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CUP-PRODUCTS IN BOUNDED COHOMOLOGY

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I BOUNDED COHOMOLOGY

II CUP-PRODUCTS IN BC: WHY?

III " : COMPUTATIONS

I BOUNDED COHOMOLOGY

Def. (bounded cohomology)

$$\Gamma \text{ group } \rightsquigarrow C_b^*(\Gamma) := \ell^\infty(\Gamma^{*+1}, \mathbb{R})^\Gamma$$

+ simplicial coboundary op

$$\rightsquigarrow H_b^*(\Gamma; \mathbb{R}) := H^*(C_b^*(\Gamma)).$$

functorial wrt group homs

forgetting boundedness leads to a natural transformation

$$c^* : H_b^*(\cdot ; \mathbb{R}) \Longrightarrow H^*(\cdot ; \mathbb{R}),$$

the so-called <u>comparison map</u>.

Classical examples:

Amenability: • If Γ is amenable, then

$$H_b^*(\Gamma;\mathbb{R}) \cong H_b^*(1;\mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } *=0 \\ 0 & \text{if } *>0 \end{cases}$$

$$\cong$$

• $H_b^*(\text{Thompson } F;\mathbb{R})$ [Monod]

Negative curvature

• $\dim_{\mathbb{R}} H_b^2(F_2;\mathbb{R}) = \infty$ (see below)

$\dim_{\mathbb{R}} H_b^3(F_2;\mathbb{R}) = \infty$ (3-mfd top)

• If M is an occ n-mfd with negative sectional curvature, then

volume cocycle

$$H_b^n(\pi_1(M);\mathbb{R}) \not\cong 0.$$

If Γ is a hyperbolic group, then for all $n \geq 2$, the comp. map

$$c_\Gamma^n : H_b^n(\Gamma;\mathbb{R}) \longrightarrow H^n(\Gamma;\mathbb{R})$$

is surjective [Mineyev].

Some applications:

- The following seq. is exact:

$$f \longmapsto [\delta f]$$

$$0 \longrightarrow QH(\Gamma)/\mathrm{Hom}(\Gamma,\mathbb{R}) \longrightarrow H_b^2(\Gamma;\mathbb{R}) \xrightarrow{c_\Gamma^2} H^2(\Gamma;\mathbb{R})$$

$\updownarrow$ [Bavard]

scl_Γ

on F_2: counting quasi-morphisms

- computations/properties of simplicial volume [Gromov], leading to geometric results

- estimates for cat_{Am}:

If $\mathrm{cat}_{Am} X \leq n$, then

$\exists\, U_1, \dots, U_n \subset X$ open s.t. $X = \bigcup_{j=1}^{n} U_j$ and all U_j are amenable in X

$$c^n_{\pi_1(X)} : H^n_b(\pi_1(X);\mathbb{R}) \longrightarrow H^n(\pi_1(X);\mathbb{R})$$

is zero [Gromov, Ivanov]

II Cup-Product: Why?

Def. ($\cdot \cup \cdot$ in H_b^*). If Γ is a group, $m, n \in \mathbb{N}$, then we set:

$$\cdot \cup \cdot : H_b^m(\Gamma;\mathbb{R}) \otimes_{\mathbb{R}} H_b^n(\Gamma;\mathbb{R}) \longrightarrow H_b^{m+n}(\Gamma;\mathbb{R})$$

$$[f] \otimes [g] \longmapsto \Big((\gamma_0, \dots, \gamma_{m+n}) \mapsto f(\gamma_0, \dots, \gamma_m) \cdot g(\gamma_m, \dots, \gamma_{m+n}) \Big)$$

Sample problems:

① Question: Let $n \in \mathbb{N}_{\geq 4}$. Is $H_b^n(F_2;\mathbb{R}) \cong 0$?!

①' Question: can low-degree classes lead to non-trivial classes in $H_b^{\geq 4}(F_2,\mathbb{R})$ via cup-products ?!

