Margulis's normal subgroup theorem
A short introduction

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The normal subgroup theorem *of Margulis expresses that many lattices in semi-simple Lie groups are simple up to finite error.*
In this talk, we give a short introduction to the normal subgroup theorem, including a brief review of the relevant notions about lattices, amenability, and property (T), as well as a short overview of the proof of the normal subgroup theorem.

1 Margulis's normal subgroup theorem

The normal subgroup theorem of Margulis expresses that many lattices in semi-simple Lie group are simple groups up to finite error:

Theorem 1.1 (Normal subgroup theorem)**.** *Let G be a connected semi-simple Lie group with finite centre with $\operatorname{rk}_{\mathbf{R}}(G) \geq 2$, and let $\Gamma \subset G$ be an irreducible lattice. If $N \subset \Gamma$ is a normal subgroup of Γ, then either N lies in the centre of G (and hence Γ is finite) or the quotient Γ/N is finite.*

Remark 1.2. In fact, Margulis proved the normal subgroup theorem for a more general class of lattices – namely, lattices in certain ambient groups that are "Lie groups" over local fields [9, Chapter IV].

After explaining the occuring terminology in detail in Section 2, we discuss the statement and the consequences of the normal subgroup theorem in Section 3. Sections 4 and 5 give a short introduction into amenability and property (T), which lie at the heart of the proof of the normal subgroup theorem. Finally, Section 6 is devoted to an overview of the proof of the normal subgroup theorem.

For a more extensive treatment of the normal subgroup theorem and related topics, we refer the reader to the books by Margulis [9, Chapter IV], Zimmer [16, Chapter 8], and Witte Morris [15, Chapter 13].

2 Intermezzo: Lattices in a nutshell

In this section, we collect the basic notions about Lie groups and their lattices required for understanding the normal subgroup theorem and its proof.

Convention 2.1 (Topological groups)**.** *Unless stated otherwise, in this text* topological *groups are always assumed to be locally compact second countable Hausdorff groups.*

Definition 2.2 (Lattice)**.** Let G be a topological group.

- A *lattice in* G is a discrete subgroup Γ of G with finite covolume, i.e., the measure on G/Γ induced by the Haar measure on G is finite. A lattice Γ in G is *uniform* if the quotient space G/Γ is compact.
- A lattice Γ in G is called *irreducible* if for every closed non-discrete normal subgroup H of G the image of Γ under the canonical projection $G \longrightarrow G/H$ is dense.

Caveat 2.3 (Irreducible lattices)**.** Unfortunately, there are different definitions of irreducibility of lattices commonly used in the literature.

Lattices give rise to interesting examples in rigidity theory (when considering quasi-isometry, measure equivalence or orbit equivalence of groups), and lattices naturally arise in the study of locally symmetric spaces.

2.1 Examples of lattices

Example 2.4 (Lattices in discrete groups)**.** Of course a subgroup of a countable discrete group is a (uniform) lattice if and only if it has finite index.

Example 2.5 (Free Abelian groups)**.** For all $n \in \mathbf{N}$ the free Abelian group $\mathbf{Z}^n$ is a uniform lattice in $\mathbf{R}^n$. This lattice is irreducible if and only if $n \leq 1$.

Example 2.6 (Products)**.** More generally, products of lattices are lattices in the product of their ambient groups; whenever one of the ambient groups is not discrete, such lattices are reducible.

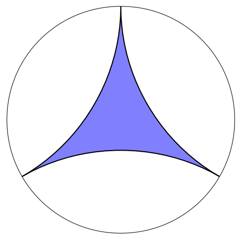

Figure 1: A measurable fundamental domain for $\mathrm{SL}(2, \mathbf{Z})$ acting on the hyperbolic plane

Example 2.7 (Fundamental groups of Riemannian manifolds)**.** Let M be a connected complete Riemannian manifold without boundary. Then the fundamental group $\pi_1(M)$ of M is a discrete subgroup of the isometry group $\mathrm{Isom}(\widetilde{M})$ of the universal covering $\widetilde{M}$ of M. If M has finite volume, then $\pi_1(M)$ is a lattice in $\mathrm{Isom}(\widetilde{M})$; if M is compact, then $\pi_1(M)$ is a uniform lattice in $\mathrm{Isom}(\widetilde{M})$ [8, Lemma 4.2].

If M has non-positive sectional curvature, then in view of the Gromoll-Wolf splitting theorem [4, 2] the irreducibility of $\pi_1(M)$ is closely related to the irreducibility of M in the sense that no finite covering of M splits as a direct product of non-trivial manifolds.

