http://www.mathematik.uni-r.de/loeh/talk.pdf

04.05.2021

OPEN COVERS WITH GEOMETRIC π_1-CONSTRAINTS

I LS-CAT   II AMENABLE/POLY CATEGORY   III OBSTRUCTIONS

Question. "How difficult" is it to cover a space by "small" open subsets?

I

Def (Lusternik–Schnirelmann category). Let X be a top. space. Then

$$\mathrm{cat}_{LS}(X) := \min\Big\{ n \in \mathbb{N} \;\Big|\; \exists\, U_1, \dots, U_n \subset X \text{ open } \bigcup_{j=1}^{n} U_j = X \text{ and } \forall_j \; U_j \hookrightarrow X \text{ is null-homotopic}\Big\} \in \mathbb{N} \cup \{\infty\}$$

Examples. • $\mathrm{cat}_{LS}(S^1) = 2$

• $\mathrm{cat}_{LS}(\mathbb{R}P^n) = n+1$ (cup-length)

Modification: Relax the conditions on U_j.

Def. Let G be a class of groups, let X be a top. space.

- A subset $U \subset X$ is a <u>G-subset of X</u> if

$$\forall_{x \in U} \quad \underbrace{\pi_1(U \hookrightarrow X)(\pi_1(U,x))}_{\in \pi_1(X,x)} \in G$$

- $\mathrm{cat}_G(X) := \min\{n \in \mathbb{N} \mid \exists\, U_1, \dots, U_n \subset X$ open, $\bigcup_{j=1}^{n} U_j = X$ and $\forall_j\ U_j$ is a G-subset of $X\}$

$\in \mathbb{N} \cup \{\infty\}$.

First example: $G := 1 :=$ class of all trivial groups.

[Eilenberg-Ganea]

Observation: If $1 \subset G$, then

$$? \leq \mathrm{cat}_G(X) \leq \mathrm{cat}_1(X) \leq \mathrm{cat}_{LS}(X)$$

$\leq \dim X + 1$

if X is a conn. simplicial complex

II Amenable/Poly covers

Recall: A group Γ is <u>amenable</u> if there ex. a Γ-inv. mean $m: \ell^\infty(\Gamma, \mathbb{R}) \to \mathbb{R}$

- m is $\mathbb{R}$-linear
- $m(1) = 1$
- $m(f) \geq 0$ if $f \geq 0$
- $\forall_{\gamma \in \Gamma} \forall_{f \in \ell^\infty(\Gamma, \mathbb{R})} \; m(\gamma \cdot f) = m(f)$.

- All finite groups are amenable.
- All Abelian groups are amenable.
- The class Am of all amenable gps is closed wrt subgroups, quotients, and extensions.

$\Rightarrow$ all virtually solvable groups are amenable.

- Let Poly denote the class of all fin. gen. groups of polynomial growth. (= class of fin. gen. virtually nilpotent groups by Gromov's polynomial growth thm).

- non-cyclic free groups are <u>not</u> amenable!

Examples. • $\mathrm{cat}_{Am}(S^1) = 1 = \mathrm{cat}_{Poly}(S^1)$

• $\mathrm{cat}_{Am}\big((S^1)^{\times n}\big) = 1 = \mathrm{cat}_{Poly}\big((S^1)^{\times n}\big)$

• $\mathrm{cat}_{Am}(\mathbb{R}P^n) = 1 = \mathrm{cat}_{Poly}(\mathbb{R}P^n)$

• $1 < \mathrm{cat}_{Am}(\Sigma_2) \leq 2+1 = 3$

exact value ?

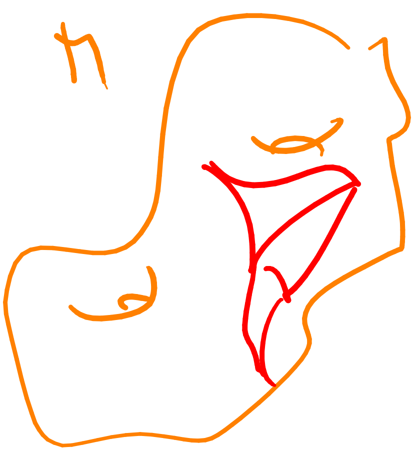
III OBSTRUCTIONS

1. SIMPLICIAL VOLUME — Am
2. MINIMAL VOLUME ENTROPY — Poly, ...
3. (L²-BETTI NUMBERS) — Am

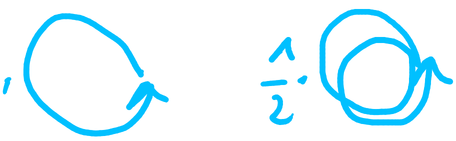
1 **Def.** [Gromov] (simplicial volume). Let M be an occ mfd of dim n. Then

singular chains!

$$\|M\| := \|[M]_{\mathbb{R}}\|_1 := \inf\{\, |c|_1 \mid c \in C_n(M;\mathbb{R}) \text{ fund. cycle} \,\} \in \mathbb{R}_{\geq 0}$$



Examples.

