \cdot (\alpha + 2n - j + 1) \cdot Y^{j-1}(v) (\in W)$$

for all $j \in \{0, \dots, n\}$.

Obviously,

$$X\big(Y^0(v)\big) = X(v) = 0 \in W,$$

and

$$\begin{aligned} X\big(Y(v)\big) &= [X,Y](v) + Y\big(X(v)\big) \\ &= H(v) + 0 \\ &= (\alpha + 2n) \cdot v. \end{aligned}$$

For the induction step, we compute for all $j \in \{1, \dots, n-1\}$

$$\begin{aligned} X\big(Y^{j+1}(v)\big) &= X\big(Y(Y^j(v))\big) \\ &= [X,Y]\big(Y^j(v)\big) + Y\big(X(Y^j(v))\big) \\ &= H\big(Y^j(v)\big) + Y\big(X(Y^j(v))\big) \\ &= (\alpha + 2n - 2j) \cdot Y^j(v) + Y\big(j \cdot (\alpha + 2n - j + 1) \cdot Y^{j-1}(v)\big) \\ &= (j+1) \cdot (\alpha + 2n - j) \cdot Y^j(v). \end{aligned}$$

Therefore, W is also closed under the action of X, and hence $V = W$, as was to be shown. ☐

Corollary (2.5). *In particular: All eigenspaces of H of an irreducible $\mathsf{sl}_2(\mathbf{C})$-representation are one dimensional, and each irreducible representation of $\mathsf{sl}_2(\mathbf{C})$ is uniquely determined by the set of eigenvalues of H.* ☐

Moreover, we see that the set of eigenvalues of H on an irreducible representation of $\mathsf{sl}_2(\mathbf{C})$ consists of integers, symmetric with respect to 0:

Let $v \in V_{\alpha+2n} \setminus \{0\}$ (where we use the same notation as above). By Proposition (2.4) we know that

$$Y^{n+1}(v) = 0, \text{ but } Y^n \neq 0.$$

Then the second part of Proposition (2.4) yields

$$0 = X\big(Y^{n+1}(v)\big) = (n+1) \cdot (\alpha - n) \cdot Y^n(v),$$

which implies $\alpha = n$. In particular, α must be an integer and the set $\{\alpha, \alpha + 2, \dots, \alpha + 2n\}$ is symmetric with respect to 0.

On the other hand, one checks that for every $n \in \mathbf{N}$ there exists an $(n+1)$-dimensional irreducible representation of $\mathsf{sl}_2(\mathbf{C})$ on which H acts with the eigenvalues $\{-n, -n+2, \dots, n\}$, namely the representation generated by the relations given in Proposition (2.4).

The combinatorics of general representations

Keeping the classification of representations of $\mathsf{sl}_2(\mathbf{C})$ in mind, we now proceed to the general case. Along with the general theory, we have a look at the representations of $\mathsf{sl}_3(\mathbf{C})$.

Inspired by the case of $\mathsf{sl}_2(\mathbf{C})$, we apply the following strategy:

- Find a replacement for $H \in \mathsf{sl}_2(\mathbf{C})$.
- Generalise the concept of eigenvalues accordingly.
- Study the action on the eigenspace decomposition.
- Analyse the geometry of the set of eigenvalues with help of copies of $\mathsf{sl}_2(\mathbf{C})$ located in the given Lie algebra.
- More precisely: Find the right symmetries and distinguished eigenvalues and associated cyclic representations.

3.1 Cartan subalgebras and generalised eigenvalues

The key to the analysis of irreducible representations of the Lie algebra $\mathsf{sl}_2(\mathbf{C})$ was the study of the action of the diagonal matrix H on the representations. In general, we cannot hope that there will be a similar single element of the Lie algebra that influences the structure of representations in the same measure. However, there is a suitable replacement for the element H, namely Abelian subalgebras. Of course, the larger the Abelian subalgebra in question, the larger the impact on the structure of representations. Hence, we are led to consider maximal Abelian subalgebras:

Definition (3.1). Let $\mathfrak{g}$ be a semisimple Lie algebra. A (with respect to inclusion) maximal Abelian subalgebra of (under the adjoint representation) diagonalisable elements of $\mathfrak{g}$ is called a **Cartan subalgebra of** $\mathfrak{g}$. ◇

Example (3.2). The subalgebra $\mathbf{C} \cdot H$ of $\mathsf{sl}_2(\mathbf{C})$ is a Cartan subalgebra of $\mathsf{sl}_2(\mathbf{C})$.

More general: Let $n \in \mathbf{N}$. *Then the set* $\mathfrak{h}$ *of diagonal matrices in* $\mathsf{sl}_n(\mathbf{C})$ *is a Cartan subalgebra of* $\mathsf{sl}_n(\mathbf{C})$*:* Clearly, $\mathfrak{h}$ is an Abelian subalgebra of diagonalisable elements. A straightforward calculation shows that an element of $\mathsf{sl}_n(\mathbf{C})$ that commutes with *all* diagonal matrices must be diagonal itself. Hence $\mathfrak{h}$ is maximal and therefore a Cartan subalgebra. ■

Theorem (3.3) (Existence of Cartan subalgebras). *Any semisimple Lie algebra possesses a Cartan subalgebra.*

Proof. To show this one can either use the existence of maximal tori in compact Lie groups, or one makes use of the non-degeneracy of the Killing form [2; Section D.1]. □

In the case of $\mathsf{sl}_2(\mathbf{C})$, any representation splits up into a sum of eigenspaces of the diagonal matrix H. In general, the elements of a chosen Cartan subalgebra h usually do not all act with the same eigenvalue on a given representation. But the dependece of the eigenvalue of elements of h is obviously linear, and hence they give rise to elements in the dual space h^*.

Definition (3.4). Let $\varrho\colon \mathsf{g} \longrightarrow \mathsf{gl}(V)$ be a representation of the semisimple Lie algebra g and let h be a Cartan subalgebra of g.

