http://www.mathematik.uni-r.de/loeh/talk.pdf 15.04.2021

AMENABLE COVERS OF MANIFOLDS

I. AMENABLE COVERS II. OBSTRUCTIONS

http://www.mathematik.uni-r.de/loeh/talk.pdf

Question: How difficult is it to cover a space by small open sets?

I

Classical example: Lusternik-Schnirelmann category

Def: Let X be a top. space. Then

$$\operatorname{cat}_{LS}(X) := \min \Big\{ n \in \mathbb{N} \;\Big|\; \exists_{U_1,\dots,U_n \underset{\text{open}}{\subset} X} \ \forall_{j \in \{1,\dots,n\}} \ U_j \hookrightarrow X \text{ null-homotopic and } \bigcup_{j=1}^{n} U_j = X \Big\} \in \mathbb{N} \cup \{\infty\}.$$

Examples: $\cdot \operatorname{cat}_{LS}(S^n) = 2$

$\cdot \operatorname{cat}_{LS}(\mathbb{R}P^n) = n+1$ (cup-length)

Modification: Relax the conditions on U_j.

Def: Let G be a class of groups, let X be a top. space.

- A subset $U \subset X$ is a <u>G-subset of X</u> if
$$\forall_{x \in U} \quad \underbrace{\pi_1(U \hookrightarrow X)\left(\pi_1(U,x)\right)}_{< \pi_1(X,x)} \in G$$

- let
$$\mathrm{cat}_G(X) := \min\{n \in \mathbb{N} \mid \exists_{\substack{U_1,\dots,U_n \subset X \\ \text{open}}} \forall_{j \in \{1,\dots,n\}} \; U_j \text{ is a } G\text{-subset of } X \text{ and } \bigcup_{j=1}^{n} U_j = X\} \in \mathbb{N} \cup \{\infty\}.$$

Example: $G := 1 :=$ all trivial groups
[Eilenberg-Ganea]

- $G := \mathrm{Am} =$ all amenable groups.

Observation:
$$? \leq \mathrm{cat}_{\mathrm{Am}}(X) \leq \mathrm{cat}_1(X) \leq \mathrm{cat}_{LS}(X)$$

If X is a simplicial complex $\longrightarrow \leq \dim X + 1$

or obstructions ?

II Examples of obstructions:

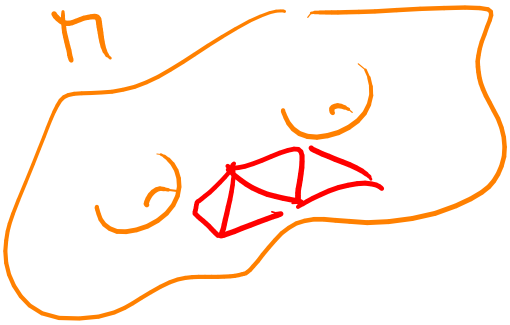
① simplicial volume

② L^2-Betti numbers

singular chain complex

① Def: [Gromov] (simplicial volume). Let M be an occ manifold of dim n. Then

$$\|M\| := \|[M]_{\mathbb{R}}\|_1 := \inf\{|c|_1 \mid c \in C_n(M;\mathbb{R}) \text{ fund. cycle}\}$$

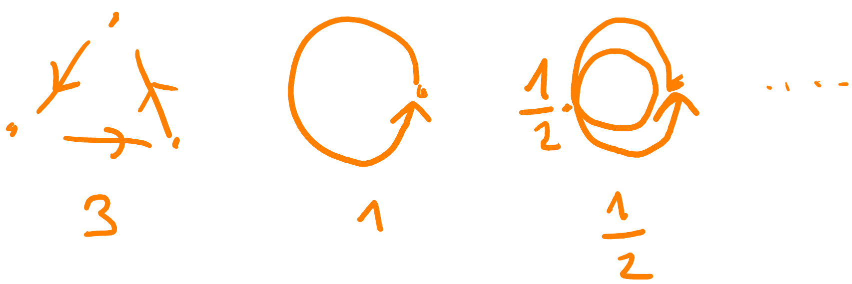


$$\Big|\sum_{j=1}^{k} a_j \cdot \sigma_j\Big|_1 := \sum_{j=1}^{k} |a_j|.$$

is a homotopy invariant.