Example 2.8 (Matrix groups)**.** Another prominent source of lattices is provided by matrix groups: For example, for every $n \in \mathbf{N}$ the integral special linear group $\mathrm{SL}(n, \mathbf{Z})$ is a lattice in $\mathrm{SL}(n, \mathbf{R})$, which is non-uniform if and only if $n > 1$: Obviously, $\mathrm{SL}(n, \mathbf{Z})$ is a discrete subgroup of $\mathrm{SL}(n, \mathbf{R})$, but showing that $\mathrm{SL}(n, \mathbf{Z})$ has indeed finite covolume is non-trivial and requires geometric input [12, 15, Chapter I.3.2, Theorem 5.10]; for instance, for $\mathrm{SL}(2, \mathbf{Z})$ we can use the standard measurable fundamental domain of the $\mathrm{SL}(2, \mathbf{Z})$-action on the hyperbolic plane (Figure 1) and apply the fact that the hyperbolic area of ideal hyperbolic simplices is uniformly bounded from above. Moreover, $\mathrm{SL}(n, \mathbf{Z})$ is an irreducible lattice of $\mathrm{SL}(n, \mathbf{R})$ whenever $n > 1$ because $\mathrm{SL}(n, \mathbf{R})$ is almost a simple group.

Notice that the free group on two generators is a subgroup of $\mathrm{SL}(2, \mathbf{Z})$ of finite index; hence, the free group on two generators is an irreducible non-uniform lattice in $\mathrm{SL}(2, \mathbf{R})$.

More surprsingly, also products of groups can contain irreducible lattices: For instance, the group $\mathrm{SL}(2, \mathbf{Z}[\sqrt{2}])$ is a lattice in $\mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R})$ via the embedding induced by the two embeddings $\mathbf{Z}[\sqrt{2}] \longrightarrow \mathbf{R}$ given by $\sqrt{2} \longmapsto \sqrt{2}$ and $\sqrt{2} \longmapsto -\sqrt{2}$ respectively. It is a nice exercise in algebra to prove that this is indeed a lattice in $\mathrm{SL}(2, \mathbf{R}) \times \mathrm{SL}(2, \mathbf{R})$ and that it is irreducible [15, Proposition 5.40].

2.2 *Basic properties of Lie groups*

We now recall some basic properties of Lie groups needed for the formulation and the outline of the proof of the normal subgroup theorem; an overview of the examples of Lie groups and their lattices is given in Table 1.

Definition 2.9 (Semi-simplicity)**.** A connected Lie group is *semi-simple* if it does not contain a non-trivial connected normal Abelian subgroup.

In terms of Lie algebras this definition can be rephrased as follows: a connected Lie group is semi-simple if and only if its Lie algebra is semi-simple (meaning that it does not contain a non-trivial Abelian ideal).

Caveat 2.10 (Semi-simplicity)**.** Some authors put additional constraints on semi-simple Lie groups such as the existence of a faithful finite dimensional complex representation.

Example 2.11 (Semi-simple Lie groups)**.** Prominent examples of semi-simple Lie groups are the special linear groups $\mathrm{SL}(n, \mathbf{R})$ for $n \in \mathbf{N}$; they all have finite centre. An example of a semi-simple Lie group with infinite centre is the universal covering of $\mathrm{SL}(2, \mathbf{R})$. In addition, looking at normal subgroups in products of simple groups we see that finite products of semi-simple Lie groups are also semi-simple Lie groups.

If G is a semi-simple Lie group, then the centre $C(G)$ of G is discrete and the quotient $G/C(G)$ is a centreless(!) semi-simple Lie group. Every centreless semi-simple Lie group can (essentially uniquely) be decomposed into a finite direct product of connected *simple* Lie groups.

Definition 2.12 ($\mathbf{R}$-Rank)**.** Let G be a semi-simple Lie group that is (for some $n \in \mathbf{N}$) a closed subgroup of $\mathrm{SL}(n, \mathbf{R})$. The $\mathbf{R}$-*rank* of G is the maximal

Lattice	ambient group	uniform?	irreducible?	**R**-rank
$\mathbf{Z}^n$	$\mathbf{R}^n$	$+$	iff $n \leq 1$	n
$\pi_1(M)$ [1]	$\operatorname{Isom}(\widetilde{M})$ [1]	$\circ$ [1]	$\circ$ [1]	$\circ$ [1]
$\mathrm{SL}(2,\mathbf{Z})$	$\mathrm{SL}(2,\mathbf{R})$	$-$	$+$	1
$\mathbf{Z} * \mathbf{Z}$	$\mathrm{SL}(2,\mathbf{R})$	$-$	$+$	1
$\mathrm{SL}(2,\mathbf{Z}) \times \mathrm{SL}(2,\mathbf{Z})$	$\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R})$	$-$	$-$	2
$\mathrm{SL}(2,\mathbf{Z}[\sqrt{2}])$	$\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R})$	$-$	$+$	2
$\mathrm{SL}(3,\mathbf{Z})$	$\mathrm{SL}(3,\mathbf{R})$	$-$	$+$	2

[1] here M is a complete connected Riemannian manifolds without boundary and with finite volume (see Examples 2.7 and 2.13)

Table 1: Overview of the examples for lattices

number $k \in \mathbf{N}$ such that G contains a k-dimensional Abelian subgroup that is conjugate to a subgroup of the diagonal matrices in $\mathrm{SL}(n,\mathbf{R})$. The **R**-rank of G is denoted by $\operatorname{rk}_{\mathbf{R}}(G)$.