- A **weight of** ϱ is a linear functional $\alpha \in \mathsf{h}^*$ such that there is a vector $v \in V \setminus \{0\}$ satisfying
$$\forall_{x \in \mathsf{h}} \quad \big(\varrho(x)\big)(v) = \alpha(x) \cdot v.$$
- The set of all $v \in V$ satisfying the above relation is called the **weight space** associated to α, and usually is denoted by V_α. The dimension of V_α is the **multiplicity of** α in ϱ.
- The non-zero weights of the adjoint representation of g are the **roots of** g.
- The weight space corresponding to a root is called the **root space of** α of g.
- The $\mathbf{Z}$-submodule of h^* generated by the roots of g is called the **root lattice of** g. ◇

All the notions in the above definition seem to depend on the choice of a Cartan subalgebra. As in the case of maximal tori in compact Lie groups, which are all conjugate, this ambiguity can be eliminated [2; Theorem D.22].

Example (3.5). The relations between the generators X, Y, and H of $\mathsf{sl}_2(\mathbf{C})$ show that 2 and -2 are the roots of $\mathsf{sl}_2(\mathbf{C})$. ■

Example (3.6). What are the roots of $\mathsf{sl}_3(\mathbf{C})$? For $j,k \in \{1,2,3\}$ let $E_{jk} \in \mathbf{C}^{3\times 3}$ be the matrix whose (j,k)-coefficient is equal to 1 and whose other entries are 0. Moreover, let $H_{jk} := E_{jj} - E_{kk}$. Then

$$\mathsf{h} := \mathbf{C} \cdot H_{12} + \mathbf{C} \cdot H_{13} + \mathbf{C} \cdot H_{23}$$

is a Cartan subalgebra of $\mathsf{sl}_3(\mathbf{C})$ (cf. Example (3.2)). How does h act on $\mathsf{sl}_3(\mathbf{C})$?

For $j \in \{1,2,3\}$ let

$$\begin{aligned} L_j\colon \mathsf{h} &\longrightarrow \mathbf{C} \\ \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} &\longmapsto x_j. \end{aligned}$$

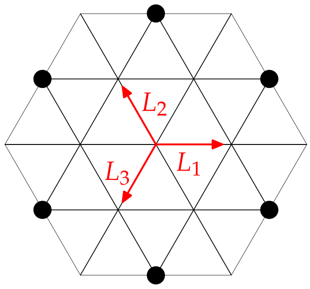

Figure 2: The roots of $\mathfrak{sl}_3(\mathbf{C})$

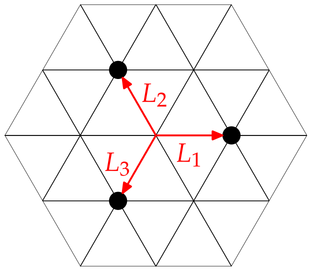

Figure 3: The weights of the standard representation of $\mathfrak{sl}_3(\mathbf{C})$ on $\mathbf{C}^3$

With this notation we obtain

$$[H_{jk}, E_{rs}] = \big((L_r - L_s)(H_{jk})\big) \cdot E_{rs}$$

for all $j,k,r,s \in \{1,2,3\}$ with $j \neq k$ and $r \neq s$. Hence, the $L_j - L_k$ are the roots of $\mathfrak{sl}_3(\mathbf{C})$. A graphical representation of this fact is given in Figure 2 (this figure is drawn in a certain *real* subspace of $\mathfrak{h}$, namely $\mathbf{R} \cdot L_1 + \mathbf{R} \cdot L_2 \cdot L_3$). ■

Example (3.7). In this example, we compute the weights of the standard representation of $\mathfrak{sl}_3(\mathbf{C})$ on $\mathbf{C}^3$ and of its dual representation: For $j \in \{1,2,3\}$ let $e_j \in \mathbf{C}^3$ be the j-th unit vector. Then for all $k \in \{1,2,3\}$

$$H_j k \cdot e_r = \delta_{jr} \cdot e_r - \delta_{kr} \cdot e_r = L_r(H_{jk}) \cdot e_r.$$

Thus, L_1, L_2, and L_3 are the weights of the standard representation (cf. Figure 3). Now Proposition (3.8) implies that $-L_1$, $-L_2$, and $-L_3$ are the weights of the dual of the standard representation of $\mathfrak{sl}_3(\mathbf{C})$ (cf. Figure 4). ■

As in the case of the Lie algebra $\mathfrak{sl}_2(\mathbf{C})$ the analysis of the roots and weights will be the first step towards a classification of the irreducible representations.

Proposition (3.8) (Weights of composite representations). *Let $\varrho_V \colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ and $\varrho_W \colon \mathfrak{g} \longrightarrow \mathfrak{gl}(W)$ be two representations of the Lie algebra $\mathfrak{g}$.*

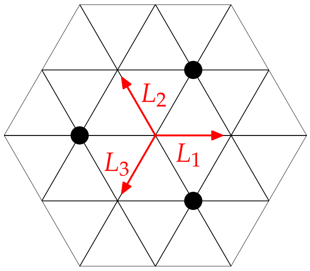

Figure 4: The weights of the dual of the standard representation of $\mathfrak{sl}_3(\mathbf{C})$ on $\mathbf{C}^3$

1. *If α and β are weights of ϱ_V and ϱ_W respectively, then $\alpha + \beta$ is a weight of $\varrho_V \otimes \varrho_W$.*
2. *If α is a weight of both ϱ_V and ϱ_W, then α is also a weight of $\varrho_V \oplus \varrho_W$.*
3. *If α is a weight of ϱ_V, then $-\alpha$ is a weight of ϱ_V^*.*

Proof. These all are straightforward computations. □

3.2 Decomposing representations into weight spaces

We continue our analysis with a brief look at the adjoint representation. Applying the Jordan decomposition theorem (Theorem (2.1)) to the restricition of the adjoint representation to a Cartan subalgebra and keeping in mind that commuting diagonalisable elements can be jointly diagonalised, we obtain the Cartan decomposition of the Lie algebra:

Theorem (3.9) (Cartan decomposition). *Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra and let A be the set of roots of $\mathfrak{g}$ (with respect to $\mathfrak{h}$). Then the Lie algebra $\mathfrak{g}$ splits up as*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in A} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the root space associated to α.