• $\|S^1\| = 0$ $\quad 1\cdot$  $\quad \frac{1}{2}\cdot$ ...

$1 \qquad \frac{1}{2}$

spheres, tori, ...

• More generally: If $\pi_1(M)$ is amenable (and $\dim M \neq 0$), then $\|M\| = 0$. [Gromov, Ivanov]

Idea of proof:

$$\|M\| = \|[M]_{\mathbb{R}}\|_1 = \underset{=0}{\sup} \Big\{ \frac{1}{\|\varphi\|_\infty} \;\Big|\; \varphi \in \overbrace{H_b^n(M;\mathbb{R})}^{=0}, \ \langle \varphi, [M]_{\mathbb{R}} \rangle = 1 \Big\}$$

$$H_b^*(\,\cdot\,;\mathbb{R}) := H^*\big(B(C_*^{|\cdot|_1}(\,\cdot\,;\mathbb{R}), \mathbb{R})\big)$$

bounded cohomology

If $\pi_1(M)$ is amenable, then $H_b^n(M;\mathbb{R}) \cong 0$ for all $n \geq 1$.

(by "amenable transfer") "□"

• If M is hyperbolic, then

$$\underbrace{\|M\|}_{\text{homotopy invariant!}} = \frac{\mathrm{vol}(M)}{v_{\dim M}} > 0 \quad \text{[Gromov, Thurston]}$$

Theorem. [Gromov, Ivanov] If M is an occ mfd, then:

$$\mathrm{cat}_{Am} M \leq \dim M \Longrightarrow \|M\| = 0.$$
$$\overset{?}{\Longleftarrow}$$

(i.e. $\|M\| > 0 \Longrightarrow \mathrm{cat}_{Am} M = \dim M + 1$)

Idea of proof: Let $\mathcal{U}$ be an open (convex) Am-cover of M. Let $n := \dim M$. Then:

$$\begin{tikzcd} H^n_b(M;\mathbb{R}) \arrow[rr, "\text{comparison map}"] \arrow[dr, dashed, "\exists"'] \arrow[rr, phantom, shift right=6ex, "G"] & & H^n(M;\mathbb{R}) \\ & \underbrace{H^n(\operatorname{nerve}(\mathcal{U});\mathbb{R})}_{\cong 0 \text{ if } n \geq \operatorname{mult}(\mathcal{U})} \arrow[ur, "H^n(\text{nerve map};\mathbb{R})"'] & \end{tikzcd}$$

proofs:

- multicomplexes [Gromov]
- spectral seq's [Ivanov]
- classifying spaces for the family Am [L, Sauer]

$\square$

(2) Theorem [Babenko, Sabourau] If X is a conn. finite simplicial complex, then

$$\operatorname{cat}_{\mathrm{Poly}} X \leq \dim X \implies \underbrace{\operatorname{minent}(X)}_{\substack{\text{is about log vol. growth}\\ \text{rate of balls in } \tilde{X}}} = 0.$$

Observation [L, Moraschini] Let $F \to E \to B$ be a fibre bundle of ~~oc~~ mfds and let G be an ISQ class of groups. Then

$$\operatorname{cat}_G F \leq \frac{\dim E}{\operatorname{cat}_{LS} B} \implies \operatorname{cat}_G E \leq \dim E.$$

- If $G = \mathrm{Am}$ $\rightsquigarrow$ new vanishing results for $\|\cdot\|$

- If $G = \overline{\mathrm{Poly}}$ $\rightsquigarrow$ new vanishing results for minent.

Why interesting?

- M smooth mfd $\rightsquigarrow$ $\|M\| = 0 \overset{?}{\Longleftrightarrow} \text{minent } M = 0$

  $\Longleftarrow$ ✓

  $\Longrightarrow$??

- M occ aspherical mfd

  $\rightsquigarrow$ $\|M\| = 0 \overset{?}{\Longrightarrow} \underbrace{\chi(M) = 0}$

  ?? $\Downarrow$ $\Uparrow$ $= \sum_{j \in \mathbb{N}} (-1)^j \, b_j^{(2)}(M)$

  [Gromov, Sauer]

  $\mathrm{cat}_{\mathrm{Am}} M \leq \dim M$