http://www.mathematik.uni-r.de/loeh/talk.pdf

AMENABLE COVERS (joint work with R. Sauer)

I AMENABLE COVERS & CATEGORY II OBSTRUCTIONS/ESTIMATES

III GROMOV'S VANISHING THM REVISITED

I AMENABLE COVERS & CATEGORY

Definition. A group Γ is amenable if it admits an invariant mean, i.e., there ex. an $\mathbb{R}$-lin. map $m : \ell^\infty(\Gamma,\mathbb{R}) \longrightarrow \mathbb{R}$ s.t.

$m(1) = 1$ and $\forall_{f \in \ell^\infty(\Gamma,\mathbb{R})} \; f \geq 0 \Rightarrow m(f) \geq 0$

and $\forall_{f \in \ell^\infty(\Gamma,\mathbb{R})} \forall_{\gamma \in \Gamma} \; m(\gamma \cdot f) = m(f)$.

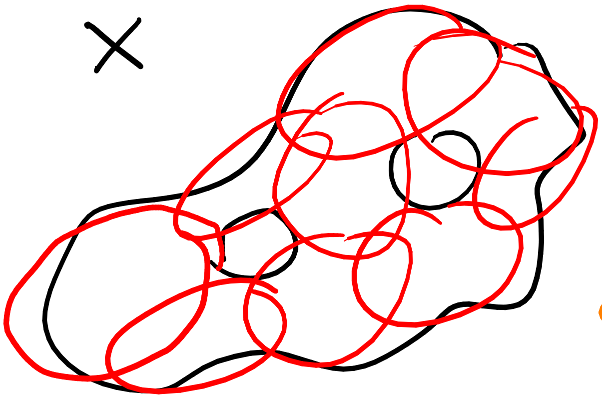
Examples:
- finite groups are amenable: $m(f) = \frac{1}{|\Gamma|} \cdot \sum_{\gamma \in \Gamma} f(\gamma)$
- Abelian groups are amenable
- amenable groups are closed wrt. subgroups, quotients, extensions

all virtually solvable gps are amenable

- F_2 is not amenable

"antagonist of negative curvature"

Definition. [Lusternik-Schnirelmann category] Let X be a top. space. Then

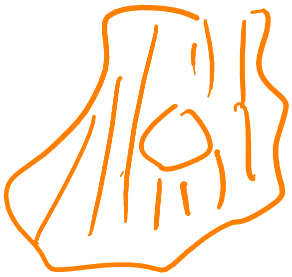


$\mathrm{cat}_{\cancel{LS}\,Am} X := \min\{n \in \mathbb{N} \mid$ there ex. open $\cancel{LS}$ Am-subsets $U_1, \dots, U_n$ of X : $X = \bigcup_{j=1}^n U_j\}$ $\in \mathbb{N} \cup \{\infty\}$

$U \subset X$ is an LS-subset if $U \hookrightarrow X$ is nullhomotopic

possible applications: top. robotics, sensor coverage

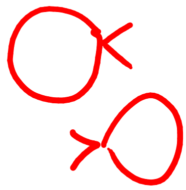
· $U \subset X$ is an <u>Am-subset</u> if
$\forall_{x \in U}$ im $\pi_1(U \hookrightarrow X, x) \subset \pi_1(X,x)$
is an amenable group.

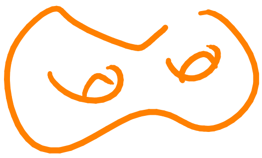


Examples: · $\mathrm{cat}_{Am} \leq \mathrm{cat}_{LS}$

· $\mathrm{cat}_{Am}\, S^1 \vee S^1 = 2$

≥ 2: $\pi_1 \cong F_2$

≤ 2: 



· $\mathrm{cat}_{Am}(\Sigma_2) = 2$ or 3

· $\mathrm{cat}_{Am}(n\text{-mfd}) \leq n+1$

II OBSTRUCTIONS/ESTIMATES

Theorem. [Gromov, Sauer] Let M be an or. cl. connected n-mfd with $\widetilde{M} \simeq *$. Then

$$\mathrm{cat}_{Am} M \leq n \implies \underbrace{\chi(M)}_{= \sum_k (-1)^k b_k^{(2)}(M)} = 0.$$

$\hookrightarrow \mathrm{cat}_{Am} \Sigma_2 = 3$

Theorem $\boxtimes$ [Gromov, Ivanov, Frigerio–Moraschini, L–Sauer] Let M be an occ n-mfd. Then