Example (3.10). The relations $[X, Y] = H$, $[H, X] = 2 \cdot X$, and $[H, Y] = -2 \cdot Y$ show that

$$\mathfrak{sl}_2(\mathbf{C}) = \mathbf{C} \cdot H \oplus \mathbf{C} \cdot X \oplus \mathbf{C} \cdot Y$$

is the Cartan decomposition of $\mathfrak{sl}_2(\mathbf{C})$. ■

Example (3.11). The calculations of Example (3.2) and Example (3.6) show that

$$\mathfrak{sl}_3(\mathbf{C}) = \mathfrak{h} \oplus \bigoplus_{j \in \{1,2,3\}} \bigoplus_{k \in \{1,2,3\} \setminus \{j\}} \mathbf{C} \cdot E_{jk}$$

is the Cartan decomposition of $\mathfrak{sl}_3(\mathbf{C})$ with respect to the Cartan subalgebra

$$\mathfrak{h} := \mathbf{C} \cdot H_{12} \oplus \mathbf{C} \cdot H_{23}.$$ ■

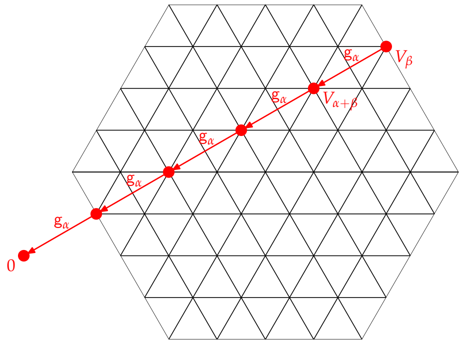

Figure 5: Graphical representation of the action on the weight space decomposition

More general, by the same argument as in Theorem (3.9), any representation can be split up into the weight spaces:

Theorem (3.12) (Weight space decomposition). *Let $\varrho \colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation of the semisimple Lie algebra $\mathfrak{g}$, and let A_ϱ be the set of weights of ϱ. Then*

$$V = \bigoplus_{\alpha \in A_\varrho} V_\alpha.$$

3.3 Interaction of the root and weight spaces

As in the case of the Lie algebra $\mathfrak{sl}_2(\mathbf{C})$, the root spaces act nicely on the decomposition of a representation into weight spaces:

If $\alpha + \beta$ is not a weight of the representation, we define $V_{\alpha+\beta} := 0$.

Proposition (3.13) (Action on the weight space decomposition). *Let $\varrho \colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation of the semisimple Lie algebra $\mathfrak{g}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Suppose $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in A} \mathfrak{g}_\alpha$ is the Cartan decomposition and suppose that $V = \bigoplus_{\alpha \in A_\varrho} V_\alpha$ is the weight space decomposition of V.*

1. *Then for all roots $\alpha \in A$ and all weights $\beta \in A_\varrho$ we obtain*

$$\mathfrak{g}_\alpha \cdot V_\beta \subset V_{\alpha+\beta}.$$

2. *In particular: If ϱ is irreducible, there is a $\beta \in \mathfrak{h}^*$ such that*

$$V = \bigoplus_{\alpha \in A} V_{\beta+\alpha}.$$

We can represent this proposition graphically as in Figure 5.

Proof. Let $y \in \mathfrak{g}_\alpha$ and $z \in V_\beta$. It suffices to show

$$\varrho(x)\bigl(\varrho(y)z\bigr) = (\alpha + \beta)(x) \cdot \varrho(y)z$$

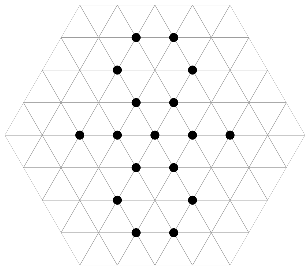

Figure 6: An impossible configuration

for all $x \in \mathfrak{h}$. By definition of the Lie bracket on $\mathfrak{gl}(V)$, we obtain

$$\begin{aligned}
\varrho(x)\bigl(\varrho(y)z\bigr) &= \bigl[\varrho(x), \varrho(y)\bigr]_{\mathfrak{gl}(V)}(z) + \varrho(y)\bigl(\varrho(x)z\bigr) \\
&= \varrho\bigl([x,y]_{\mathfrak{g}}\bigr)(z) + \varrho(y)(\beta(x)\cdot z) \\
&= \varrho\bigl(\mathrm{ad}(x)y\bigr)z + \beta(x)\cdot\varrho(y)z \\
&= \alpha(x)\cdot\varrho(y)z + \beta(x)\cdot\varrho(y)z
\end{aligned}$$

for all $x \in \mathfrak{h}$, as desired. □

Proposition (3.13) shows that the weight decomposition might be a good tool for the study of representations.

Definition (3.14). Let $\varrho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation of the semisimple Lie algebra $\mathfrak{g}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. The **weight diagram of** ϱ is the set of weights of ϱ in the dual space $\mathfrak{h}$. ◇

In the following steps, we analyse the beautiful geometric structure lying at the heart of weight diagrams of (irreducible) representations.