This definition indeed does not depend on the chosen embedding of the Lie group in question into a special linear group [5, Section 21.3].

Example 2.13 (Ranks of Lie groups)**.** The **R**-rank of $\mathrm{SL}(n,\mathbf{R})$ equals $n-1$ for all $n \in \mathbf{N}_{>0}$. The **R**-rank is additive with respect to direct products, so $\operatorname{rk}_{\mathbf{R}}(\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R})) = 2$. Compact Lie groups have **R**-rank equal to 0; this should not be confused with other types of ranks used for compact Lie groups!

Notice that the rank of a Riemannian symmetric space defined in terms of totally geodesic flat submanifolds coincides with the **R**-rank of the corresponding semi-simple Lie group [2, 15, Section 2.7, Chapter 2].

During the proof of the normal subgroup theorem, certain subgroups of the semi-simple Lie groups under consideration play a decisive rôle:

Definition 2.14 (Borel subgroups)**.** Let G be a Lie group. A maximal connected solvable subgroup of G is called a *Borel subgroup*.

Example 2.15 (Borel subgroup of $\mathrm{SL}(n, \mathbf{R})$)**.** For $n \in \mathbf{N}$ the subgroup of upper triangular matrices in $\mathrm{SL}(n, \mathbf{R})$ is a Borel subgroup of $\mathrm{SL}(n, \mathbf{R})$. Checking that the upper triangular matrices indeed are a *maximal* solvable subgroup of $\mathrm{SL}(n, \mathbf{R})$ is non-trivial and best done on the level of Lie algebras [5, Section 21.3].

Notice that, by definition, Borel subgroups are amenable (see Section 4).

One can show that all Borel subgroups in a connected Lie group are conjugate. Moreover, algebraic subgroups of connected algebraic groups are Borel subgroups if and only if they are minimal parabolic subgroups; recall that an algebraic group is nothing but a group object in the category of varieties (and regular maps) and that a subgroup is parabolic if the corresponding quotient is a projective variety.

3 The normal subgroup theorem revisited

In the subsequent paragraphs, we have a closer look at the statement and the proof of the normal subgroup theorem.

3.1 The statement of the normal subgroup theorem

Key examples of lattices that satisfy the assumptions of the normal subgroup theorem are $\mathrm{SL}(3, \mathbf{Z}) \subset \mathrm{SL}(3, \mathbf{R})$ and $\mathrm{SL}(2, \mathbf{Z}[\sqrt{2}]) \subset \mathrm{SL}(2, \mathbf{R})^2$.

On the other hand, standard examples show that the two main conditions posed in the normal subgroup theorem cannot easily be relaxed:

- *Irreducibility.* Obviously, product lattices in products of Lie groups in general do not satisfy the conclusion of the normal subgroup theorem.
- *Higher rank.* For example, $\mathrm{SL}(2, \mathbf{R})$ has $\mathbf{R}$-rank equal to 1 and we can view $\mathbf{Z} * \mathbf{Z}$ as an irreducible lattice in $\mathrm{SL}(2, \mathbf{R})$; but clearly, $\mathbf{Z} * \mathbf{Z}$ does not satisfy the conclusion of the normal subgroup theorem.

3.2 The proof of the normal subgroup theorem

We now explain the basic strategy for the proof of the normal subgroup theorem (Theorem 1.1): Let us first recall the setup of the normal subgroup theorem; let G be a connected semi-simple Lie group with finite centre and with $\operatorname{rk}_{\mathbf{R}}(G) \geq 2$, let $\Gamma \subset G$ be an irreducible lattice, and let $N \subset \Gamma$ be a normal subgroup of Γ.

Because G has higher **R**-rank, it must be non-compact. In particular, N cannot both be contained in the centre of G and have finite index in Γ. So, in the following, we suppose that N is not contained in the centre of G.

In order to prove the normal subgroup theorem, it then suffices to show that Γ/N is finite. Basic examples such as congruence subgroups in integral special linear groups show that in general Γ will have many normal subgroups of finite index and that these indices are not uniformly bounded.

The knight in shining armour for proving finiteness of the quotient Γ/N is the interplay between two competing properties of groups – amenability and property (T). Thus, the basic strategy of proof for the normal subgroup theorem consists of the following three steps:

1. We show that the quotient Γ/N is amenable.
2. We show that the quotient Γ/N has property (T).
3. We conclude that the quotient Γ/N is finite.

In Section 4, we discuss amenability in detail. Section 5 gives a short overview of property (T). Finally, Section 6 is concerned with the above three steps, mainly with the first step.

4 What is amenability?

A geometric and quite concrete approach to amenability is provided by the notion of a Følner sequence, which is a precise way of saying that a group is amenable if it contains subsets of finite non-zero measure that are almost invariant under translation.