② Question [Gromov]. Does every epi: $\Gamma \to \Lambda$ induce a mono $H_b^*(\Lambda;\mathbb{R}) \to H_b^*(\Gamma;\mathbb{R})$?!

(true in degree 2 [Bouarich])

if $\cdot \cup \cdot : H_b^2(F_2;\mathbb{R}) \otimes H_b^2(F_2;\mathbb{R}) \to H_b^4(F_2;\mathbb{R})$ is zero, then: counterexamples in deg 4.

③ Question. [Capovilla, L, M] Does the cup-length on $H_b^*(\cdot;\mathbb{R})$ give a lower bound for cat_{Am} ?!

(if yes, then $\cdot \cup \cdot : H_b^2(F_2;\mathbb{R}) \otimes H_b^2(F_2;\mathbb{R})) \to H_b^4(F_2,\mathbb{R})$ would be zero)

④ Question. Which graded commutative $\mathbb{R}$-algebras can be realised as $H_b^*(\cdot;\mathbb{R})$?!

III CUP-PRODUCT: COMPUTATIONS

First steps towards ④ :

- $\mathbb{R}[x] \cong H_b^*(T;\mathbb{R})$ — Thompson — [Monod, Nariman] [F, L, M]

  $|x|=2$

  $x \longmapsto eu_b^T$

  idea of pf: $T \curvearrowright S^1$ is "highly transitive" and has boundedly acyclic stabilisers (F! [Monod])

$\rightsquigarrow H_b^*(T;\mathbb{R}) \cong H^*(\ell^\infty_{alt}(S^{*+1}_{\cdot}, \mathbb{R}))$ — orbit of $1 \in S^1$

$\overset{(\dots)}{\rightsquigarrow}$ $\underbrace{\mathbb{R}[x_1,\dots,x_d]}_{\deg 2} \cong H^*_b(T^{\times d};\mathbb{R})$

Problem: (How) can we get $\mathbb{R}[x]/(x^4)$??

Towards ①':

Theorem. [Bucher, Monod] [Heuer] [Amontova, Bucher]

For many $\underbrace{\text{"local"}}_{\text{eg. counting qm's}}$ qms $f : F_2 \to \mathbb{R}$, we have

$$\forall_{\varphi \in H^{k>0}_b(F_2;\mathbb{R})} \quad [\delta f] \cup \varphi = 0 \in H^{2+k}_b(F_2;\mathbb{R}).$$

Theorem. [BFL] Let $\Gamma \curvearrowright X$ on a fin-dim (CAT(0)-)cube complex X. Let s be an $\mathcal{H}$-segment in X and let $\varphi \in H^n_{\Gamma,b}(X;\mathbb{R})$ be non-transverse to $\Gamma . s$. Then

$$[\delta f_s] \cup \varphi = 0 \in H^{2+n}_{\Gamma,b}(X;\mathbb{R}).$$

(e.g: RAAG $\curvearrowright$ univ. cov. of the Salvetti complex)

Sketch of proof: let $\kappa \in C^n_{\Gamma,b}(X;\mathbb{R})$ be a weight map φ and is non-transverse to Γ.s.

let

$$\beta := \underbrace{f_s}_{\substack{\text{primitive of } \delta f_s \cup \kappa, \\ \text{but in general } \underline{\text{not}} \text{ bdd}}} \cup \kappa + \delta\eta$$

where

$$\eta : X(0)^{n+1} \longrightarrow \mathbb{R}$$

$$(x_0, \dots, x_n) \longmapsto \sum_{t \in [x_0, x_1]^{|s|}_{\pm t}} \varepsilon_s(t) \cdot \frac{1}{2} \cdot \Big(\kappa(\overset{\text{a head of } t}{\alpha}, x_1, \dots, x_n) + \kappa(\underset{\text{a tail of } t}{\omega}, x_1, \dots, x_n) \Big)$$

$\square$