3.4 Looking for atoms

It is plausible that not all configurations of weights of a given Lie algebra belong to some irreducible representation. But which configurations are possible? In some sense every semisimple Lie algebra is built up out of copies of $\mathfrak{sl}_2(\mathbf{C})$ and the representation theory of $\mathfrak{sl}_2(\mathbf{C})$ helps us to discover some of the rules of the geometry of possible weight diagrams. For example, these rules imply that the configuration in Figure 6 cannot occur as a weight diagram of $\mathfrak{sl}_3(\mathbf{C})$.

Proposition (3.15) (Locating copies of $\mathfrak{sl}_2(\mathbf{C})$). *Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in A} \mathfrak{g}_\alpha$. For each root $\alpha \in A$, the direct sum*

$$\mathfrak{s}_\alpha := \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$$

with the induced Lie bracket is a subalgebra of $\mathfrak{g}$ that is isomorphic to the Lie algebra $\mathfrak{sl}_2(\mathbf{C})$.

All cited proofs rely on the non-degeneracy of the Killing form on $\mathfrak{h}^*$.

Proof. If α is a root of $\mathfrak{g}$, then $-\alpha$ is also a root of $\mathfrak{g}$ [2; D.13]. Moreover, the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are both one dimensional [2; D.20] and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-alpha}] \neq 0$, as well as $\bigl[[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}], \mathfrak{g}_\alpha\bigr] \neq 0$ [2; D.16, D.19].

Hence, we can find elements $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$ and $H \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ satisfying the relations $[H, X] = 2 \cdot X$, $[H, Y] = -2 \cdot Y$, and $[X, Y] = H$. Therefore, $\mathfrak{s}_\alpha$ must be ismorphic to $\mathfrak{sl}_(\mathbf{C})$. □

In the rest of this section, we mainly stick to the following notation:

Setup (3.16). Let $\mathfrak{g}$ be a semisimple Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in A} \mathfrak{g}_\alpha$$

be the corresponding Cartan decomposition. For each $\alpha \in A$, we write

$$\mathfrak{s}_\alpha := \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$$

and we choose generators $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$, and $H_\alpha \in [\mathfrak{g}_\alpha . \mathfrak{g}_{-\alpha}]$ satisfying the relations

$$[H_\alpha, X_\alpha] = 2 \cdot X_\alpha, \qquad [H_\alpha, Y_\alpha] = -2 \cdot Y_\alpha, \qquad [X_\alpha, Y_\alpha] = H_\alpha,$$

which is possible by Proposition (3.15). ◇

The results of Section 2 imply that the eigenvalues of H_α on any representation of $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2(\mathbf{C})$ must be integral. In particular, this must be true for all the restrictions $\varrho|_{\mathfrak{s}_\alpha}$. Thus:

Proposition (3.17). *We assume the notation of Setup (3.16). If $\beta \in \mathfrak{h}^*$ is a weight of a representation of the semisimple Lie algebra $\mathfrak{g}$, then $\beta(H_\alpha)$ is integral for all $\alpha \in A$.* □

Definition (3.18). Assume the notation of Setup (3.16). The set of functionals $\beta \in \mathfrak{h}^*$ satisfying

$$\forall_{\alpha \in A} \quad \beta(H_\alpha) \in \mathbf{Z}$$

is called the **weight lattice of** $\mathfrak{g}$. ◇

In particular, all the weights of a representation of a semisimple Lie algebra must be contained in its weight lattice.

3.5 Discovering symmetries

The weight diagrams of $\mathfrak{sl}_2(\mathbf{C})$ all are symmetric with respect to 0. The subalgebras $\mathfrak{s}_\alpha$ carry these symmetries over to the general case. The symmetry group then of course does not only consist of two elements, but it is still generated by reflections.

Definition (3.19). We assume the notation of Setup (3.16).

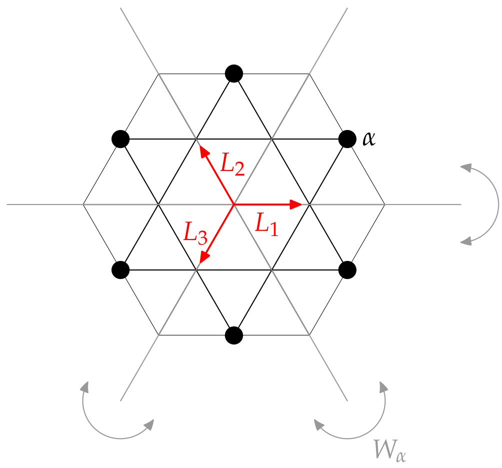

Figure 7: The Weyl group of $\mathfrak{sl}_3(\mathbf{C})$

- For any root $\alpha \in \mathfrak{h}^*$ let W_α be the "reflection"

$$\begin{aligned} W_\alpha \colon \mathfrak{h}^* &\longrightarrow \mathfrak{h}^* \\ \beta &\longmapsto \beta - \frac{2 \cdot \beta(H_\alpha)}{\alpha(H_\alpha)} \cdot \alpha = \beta - \beta(H_\alpha) \cdot \alpha \end{aligned}$$

  at the hyperplane "orthogonal" to α.

- The **Weyl group of** $\mathfrak{g}$, denoted by $W(\mathfrak{g})$, is the subgroup of $\operatorname{End}(\mathfrak{h}^*)$ generated by the "reflections" $(W_\alpha)_{\alpha \in A}$. ◇

The term "reflection" can be made precise by introducing a suitable inner product on the dual space $\mathfrak{h}^*$, for example the Killing form [2; Section 14.2].

A priori, the definition of the Weyl group depends on a choice of a Cartan subalgebra, but it can be shown that this group is indeed independent of the chosen Cartan subalgebra [2; Theorem D.22].