Definition 4.1 (Amenability)**.**

1. A topological group is *amenable* if it admits a Følner sequence.

2. Let G be a topological group and let μ be a Haar measure on G. A *Følner sequence* for G is a sequence $(S_n)_{n\in\mathbf{N}}$ of measurable subsets of G of finite μ-measure such that for every compact subset $K \subset G$ we have

$$\lim_{n\to\infty} \sup_{g\in K} \frac{\mu(S_n \bigtriangleup g \cdot S_n)}{\mu(S_n)} = 0.$$

(Here, $\bigtriangleup$ stands for the symmetric difference of sets.)

Before giving examples of (non-)amenable groups in Section 4.3, we first present a list of standard characterisations of amenable groups and study the inheritance properties of amenable groups.

4.1 Characterising amenability

Translating the Følner condition into different contexts – such as measure theory, ergodic theory or representation theory – leads to characterisations of amenable groups in terms of the existence of certain averaging operations or weak fixed point properties (Theorem 4.2). Here, we restrict ourselves to descriptions of amenability that are related to the (proof of the) normal subgroup theorem; other interesting aspects of amenability include the characterisation via paradoxical decompositions and the Banach-Tarski paradoxon [14, Section 3.12ff] or the characterisation via bounded cohomology [6].

Theorem 4.2 (Characterising amenability)**.** *Let G be a topological group. Then the following are equivalent:*

1. Geometry. *The group G is amenable, i.e., it possesses a Følner sequence.*
2. Averaging. *There exists a G-invariant mean on $L^\infty(G,\mathbf{R})$, i.e., there is a linear function $m\colon L^\infty(G,\mathbf{R}) \longrightarrow \mathbf{R}$ satisfying*

$$\inf_{x\in G} f(x) \leq m(f) \leq \sup_{x\in G} f(x),$$
$$m(g\cdot f) = m(f)$$

for all $f \in L^\infty(G,\mathbf{R})$ and all $g \in G$; here, the function $g\cdot f$ is defined via $x \longmapsto f(g^{-1}\cdot x)$.

3. Ergodic theory. *For every continuous G-action on a non-empty compact metrisable space X there exists a G-invariant probability measure on X.*
4. Representation theory. *The regular representation of G on $L^2(G, \mathbf{C})$ has almost invariant vectors.*

For the sake of completeness we briefly recall the occuring terminology from representation theory:

Definition 4.3 (Unitary representations and (almost) invariant vectors)**.** Let G be a topological group.

– A *unitary representation* of G on a (complex) Hilbert space H is a homomorphism $G \longrightarrow U(H)$ that is continuous in the strong topology.
– An *invariant vector* of a unitary representation π of G on H is a vector $x \in H \setminus \{0\}$ such that $\pi(g)(x) = x$ for all $g \in G$.
– Let $K \subset G$ be a compact subset and let $\varepsilon \in \mathbf{R}_{>0}$. A vector $x \in H$ is a (K, ε)*-invariant vector* of a unitary representation π of G on H if
$$\big\|\pi(g)(x) - x\big\| < \varepsilon \cdot \|x\|$$
holds for all $g \in K$. (Notice in particular that $x \neq 0$).
– A unitary representation of G *has almost invariant vectors* if it has for every compact subset $K \subset G$ and every $\varepsilon \in \mathbf{R}_{>0}$ a (K, ε)-invariant vector.

For the purpose of this talk, we do not need the equivalence of the characterisations given in Theorem 4.2 – therefore, we only provide proofs for the implications needed in the sequel. The proof of the remaining implications can, for example, be found in the books of Paterson [14] and Witte Morris [15, Chapter 10].

Partial proof of Theorem 4.2.

$1 \Rightarrow 2$ Let μ be a Haar measure on G, let $(S_n)_{n\in\mathbf{N}}$ be a Følner sequence for G, and let ω be some non-principal ultrafilter on $\mathbf{N}$. Then
$$\begin{aligned} L^\infty(G, \mathbf{R}) &\longrightarrow \mathbf{R} \\ f &\longmapsto \lim_{n\in\omega} \frac{1}{\mu(S_n)} \cdot \int_G f \cdot \chi_{S_n} \, d\mu \end{aligned}$$
is a mean on $L^\infty(G, \mathbf{R})$; that this mean is G-invariant follows easily from the almost translation invariance of the Følner sets.

2 ⇒ 3 Let $m\colon L^\infty(G,\mathbf{R}) \longrightarrow \mathbf{R}$ be a G-invariant mean, and suppose that G acts continuously on a non-empty compact metrisable space X. We choose a point $x \in X$. Combining the mean m with the map

$$\begin{aligned} C(X,\mathbf{R}) &\longrightarrow L^\infty(G,\mathbf{R}) \\ f &\longmapsto \bigl(g \mapsto f(g\cdot x)\bigr), \end{aligned}$$

we obtain a G-invariant non-trivial bounded linear functional m_X on $C(X,\mathbf{R})$; notice that the map above is well-defined because X is compact.