Basic examples:

- $\|S^1\| = 0$



similarly: tori, S^n with $n \geq 1$, [Gromov, Ivanov]

If $\pi_1(M)$ is amenable, then $\|M\| = 0$.

idea of proof:

- $\|M\| = \|[M]_{\mathbb{R}}\|_1$

$$= \sup \left\{ \frac{1}{\|\varphi\|_\infty} \;\middle|\; \varphi \in H^n_{(b)}(M;\mathbb{R}),\ \langle \varphi, [M]_\mathbb{R} \rangle = 1 \right\}$$

$H^*_b := H^*(B(C_*, \mathbb{R}))$

bounded cohomology

- $H^n_b(X;\mathbb{R}) \cong 0$ if $n>0$ and $\pi_1(X)$ is amenable $\square$

transfer via invariant means

- $\|M\| = \frac{\mathrm{vol}(M)}{v_n}$ if M is occ hyperbolic [Gromov, Thurston].

homotopy invariant!

Theorem. [Gromov, Ivanov] If M is an occ mfd, then

$$\mathrm{cat}_{Am} M \le \dim M \implies \|M\| = 0.$$

$\Leftarrow$?

(i.e.: $\|M\| > 0 \implies \mathrm{cat}_{Am} M = \dim M + 1$)

e.g.: If M is an occ hyp mfd, then

$\mathrm{cat}_{Am} M = \dim M + 1$.

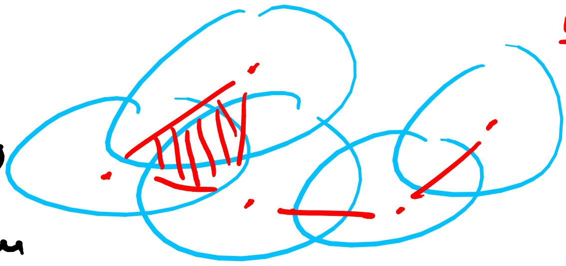
idea of proof: let U be an amenable (convex) open cover of M. Let $n := \dim M$.

Consider:

$$\begin{tikzcd}
H^n_b(M;\mathbb{R}) \arrow[rr, "\text{comparison map}"] \arrow[dr, dashed, "\exists"'] & \circlearrowleft & H^n(M;\mathbb{R}) \\
& \underbrace{H^n(\mathrm{nerve}(\mathcal{U});\mathbb{R})}_{\cong 0 \text{ if } \mathrm{mult}(\mathcal{U}) \leq n} \arrow[ur, "H^n(\text{nerve map};\mathbb{R})"'] &
\end{tikzcd}$$

proofs:

- multicomplexes [Gromov]
- spectral seq's [Ivanov]
- classif. spaces wrt family Am [Sauer-L]



$\square$

Most of the known vanishing results for simplicial volume follow from this thm.

Recent example: [Moraschini, L]

Let $F \to E \to B$ be a fibre bundle of occ mfds. If

$$\mathrm{cat}_{Am} F \leq \frac{\dim E}{\mathrm{cat}_{LS} B},$$

then $\|E\| = 0$.

Other interesting families: e.g.: Poly $\leadsto$ vanishing results for minvolume

② L^2-Betti numbers:

Def [Atiyah, ...] Let M be an occ mfd, let $k \in \mathbb{N}$. Then

$$b_k^{(2)}(M) := b_k^{(2)}(\pi_1(M) \curvearrowright \widetilde{M})$$

$$:= \dim_{L\pi_1(M)} H_k(M; L\pi_1(M))$$

twisted coeffs!

$$\in \mathbb{R}_{\geq 0}.$$

Basic examples:

- If M is aspherical ($\widetilde{M} \simeq *$) and $\pi_1(M)$ is amenable, then

$$\forall_{k \in \mathbb{N}} \quad b_k^{(2)}(M) = 0$$

[Cheeger Gromov]

- If M is occ hyperbolic, then

$$\forall_{k \in \mathbb{N},\ k \neq \frac{\dim M}{2}} \quad b_k^{(2)}(M) = 0.$$

[...]

- $$\chi(M) = \sum_{j \in \mathbb{N}} (-1)^j \cdot b_j^{(2)}(M)$$

Theorem [Sauer] Let M be an occ aspherical mfd. Then

$$\mathrm{cat}_{\mathrm{Am}} M \le \dim M \Longrightarrow \forall_{k \in \mathbb{N}}\, b_k^{(2)}(M) = 0.$$

Question. [Gromov] Let M be an occ aspherical mfd. Then

$$\|M\| = 0 \overset{?}{\Longrightarrow} \chi(M) = 0.$$

$$\begin{tikzcd}
\|M\| = 0 \arrow[r, Rightarrow, "?"] \arrow[d, Rightarrow, "?"'] \arrow[dr, Rightarrow, "?"] & \chi(M) = 0 \\
\mathrm{cat}_{\mathrm{Am}} M \le \dim M \arrow[r, Rightarrow, "\text{[Sauer]}"'] & \forall_{k \in \mathbb{N}}\, b_k^{(2)}(M) = 0 \arrow[u, Rightarrow]
\end{tikzcd}$$