Example (3.20). The Weyl group of $\mathfrak{sl}_3(\mathbf{C})$ and its action on the Cartan decomposition of $\mathfrak{sl}_3(\mathbf{C})$ are illustrated in Figure 7. ■

Proposition (3.21) (Symmetries of the weights). *The weights of any representation of a semisimple Lie algebra $\mathfrak{g}$ are invariant under the action of the Weyl group $W(\mathfrak{g})$. Moreover, also the multiplicities of weights of any representation are invariant under the Weyl group.*

Proof. Let $\varrho \colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. We now make use of the notation esatblished in Setup (3.16). Let $\alpha \in \mathfrak{h}^*$ be some root of $\mathfrak{g}$, and let $\beta \in \mathfrak{h}^*$ be a weight of ϱ. Then

$$V_{[\beta]} := \bigoplus_{n \in \mathbf{Z}} V_{\beta + n \cdot \alpha}$$

is a subrepresentation of the $\mathfrak{s}_\alpha$-representation $\operatorname{res}^{\mathfrak{g}}_{\mathfrak{s}_\alpha}$ on V (cf. Proposition (3.13)).

By the classification of $\mathfrak{sl}_2(\mathbf{C})$-representations we know that the set of eigenvalues of H_α on $V_{[\beta]}$, i.e., the set $S(H_\alpha)$ with

$$S := \{\beta + n \cdot \alpha \mid n \in \mathbf{Z}, V_{\beta+n\cdot\alpha} \neq 0\},$$

is symmetric about the origin. By replacing β with a translate by a multiple of α, we may assume that

$$S = \{\beta, \beta + \alpha, \dots, \beta + n \cdot \alpha\}$$

for some suitable $n \in \mathbf{N}$. Since $\alpha(H_\alpha) = 2$ and S is symmetric, we obtain

$$-\beta(H_\alpha) = \beta(H_\alpha) + 2 \cdot n$$

and thus $n = -\beta(H_\alpha)$. Therefore,

$$\begin{aligned} W_\alpha \cdot S &= \{W_\alpha(\beta + j \cdot \alpha) \mid j \in \{0, \dots, n\}\} \\ &= \{\beta + j \cdot \alpha - (\beta + j \cdot \alpha)(H_\alpha) \cdot \alpha \mid j \in \{0, \dots, n\}\} \\ &= \{\beta + (n - j) \cdot \alpha \mid j \in \{0, \dots, n\}\} \\ &= S, \end{aligned}$$

which implies that the weights of $V_{[\beta]}$ are invariant under the action of W_α. Hence also the set of all weights of V must be invariant under W_α (and thus under $W(\mathfrak{g})$). The same argument also shows that the multiplicities are invariant under the action of the Weyl group. □

3.6 Highest weights

When classifying the irreducible representations of $\mathfrak{sl}_2(\mathbf{C})$ we exploited the fact that we could assign an order to the occurring eigenvalues. Of course, in the general case, we cannot expect to have such a linear ordering of the weights, but there still are distinguished weights, the highest weights:

Definition (3.22). We assume the notation of Setup (3.16). Furthermore, let ℓ be a functional on the root lattice of the Lie algebra $\mathfrak{g}$ that is irrational with respect to the root lattice.

- A root $\alpha \in \mathfrak{h}^*$ of $\mathfrak{g}$ is called **positive**, if $\ell(\alpha) > 0$. The roots that are not positive are called **negative**. Such a decomposition of the root space is called an **ordering of the roots of** $\mathfrak{g}$.
- A **highest weight** of a representation $\varrho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a weight of ϱ admitting a highest weight vector.
- A **highest weight vector** of the representation ϱ is a vector $v \in V \setminus \{0\}$ contained in some weight space of V such that

  $$\mathfrak{g}_\alpha \cdot v = 0$$

  holds for all positive roots α of $\mathfrak{g}$. ◇

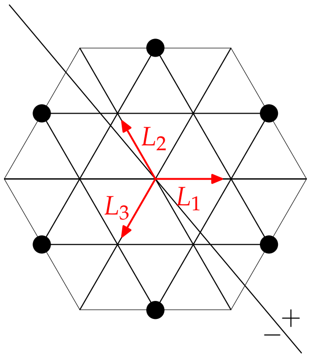

Figure 8: An ordering of the roots of $\mathfrak{sl}_3(\mathbf{C})$

Example (3.23). A possible ordering of the roots of $\mathfrak{sl}_3(\mathbf{C})$ is given in Figure 8. ■

Proposition (3.24) (Highest weights of irreducible representations). *Let $\mathfrak{g}$ be a semisimple Lie algebra and let $A = A^+ \sqcup A^-$ be an ordering of the roots of $\mathfrak{g}$.*

1. *Any (finite dimensional) representation of $\mathfrak{g}$ possesses a highest weight vector.*
2. *If v is a highest weight vector of some representation of $\mathfrak{g}$, then the subspace W generated by successive applications to v of the root spaces $\mathfrak{g}_\alpha$ with $\alpha \in A^-$ is an irreducible subrepresentation of $\mathfrak{g}$.*
3. *The highest weight vectors of an irreducible representation of $\mathfrak{g}$ are unique up to scalars.*

Proof. 1. This is a direct consequence of Theorem (3.12) and finiteness of the dimension of the representation.
2. Let $\varrho \colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ and let v be a highest weight vector of ϱ. For $n \in \mathbf{N}$ let W_n be the subspace generated by all vectors of the form $w_n \cdot v$, where w_n is a word of length at most n consisting of elements of the root spaces with negative roots. Then

$$W = \bigcup_{n \in \mathbf{N}} W_n.$$

First we show that W is a subrepresentation of ϱ: For this it suffices to show that W is invariant under the action of all elements in the positive root spaces (because the Cartan algebra acts diagonally on W). To this end we verify inductively that

$$x \cdot W_n \subset W_{n+1}$$

holds for all $n \in \mathbf{N}$ and all $x \in \mathfrak{g}_\alpha$ with $\alpha \in A^+$:

Since v is a highest weight vector of ϱ, this is true in the case $n = 0$. For the induction step let $x \in \mathfrak{g}_\alpha$ with $\alpha \in A^+$ and let $w \in W_n$. By definition of W_n, we can write $w = y \cdot w'$ with $w' \in W_{n-1}$ and $y \in \mathfrak{g}_\beta$ with $\beta \in A^-$. Then

$$\begin{aligned} x \cdot w &= x \cdot y \cdot w' \\ &= [x,y] \cdot w' + y \cdot x \cdot w', \end{aligned}$$

where $[x,y] \in \mathfrak{g}_{\alpha+\beta}$ (Proposition (3.13)). If $\alpha + \beta \in A^-$, then $[x,y] \cdot w' \in W_n \subset W_{n+1}$ by definition of W_n. If $\alpha + \beta \in A^+$, then $[x,y] \cdot w' \in W_n \subset W_{n+1}$ by induction. If $\alpha + \beta = 0$, then $[x,y]$ lies in the Cartan algebra and hence $[x,y] \cdot w' \in W_{n-1} \subset W_{n+1}$. Moreover, the second summand is contained in $W_n \subset W_{n+1}$ by induction.

Therefore, W is a subrepresentation of ϱ. Since the highest weight space of W, i.e., W_0, is one dimensional, Proposition (3.8) yields that W is irreducible.
3. This follows from the second part and from Proposition (3.13). □

3.7 Weight polytopes of representations

Definition (3.25). Let $\varrho \colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation of the semisimple Lie algebra $\mathfrak{g}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. The **weight polytope of** ϱ is the convex hull in $\mathfrak{h}^*$ of all weights of ϱ. ◇

Proposition (3.26) (The weight polytope). *Let $\varrho \colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be an irreducible representation of the semisimple Lie algebra $\mathfrak{g}$ and let $\alpha \in \mathfrak{h}^*$ be a highest weight of ϱ (with respect to some chosen ordering $A = A^+ \sqcup A^-$ of the roots of $\mathfrak{g}$).*

1. *Every vertex of the weight polytope of ϱ is conjugate to α under the action of the Weyl group $W(\mathfrak{g})$.*

2. *The set of weights of ϱ is convex in the following sense: If $\beta \in \mathfrak{h}^*$ and γ is any root of $\mathfrak{g}$, then the intersection of the set of weights with the (discrete) line $\{\beta + n \cdot \gamma | n \in \mathbf{Z}\}$ is a connected string.*

Proof. 1. By Proposition (3.24), the weights of ϱ must be contained in the cone

$$\alpha + \{n \cdot \beta \mid n \in \mathbf{N}, \alpha \in A^+\}.$$

Moreover, we know that for any $\beta \in A^-$, the functionals $\alpha, \alpha + \beta, \dots, \alpha + (-\alpha(H_\beta)) \cdot \beta$ all must be weights of ϱ (proof of Proposition (3.21)). Hence, any vertex of the weight polytope adjacent to α must be of the form

$$\alpha - \alpha(H_\beta) \cdot \beta = W_\beta(\alpha) \in W(\mathfrak{g}) \cdot \alpha.$$

Inductively we see that all vertices of the weight polytope must lie in $W(\mathfrak{g}) \cdot \alpha$.
2. This can be shown using the same argument as in the proof of Proposition (3.21). □

Sometimes it is convenient, to have a special fundamental domain of the action of the Weyl group on the dual of the Cartan algebra at hand.

Definition (3.27). Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$, and let $A = A^+ \sqcup A^-$ be an ordering of the roots. The **Weyl chamber** associated to this ordering is the set of all $\alpha \in \mathfrak{h}^*$ satisfying

$$\forall_{\beta \in A^+} \quad \alpha(H_\beta) \geq 0.$$ ◇

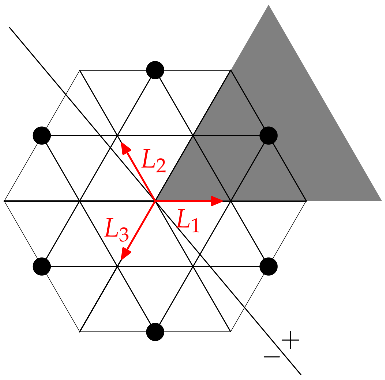

Figure 9: Weyl chamber of $\mathfrak{sl}_3(\mathbf{C})$

By the above proposition, any Weyl chamber contains exactly one vertex of the weight polytope.

Example (3.28). Corresponding to the ordering of Figure 8, we obtain the Weyl chamber shown in Figure 9. ■

3.8 The classification

We now collected all tools necessary to provide a classification of the irreducible representations of semisimple Lie algebras.

Theorem (3.29) (Classification of irreducible representations). *Let $\mathfrak{g}$ be a semisimple Lie algebra, let $A = A^+ \sqcup A^-$ be be an ordering of the roots of $\mathfrak{g}$ and let C be the corresponding Weyl chamber. Moreover, let Λ be the weight lattice of $\mathfrak{g}$.*

1. *For any $\alpha \in C \cap \Lambda$ there exists a unique (finite dimensional) representation Γ_α of $\mathfrak{g}$ with highest weight α.*
2. *Hence there is a bijection between $C \cap \Lambda$ and the set of isomorphism classes of irreducible (finite dimensional) representations of $\mathfrak{g}$.*
3. *The set of weights of Γ_α is convex in the following sense: If $\beta \in \mathfrak{h}^*$ and γ is any root of $\mathfrak{g}$, then the intersection of the set of weights with the (discrete) line $\{\beta + n \cdot \gamma | n \in \mathbf{Z}\}$ is a connected string.*
4. *In particular, any representation of $\mathfrak{g}$ is uniquely determined by the multiplicities of its weights.*

Proof. It only remains to show existence and uniqueness of the representations Γ_α.
Uniqueness. Let $\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ and $\mathfrak{g} \longrightarrow \mathfrak{gl}(W)$ be two irreducible representations of g both with the same highest weight $\alpha \in \mathfrak{h}^*$. Let $v \in V$ and $w \in W$ be corresponding highest weight vectors of these representations. Then the pair (v, w) clearly is a highest weight vector of the direct sum representation on $V \oplus W$. Let $U \subset V \oplus W$

be the (irreducible) subrepresentation generated by (v, w). We now consider the projection homomorphisms

$$\pi_V : U \longrightarrow V,$$
$$\pi_W : U \longrightarrow W,$$

which clearly are homomorphisms of $\mathfrak{g}$-representations. Since $(v, w) \in U$, we obtain $\pi_V \neq 0$ and $\pi_W \neq 0$. Now irreducibility of U, V, and W forces π_V and π_W to be isomorphisms. In particular, V and W must be isomorphic, which proves the uniqueness part.