In view of the Riesz representation theorem, we can interpret the functional $m_X \in C(X,\mathbf{R})'$ as a probability measure on X; this probability measure is G-invariant because m_X is G-invariant.

1 ⇒ 4 Let μ be a Haar measure on G and let $(S_n)_{n\in\mathbf{N}}$ be a Følner sequence for G. Moreover, let $K \subset G$ be compact and $\varepsilon \in \mathbf{R}_{>0}$. By the Følner condition, there is an $n \in \mathbf{N}$ with

$$\sup_{g\in K} \frac{\mu(S_n \bigtriangleup g\cdot S_n)}{\mu(S_n)} < \varepsilon^2.$$

Then $f := \chi_{S_n} \in L^2(G,\mathbf{C})$ is a (K,ε)-invariant vector of the regular representation of G on $L^2(G,\mathbf{C})$ because for all $g \in K$ we have

$$\begin{aligned} \|g\cdot f - f\|_2^2 = \|g\cdot\chi_{S_n} - \chi_{S_n}\|_2^2 &= \|\chi_{S_n\bigtriangleup g\cdot S_n}\|_2^2 \\ &= \int_G \chi^2_{S_n\bigtriangleup g\cdot S_n}\,d\mu \\ &= \mu(S_n \bigtriangleup g\cdot S_n) \\ &< \varepsilon\cdot\mu(S_n) \\ &= \varepsilon\cdot\|f\|_2^2. \end{aligned}$$

Hence, the regular representation of G has almost invariant vectors, as desired. □

4.2 Inheritance properties of amenability

The class of amenable groups enjoys the following inheritance properties [14, Proposition 0.16, 1.12, 1.13]:

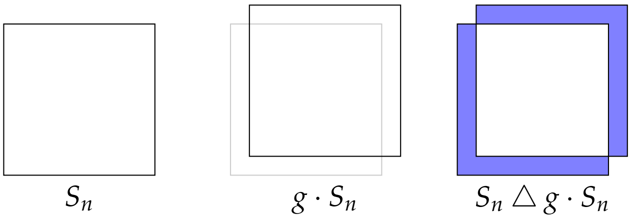

Figure 2: The Følner condition in $\mathbf{Z}^2$, schematically

Proposition 4.4 (Inheritance properties of amenability)**.**

1. *Closed subgroups of amenable groups are amenable.*
2. *Quotients of amenable groups by closed normal subgroups are amenable.*
3. *The class of amenable groups is closed with respect to taking extensions: If a topological group G contains a closed normal amenable subgroup such that the quotient group is amenable, then G is amenable.*
4. *Direct unions of amenable groups are amenable.*
5. *A countable discrete group is amenable if and only if all its finitely generated subgroups are amenable.*

Moreover, from the perspective of geometric and measurable group theory it is significant that amenability is a quasi-isometry invariant (which can be seen by carefully examining Følner sequences) and that all countable discrete amenable groups are measure equivalent [13].

4.3 Examples of (non-)amenable groups

Example 4.5 (Compact groups)**.** Obviously, all compact topological groups (and in particular all finite groups) are amenable.

Example 4.6 (Finitely generated free Abelian groups)**.** Let $d \in \mathbf{N}_{>0}$. Then a straightforward computation shows that $(\{-n, \ldots, n\}^d)_{n \in \mathbf{N}}$ is a Følner sequence for $\mathbf{Z}^d$; roughly speaking, the symmetric differences occuring in the Følner condition grow polynomially with exponent $d - 1$, but the given sequence grows polynomially with exponent d, see Figure 2.

Using the extension property for amenable groups, we can deduce that all finitely generated Abelian groups are amenable; in view of the direct union property for amenable groups, it follows that all discrete Abelian groups are amenable.

Similarly, $([-n, n]^d)_{n\in\mathbf{N}}$ is a Følner sequence for $\mathbf{R}^d$.

More generally, using the Kakutani-Markov fixed point theorem and the characterisation of amenability through ergodic theory, one can show that all Abelian topological groups are amenable [14, Proposion 0.14, 0.15]. Notice however, that it is in general not possible to give explicit invariant means for infinite Abelian groups – not even for $\mathbf{Z}$.

Example 4.7 (Solvable groups)**.** All solvable topological groups are amenable; this follows from Example 4.6 and the fact that the class of amenable groups is closed with respect to taking extensions (Proposition 4.4).