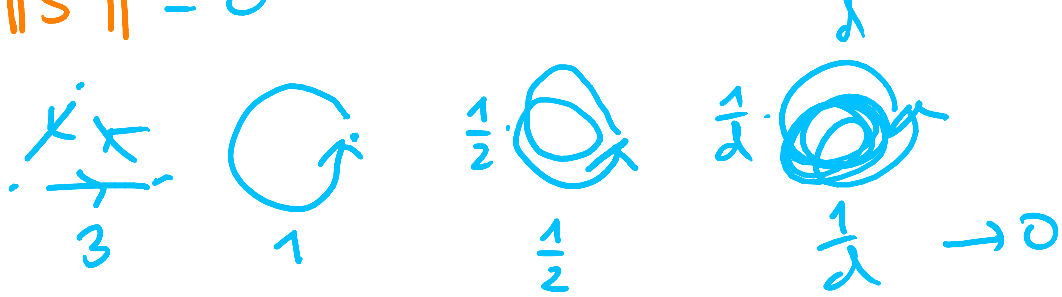
$$\mathrm{cat}_{Am} M \leq n \implies \underbrace{\|M\|} = 0.$$

$$\underbrace{\|M\|}_{\text{simplicial volume}} := \inf\Big\{ \sum_{j=1}^{m} |a_j| \;\Big|\; \underbrace{\sum_{j=1}^{m} a_j \cdot \sigma_j \in C_n(M;\mathbb{R})}_{\text{singular chains}} \;\; \underbrace{\mathbb{R}\text{-fund. cycle}}_{\text{gen. version of triangulations}} \Big\} \in \mathbb{R}_{\geq 0}$$

"min number of simplices"

Examples: • $\|M\| \leq$ min number of top-simplices in a triangulation of M

• $\|S^1\| = 0$



rigidity thm • $\|M\| = \dfrac{\operatorname{vol} M}{v_n}$ if M is hyperbolic [Gromov, Thurston]

Question. [Gromov] Let M be an occ mfd with $\tilde{M} \simeq *$.

$$\|M\| = 0 \overset{?}{\Longrightarrow} \chi(M) = 0$$

III) PROOF OF THEOREM $\boxed{*}$.

Thm $\boxed{*}$ follows from: (by duality)

Theorem $\boxed{**}$ [Gromov, Ivanov, ...] Let M be an occ mfd, and let $n \geq \operatorname{cat}_{Am}(M)$. Then $c_M = 0$:

$$\underbrace{H_b^n(M;\mathbb{R})}_{\text{bounded cohomology}} \xrightarrow{c_M} H^n(M;\mathbb{R})$$

$$:= H^n(\underbrace{C_b^*(M;\mathbb{R})}) := \Big\{ f \in C^*(M;\mathbb{R}) \;\Big|\; \sup_{\sigma \in \operatorname{map}(\Delta^*, M)} |f(\sigma)| < \infty \Big\}$$

Sketch of proof of Thm $\boxed{**}$ [L-Sauer]

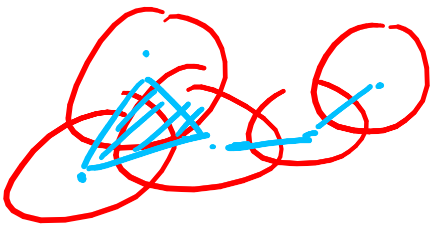
$\Gamma := \pi_1(M)$

$$\Gamma \curvearrowright \tilde{M} \xrightarrow[\Gamma]{\text{nerve map}} \Gamma \curvearrowright |\tilde{N}| \xrightarrow[\Gamma]{} E_{Am}\Gamma$$

amenable isotropy

$$H^*\big(C_b^*(E_{Am}\Gamma)^\Gamma\big) \cong H_b^*(\Gamma;\mathbb{R}) \overset{[\text{Gromov}]}{\cong} H_b^*(M;\mathbb{R})$$

optimal amenable cover of M ⇝ nerve $\tilde{N}$



⇝ $H_b^n(M;\mathbb{R}) \xrightarrow{c_M} H^n(M;\mathbb{R})$

factors over $H^*\big(C^*(|\tilde{N}|;\mathbb{R})^\Gamma\big)$. $\square$

$\cong 0$ in dim n

has no simplices in dim $\geq \operatorname{cat}_{Am} M$