Existence. Let $\alpha \in C \cap \Lambda$. In view of Proposition (3.24) it suffices to construct any finite dimensional representation of $\mathfrak{g}$ having highest weight α (because then the subrepresentation generated by a highest weight vector is an irreducible representation of $\mathfrak{g}$ with highest weight α).

By taking tensor products and duals of representations, we see that we only need to construct such representation for so-called **simple roots** (i.e., positive roots that cannot be decomposed as an integral linear combination of positive roots with positive coefficients). However, in general, it is quite difficult to construct these representations. Other approaches include the use of Verma modules or the explicit construction for all types of simple Lie algebras (and applying to the classification of simple Lie algebras).

We only show existence in the special case of $\mathfrak{sl}_3(\mathbf{C})$: There is an ordering of the roots of $\mathfrak{sl}_3(\mathbf{C})$ such that the Weyl chamber is the positive cone generated by L_1 and $-L_3$ (cf. Figure 9). For $a, b \in \mathbf{N}$ we consider the representation

$$\Gamma'_{a,b} := \varrho^{\otimes a} \otimes (\varrho^*)^{\otimes b},$$

where $\varrho\colon \mathfrak{sl}_3(\mathbf{C}) \longrightarrow \mathfrak{gl}(\mathbf{C}^3)$ is the standard representation of $\mathfrak{sl}_3(\mathbf{C})$. Clearly, the representation $\Gamma'_{a,b}$ has highest weight a times the highest weight of ϱ plus b times the highest weight of ϱ^*. Now taking the subrepresentation generated by a highest weight vector yields an irreducible representation of $\mathfrak{sl}_3(\mathbf{C})$ with highest weight

$$a \cdot L_1 - b \cdot L_3,$$

as desired. The case "$a = 6, b = 2$" is depicted in Figure 10. □

Tensor products of representations

So far, we only studied the additive structure of the representation ring. In the following, we want to describe the effect of multiplication of representations in the representation ring, i.e., we want to decompose the tensor products of irreducible representations as a direct sum of irreducible representations.

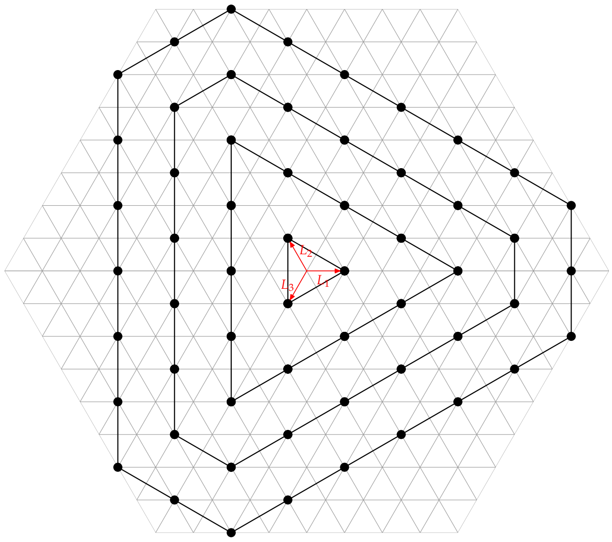

Figure 10: The weights of the irreducible representation of $\mathfrak{sl}_3(\mathbf{C})$ with highest weight $6 \cdot L_1 - 2 \cdot L_3$

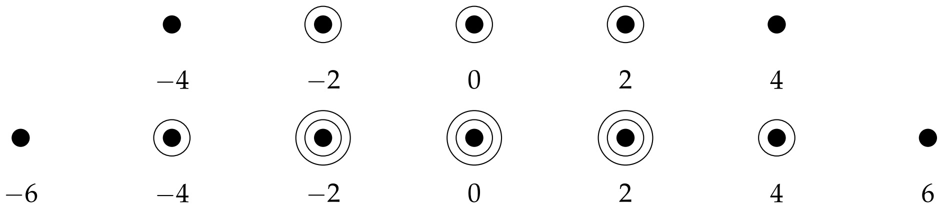

Figure 11: The tensor product representations $V^{(4)} \otimes V^{(2)}$ and $V^{(5)} \otimes V^{(3)}$ of $\mathfrak{sl}_2(\mathbf{C})$

4.1 Tensor products of $\mathfrak{sl}_2(\mathbf{C})$-representations

To find decompositions of tensor products into irreducible representations it is necessary to have some knowledge about the multiplicities of weights of irreducible representations. For example, the multiplicities of weights of irreducible representations of $\mathfrak{sl}_2(\mathbf{C})$ are all equal to 1 (see Section 2).

Let $V^{(n)}$ and $V^{(m)}$ the n-dimensional and the m-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbf{C})$ respectively. What is the decomposition of the tensor product representation $V := V^{(n)} \otimes V^{(m)}$ into irreducible representations?

The examples $V^{(4)} \otimes V^{(2)}$ and $V^{(5)} \otimes V^{(3)}$ are illustrated in Figure 11.