Example 4.8 (Non-Abelian free groups)**.** Non-Abelian free groups are not amenable: Because subgroups of amenable groups are amenable, it suffices to prove that the free group F on two generators is not amenable. In order to prove that F is not amenable, we show that F does not satisfy the ergodic theoretical condition:

To this end we consider the continuous action of F on the circle $S^1 = \mathbf{R}/\mathbf{Z}$ given by the two homeomorphisms

$$\begin{aligned} f\colon S^1 &\longrightarrow S^1 \\ [t] &\longmapsto [t^2], \\ g\colon S^1 &\longrightarrow S^1 \\ [t] &\longmapsto [t+\alpha], \end{aligned}$$

where $\alpha \in \mathbf{R}\setminus\mathbf{Z}$, i.e., g is nothing but rotation about α. *Assume* for a contradicition that there exists a F-invariant probability measure μ on S^1. For $\varepsilon \in \mathbf{R}_{>0}$ there exists a $t \in [0,1)$ with $\mu([0,t]) > 1-\varepsilon$. Using the f-invariance of μ we obtain

$$\begin{aligned} \mu\bigl(\{[0]\}\bigr) &= \mu\Bigl(\bigcap_{n\in\mathbf{N}} f^n([0,t])\Bigr) = \lim_{n\to\infty} \mu\bigl(f^n([0,t])\bigr) = \lim_{n\to\infty} \mu\bigl([0,t]\bigr) \\ &> 1-\varepsilon, \end{aligned}$$

and hence $\mu(\{[0]\}) = 1$. But now g-invariance of μ contradicts the fact that $\mu(S^1) = 1$. Therefore, there is no F-invariant probability measure on S^1, which proves that F is not amenable.

Alternatively, one can also show that F does not admit a Følner sequence or that F has a paradoxical decomposition. More advanced methods also reveal that F is not amenable – for example, the L^2-Betti numbers and the bounded cohomology of F are non-trivial.

Moreover, if one knows for some other reason that there exists a discrete non-amenable group, then using the inheritance properties of amenability one can conclude that the free group on two generators has to be non-amenable. Conversely, von Neumann asked whether all non-amenable groups have to contain the free group on two generators as a subgroup; Olshanskii was the first to prove that there are non-amenable groups that do not contain the free group on two generators – more, precisely, he produced torsion groups that are non-amenable [11].

Example 4.9 (Integral special linear groups)**.** Let $n \in \mathbf{N}_{>1}$. Then $\mathrm{SL}(n, \mathbf{Z})$ is not amenable: Notice that $\mathrm{SL}(n, \mathbf{Z})$ contains the free group on two generators. As subgroups of amenable groups are amenable but the free group on two generators is not amenable, also $\mathrm{SL}(n, \mathbf{Z})$ cannot be amenable.

Alternatively, one can study the canonical action of $\mathrm{SL}(n, \mathbf{Z})$ on projective $(n-1)$-space and refer to the Furstenberg Lemma [16, p. 62 and Lemma 3.2.1]

5 What is Kazhdan's property (T)?

Property (T) is a rigidity property for groups, in a representation theoretic sense:

Definition 5.1 (Kazhdan's property (T))**.** A topological group G has *(Kazhdan's) property (T)* if there exists a compact subset $K \subset G$ and an $\varepsilon \in \mathbf{R}_{>0}$ such that all unitary representations of G with a (K, ε)-invariant vector have already an invariant vector.

Alternative descriptions of property (T) can be given, for example in terms of the weak containment relation of representations or in terms of cohomology [1]. An interesting aspect of property (T) is that it can be used as a tool to exhibit explicit expanders [10].

5.1 Unmasking finite groups via amenability and property (T)

The coup de grace of the proof of the normal subgroup theorem consists of playing amenability and property (T) out against each other:

Proposition 5.2. *Any discrete group that both is amenable and has property (T) is finite.*

Proof. Let Γ be a discrete group that is amenable and that has property (T). The representation theoretic characterisation of amenability shows that the regular representation of Γ on $\ell^2(\Gamma)$ has almost invariant vectors. Therefore, Γ having property (T) entails that the regular representation of Γ has a (non-trivial) invariant vector. But this clearly implies that Γ is finite. □

In particular, the integers $\mathbf{Z}$ do not have property (T) and property (T) is not stable under taking subgroups.

5.2 Inheritance properties of property (T)

Curiously, property (T) does not allow groups to be too large – they have to be compactly generated [1, Section 1.3] (Proposition 5.3). For the proof of the normal subgroup theorem an important feature is that property (T) descends to quotients by normal subgroups [1, Section 1.7] (Proposition 5.4).

Proposition 5.3 (Property (T) implies compact generation)**.** *Any topological group with property (T) is compactly generated.*

Proof. For simplicity, we only consider the case of discrete groups with property (T). Let Γ be a countable discrete group with property (T), and let $F(\Gamma)$ be the set of all finitely generated subgroups of Γ. Then translation induces a unitary representation of Γ on

$$H := \bigoplus_{F \in F(\Gamma)} L^2(\Gamma/F, \mathbf{C}).$$

Let (K, ε) be a finite subset of Γ and a positive real number witnessing property (T) of Γ. Then the subgroup $\langle K \rangle$ generated by K lies in $F(\Gamma)$ and $\chi_{\langle K \rangle / \langle K \rangle} \in L^2(\Gamma/F, \mathbf{C})$ is a (K, ε)-invariant vector of the Γ-representation on H. Because Γ has property (T) this forces H to have a Γ-invariant vector; in particular, there is a finitely generated subgroup F of Γ such that $L^2(\Gamma/F, \mathbf{C})$ has a Γ-invariant vector. As Γ acts transitively on Γ/F we see that Γ/F is finite. Hence, also Γ is finitely generated. □

In particular, property (T) is not stable with respect to taking colimits of groups.