Clearly, the highest weight of V is $n - 1 + m - 1 = n + m - 2$. Thus, V must contain one copy of $V^{(n+m-2)}$. Now we can look at what remains of V after removing $V^{(n+m-2)}$. We again find some highest weight and remove the corresponding irreducible representation (with the right multiplicity) and so on …

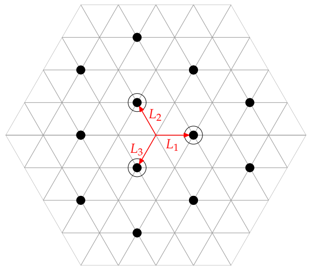

Figure 12: The irreducible representation of $\mathfrak{sl}_3(\mathbf{C})$ with highest weight $2 \cdot L_1 - L_3$

More general, a careful (inductive) analysis of V with help of Proposition (3.8) reveals for $n \geq m$ that

$$V = V^{(n+m-2)} \oplus V^{(n+m-4)} \oplus \cdots \oplus V^{(n-m)}.$$

4.2 Tensor products of $\mathfrak{sl}_3(\mathbf{C})$-representations

Let ϱ and ϱ' be two irreducible representations of $\mathfrak{sl}_3(\mathbf{C})$ corresponding to the highest weights α and α' respectively. Then the irreducible representation of $\mathfrak{sl}_3(\mathbf{C})$ with highest weight $\alpha + \alpha'$ is contained in the tensor product $\varrho \otimes \varrho'$. We now remove this part from $\varrho \otimes \varrho'$ and look at the highest weight of the remaining representation, and so on ...

I.e., if we know the multiplicities of all weights in the irreducible representations, we can inductively decompose tensor products of irreducible representations into direct sums of irreducible representations.

For example, let V be the standard representation of $\mathfrak{sl}_3(\mathbf{C})$ (see Figure 3) and let $\Gamma_{2,1}$ be the irreducible representation of $\mathfrak{sl}_3(\mathbf{C})$ with highest weight $2 \cdot L_1 - L_3$ (see Figure 12). Then the tensor product $W := V \otimes \Gamma_{2,1}$ has the weights

$3 \cdot L_j - L_k$	with multiplicity 1,
$2 \cdot L_j + L_k - L_r$	with multiplicity 2,
$2 \cdot L_j$	with multiplicity 4,
$L_j + L_k$	with multiplicity 5.

Hence, W must contain the irreducible representation $\Gamma_{3,1}$ with highest weight $3 \cdot L_j - L_k$. Removing this part from W, we obtain the representation depicted in Figure 13. This representation has highest weight

$$2 \cdot L_1 + L_2 - L_3 = L_1 - 2 \cdot L_3,$$

and hence this representation must contain $\Gamma_{1,2}$. Removing this representation yields the representation of Figure 14, which is easily recognised to be $\Gamma_{2,0}$.

Putting it all together shows that

$$W \cong \Gamma_{3,1} \oplus \Gamma_{1,2} \oplus \Gamma_{2,0}.$$

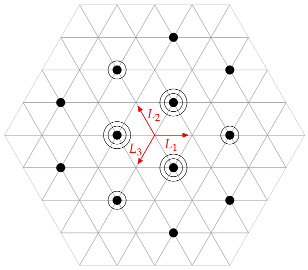

Figure 13: After reducing W the first time

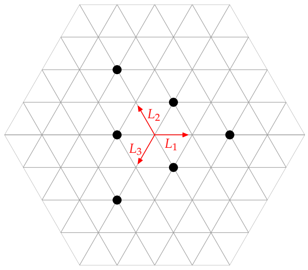

Figure 14: After reducing W the second time

4.3 The structure of representation rings

There are general techniques to compute the multiplicities of weights in the irreducible representations of semisimple Lie algebras. We will not explain these in these notes, but we present the final classification result in Theorem (4.3).

Definition (4.1). Let $\mathfrak{g}$ be a semisimple Lie algebra with weight lattice Λ. We write $\mathbf{Z}[\Lambda]$ for the integral group ring of the Abelian group Λ. Then the **character homomorphism** of the Lie algebra $\mathfrak{g}$ is given by

$$\begin{aligned} \chi\colon R(\mathfrak{g}) &\longrightarrow \mathbf{Z}[\Lambda] \\ [V] &\longmapsto \sum_{\lambda\in\Lambda} \dim V_\lambda \cdot \lambda, \end{aligned}$$

where V_λ is the weight space corresponding to λ. $\diamond$

Here, $V_\lambda = 0$, if λ is not a weight of V.

Clearly, the character of a representation can be read off the weight diagram (with multiplicities). In this sense, all our pictures of weight diagrams can be interpreted as pictures of characters of representations.

Definition (4.2). Let $\mathfrak{g}$ be a semisimple Lie algebra and let an ordering of the roots be given. The **fundamental weights** of $\mathfrak{g}$ with respect to this ordering are the first non-zero weights met along the edges of the Weyl chamber. $\diamond$

With this terminology, the classification of representations of semisimple Lie algebras can be put into the following theorem [2; Theorem 23.24].

Theorem (4.3) (Representation rings of semisimple Lie algebras). *Let* $\mathfrak{g}$ *be a semisimple Lie algebra with fundamental weights* $\omega_1, \dots, \omega_n$ *and let* $\Gamma_1, \dots, \Gamma_n$ *be the corresponding irreducible representations. Then the representation ring* $R(\mathfrak{g})$ *is a polynomial ring on the variables* $\Gamma_1, \dots, \Gamma_n$ *and the character homomorphism induces an isomorphism*

$$R(\mathfrak{g}) \cong \mathbf{Z}[\Lambda]^{W(\mathfrak{g})}$$

of rings.

References

[1] J. Frank Adams. *Lectures on Lie Groups*. Midway reprint, The University of Chicago Press, 1982.

[2] William Fulton, Joe Harris. *Representation Theory. A First Course*. Graduate Texts in Mathematics, 129, Springer Verlag, New York, 1991.