Proposition 5.4 (Property (T) descends to quotients)**.** *If a topological group has property (T) then so do all its quotients by closed normal subgroups.*

Proof. For simplicity, we only consider the case of discrete groups. So let Γ be a countable discrete group and let N be a normal subgroup of Γ.

We have to show that unitary representations of Γ/N are rigid. Let (K, ε) be a compact subset of Γ and a positive real number witnessing that Γ has property (T); we now show that $(\overline{K}, \varepsilon)$ witnesses that the quotient Γ/N has property (T), where $\overline{K}$ is the image of K under the canonical projection $\Gamma \longrightarrow \Gamma/N$:

Let π be a unitary representation of Γ/N on a Hilbert space H. Composing π with the canonical projection $\Gamma \longrightarrow \Gamma/N$, we obtain a unitary representation $\widetilde{\pi}$ of Γ on H. Suppose that π has a $(\overline{K}, \varepsilon)$-invariant vector x. Clearly, then x is also a (K, ε)-invariant vector of $\widetilde{\pi}$. Because Γ has property (T), the representation $\widetilde{\pi}$ has an invariant vector $x' \in H$; but then, by construction of $\widetilde{\pi}$, the vector x' is also invariant for the representation π of Γ/N. Hence, Γ/N has property (T) as well. □

In contrast to amenability, property (T) is not a geometric property, i.e., it is not a quasi-isometry invariant [3, Section 2.3].

5.3 Examples of groups (not) having property (T)

Example 5.5 (Compact groups)**.** All unitary representations of compact topological groups have an invariant vector, as can be seen by averaging via the Haar measure on the group in question. In particular, all finite groups have property (T).

Example 5.6 (Non-Abelian free groups)**.** Non-Abelian free groups do not have property (T): Looking at the Abelianisation of non-Abelian free groups we see that non-Abelian free groups have infinite amenable quotients. If a non-Abelian free group would have property (T), then so would all its quotients, contradicting the fact that the only discrete groups both being amenable and having property (T) are finite groups.

Example 5.7 (Lattices in higher rank Lie groups)**.** The key example of groups having property (T) are simple Lie groups of $\mathbf{R}$-rank at least 2 as well as irreducible lattices in semi-simple Lie groups of $\mathbf{R}$-rank at least 2 with finite centre (and without compact factors) [9, 16, Chapter III, Chapter 7].

6 Outline of proof for the normal subgroup theorem

In this section, we explore the steps of the basic strategy for the proof of the normal subgroup theorem (Theorem 1.1) in more detail, focusing on the amenability of the quotient group. We first recall the setup of the normal subgroup theorem:

Setup 6.1. *Let G be a connected semi-simple Lie group with finite centre and with $\operatorname{rk}_{\mathbf{R}}(G) \geq 2$, and let $\Gamma \subset G$ be an irreducible lattice; furthermore, let $N \subset \Gamma$ be a normal subgroup of Γ that is not contained in the centre of G.*

In order to prove the normal subgroup theorem, it then suffices to show that Γ/N is finite, which is achieved by the following steps:

1. We show that the quotient Γ/N is amenable.
2. We show that the quotient Γ/N has property (T).
3. We conclude that the quotient Γ/N is finite.

6.1 The first step

In order to verify that in the situation of Setup 6.1 the quotient Γ/N is amenable, we take the ergodic theory point of view of amenability. In other words, to prove that Γ/N is amenable, we have to show that whenever Γ/N acts on a non-empty compact metrisable space X then there exists a Γ/N-invariant probability measure on X.

Proposition 6.2 (First step)**.** *Let G and Γ be as in Setup 6.1. Then the quotient group Γ/N is amenable.*

Proof. By passing to the quotient by the centre of G, we may assume that G has trivial centre; hence, we may assume that N is not the trivial group.

Suppose that Γ/N acts (by homeomorphisms) on a non-empty compact metrisable space X. Of course, a Γ/N-invariant probablity measure on X is nothing but a fixed point of the Γ/N-action on $\operatorname{Prob}(X)$ induced by the action of Γ/N on X. Therefore, we study the Γ/N-action on $\operatorname{Prob}(X)$ in more detail; the fundamental idea – and the fundamental difficulty – is to link measurable Γ-actions successfully to the translation action of Γ on suitable quotients of the ambient group G.

Lemma 6.3. *Let G and Γ be as in Setup 6.1 and suppose that Γ acts on a non-empty compact metric space X. Moreover, let P be a closed amenable subgroup of G. Then there exists an essentially Γ-equivariant measurable map*

$$G/P \longrightarrow \operatorname{Prob}(X);$$

here, $\operatorname{Prob}(X)$ denotes the space of probability measures on X (with respect to the Borel σ-algebra on X) – as topology on $\operatorname{Prob}(X)$ we take the one given by total variation. The term "essentially" refers to null sets with respect to the Haar measure on G/P.

Notice that the continuous Γ-action on X induces a continuous Γ-action on $\operatorname{Prob}(X)$.

Proof of Lemma 6.3. For brevity, we write $Z := \operatorname{Prob}(X)$ in the rest of this proof.

As Z is not empty, also the space $L^\infty(G,Z)^\Gamma$ of Γ-equivariant measurable maps of type $G \longrightarrow Z$ is not empty; for example, one can choose a measurable fundamental domain for Γ in G, and then define a measurable function $G \longrightarrow Z$ by extending a constant function on the measurable fundamental domain via translation by Γ. Moreover, the space $L^\infty(G,Z)^\Gamma$ carries a G-action by translation.

Standard techniques from measure theory show that $L^\infty(G,Z)^\Gamma$ is a compact metrisable space. Because P is assumed to be amenable, there exists a

P-invariant probability measure μ on $L^\infty(G,Z)^\Gamma$. A straightforward computation shows that the integral

$$\psi_P := \int_{L^\infty(G,Z)^\Gamma} f \, d\mu(f)$$

is a well-defined element of $L^\infty(G,Z)^\Gamma$, which is P-invariant; this relies on the fact that Z and hence $L^\infty(G,Z)^\Gamma$ are convex.

Using translation by G, we obtain from ψ_P a well-defined measurable map $G/P \longrightarrow Z = \operatorname{Prob}(X)$. By construction, this map is essentially Γ-equivariant. □

The lion share of the proof of amenability of the quotient Γ/N is the following step [9, 16, Chapter IV, Chapter 8].

Black box 6.4. *Let G and Γ be as in Setup 6.1, let P be a Borel subgroup of G. Suppose that Γ acts on the non-empty compact metrisable space Y and that there is an essentially Γ-equivariant measurable map $\psi\colon G/P \longrightarrow Z$.*

1. *Then the action of Γ on Z extends almost everywhere to an action of G on Z such that ψ is essentially G-equivariant.*
2. *In particular: The Γ-space Z is measurably isomorphic to a Γ-space G/Q, where Q is some closed subgroup of G containing the Borel subgroup P and where G/Q carries the Γ-action given by translation.*

(Here, as measure on Z we use the ψ-push-forward of the measure induced by the Haar measure on G).

We now continue with the proof of Proposition 6.2: Lemma 6.3 provides us with an essentially Γ-equivariant measurable map

$$\psi\colon G/P \longrightarrow \operatorname{Prob}(X),$$

where P is some Borel subgroup of G. Standard arguments show that $\operatorname{Prob}(X)$ is a compact metrisable space; so we can pull a closed subgroup Q of G containing P out of the Black box 6.4 with the property that the Γ-spaces $\operatorname{Prob}(X)$ and G/Q are measurably Γ-isomorphic (with respect to the measure on $\operatorname{Prob}(X)$ given by the ψ-push-forward of the Haar measure on G/P).

As next step we show that the action of Γ on G/Q is trivial: By assumption, the action of N on X, and hence on $\mathrm{Prob}(X)$, is trivial. In view of the previous paragraph, we can thus conclude, that the action of N on G/Q by translation is essentially trivial; because the translation action is continuous and because the Haar-measure on non-empty open subsets of G is non-zero, it follows that N in fact acts trivially on G/Q, not only essentially trivially. Hence, the kernel K of the G-action on G/Q contains the group N.

Because G is a semi-simple Lie group without centre, we can decompose G as a finite product of simple Lie groups. Since normal subgroups of products of non-Abelian simple groups are just products of some of the factors, we know that K is a product of the simple factors of G. *Assume* for a contradiction that K is not all of G. Hence, there is a factor H of G not contained in the product decomposition of K. As N is contained in K, this factor H lies in the normaliser $N_G(N)$. On the other hand, $\Gamma \subset N_G(N)$. Because Γ is an irreducible lattice, it follows that $G = \overline{\Gamma \cdot H} \subset N_G(N)$. A standard argument shows that discrete normal subgroups of G must lie in the centre of G, which is trivial. This contradicts the assumption that N is not the trivial group. Hence, $H = G$, which means that the translation G-action on G/Q is trivial.

In particular, the action of Γ on G/Q is trivial – and hence, the actions of Γ and Γ/N on $\mathrm{Prob}(X)$ are essentially trivial. Therefore, there is a Γ/N-fixed point in $\mathrm{Prob}(X)$, which proves that Γ/N is amenable. □

6.2 The second step

If a (discrete) group has property (T), then so do all its quotients (Proposition 5.4). Hence, it suffices to verify that the lattice Γ in the situation of the normal subgroup theorem (Theorem 1.1) has property (T); this in turn is a classical result of Kazhdan [7] and Margulis [9, Chapter III].

6.3 The third step

Discrete groups that both are amenable and have property (T) are finite in view of the representation theoretic characterisation of amenability (Proposition 5.2).

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