http://www.mathematik.uni-r.de/loeh/talk.pdf 18.02.2022

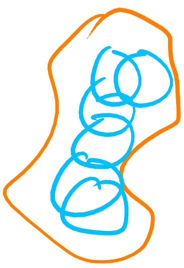
COVERING SPACES BY "SMALL" SUBSETS

I CAT II LOWER CAT BOUNDS

Question. How difficult is it to cover a top. space by small open subsets ?

I

Def. (LS-category) Let X be a top. space. Then



$$\mathrm{cat}_{LS}(X) := \text{"min"} \Big\{ n \in \mathbb{N} \;\Big|\; \exists_{U_1,\dots,U_n \subset X \text{ open}} \; \bigcup_{j=1}^{n} U_j = X \text{ and } \forall_j \; U_j \hookrightarrow X \text{ null-homotopic} \Big\}$$
$$\in \mathbb{N} \cup \{\infty\}.$$

Example. · $\mathrm{cat}_{LS}(S^n) = 2$

· $\mathrm{cat}_{LS}(\mathbb{R}P^n) = n+1$ (cup-length)

Modification: Relax the cond. on U_j.

Def. (G-subsets, cat_G). Let G be a class of groups, let X be a top. space.

· A subset $U \subset X$ is a $\underline{G\text{-subset}}$ if

$$\forall_{x \in U} \; \pi_1(U \hookrightarrow X)\big(\pi_1(U,x)\big) \in G$$

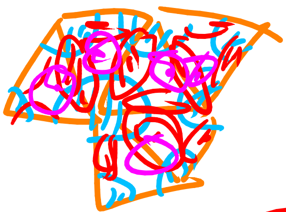
- $\mathrm{cat}_G(X) := \min\{n \in \mathbb{N} \mid \exists_{U_1,\dots,U_n \subset X \text{ open}}\ \bigcup_{j=1}^{n} U_j = X$ and $\forall_j\ U_j$ is a G-subset of $X\} \in \mathbb{N} \cup \{\infty\}$.

Examples:
- $G := 1$ [Eilenberg–Ganea]
- $G := Am$ class of all amenable groups (e.g., all virtually solvable groups)

Basic observation: if X is a complex

$$? \leq \mathrm{cat}_{Am}(X) \leq \mathrm{cat}_1(X) \leq \mathrm{cat}_{LS}(X) \leq \dim X + 1$$

"good" lower bounds ?



Example: 

$2 \leq \mathrm{cat}_{Am}(\Sigma_2) \leq 3$ is it 2 or 3 ?

⚠ Examples of obstructions: for cat_{Am}

- bounded cohomology/simplicial volume [Gromov, Ivanov]

- for aspherical spaces: L^2-Betti numbers [Gromov, Sauer]
- for asph. occ mflds: rg, cost, torsion homology growth [Mraschkin, Sauer, L]

Basic strategy: via equivariant topology [Sauer]

Setup:
- let X be a connected CW-complex
- let $\Gamma := \pi_1(X)$
- let F be a family of subgroups of Γ with $1 \in F$ (e.g.: Am_Γ)

closed under conj. and finite-n

- let $\mathcal{U}$ be an open F-cover of X, consisting of path-conn. subsets
- let $n := \mathrm{mult}(\mathcal{U})$.

Goal: say something about n.

Key ingredients: ① nerves of covers

② classifying spaces

Have:

(1) $\mathcal{U}$: $\mathcal{F}$-cover of X $\rightsquigarrow$ $\tilde{\mathcal{U}}$ lifted cover of $\tilde{X}$

$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$ isotropy in $\mathcal{F}$

nerve $N(\mathcal{U})$ $\qquad\qquad$ nerve $\Gamma \circlearrowright N(\tilde{\mathcal{U}})$

+ nerve map $X \xrightarrow[\nu]{} N(\mathcal{U})$ $\qquad\qquad$ + nerve map $\tilde{X} \xrightarrow[\tilde{\nu}]{\Gamma} N(\tilde{\mathcal{U}})$

(2) In Γ-CW:

$$\begin{tikzcd}
H(\tilde{X}) \arrow[r, equal] & H(\tilde{X}) \arrow[dd, "\tilde{\nu}"] \\
H(E\Gamma) \arrow[u] \arrow[from=u, "c"'] & \\
H(E_{\mathcal{F}}\Gamma) \arrow[u] \arrow[from=u, "c"'] \arrow[r] & H(N(\tilde{\mathcal{U}})) \arrow[l, "c", bend left] \arrow[uu, bend right]
\end{tikzcd}$$

classif. map c

universal prop.

$1 \in \mathcal{F}$

$\cong_{\Gamma}$

$H(N(\tilde{\mathcal{U}})) \cong 0$

ex.!

$\dim = \operatorname{mult}(\tilde{\mathcal{U}}) - 1$
$= \operatorname{mult}(\mathcal{U}) - 1 = n-1$

Idea: apply functors! contravariant

Wishlist for $H : \Gamma\text{-CW} \longrightarrow$ "Ab"

⓪ $\underset{\Gamma}{\simeq}$-invariant

① $H(Y) \cong 0$ if $\dim Y \leq n-1$

② $H(\tilde{X} \underset{c}{\longrightarrow} E\Gamma)$ is an iso

③ $H(E\Gamma \underset{c}{\longrightarrow} E_F\Gamma)$ is an iso

$\Rightarrow H(\tilde{X}) \cong 0$

Example. · let $k \in \mathbb{N}_{\geq n}$.

· $H := H^k_\Gamma(\cdot\,; N\Gamma)$ ⓪✓ ①✓

· X aspherical $\rightsquigarrow$ ②✓

· $F := \mathrm{Am}_\Gamma$ $\rightsquigarrow$ ③✓

$\rightsquigarrow b_k^{(2)}(\Gamma) = 0.$ [Sauer]

Concrete example:

$BF_2^{\times n} \simeq (\bigvee_2 S^1)^{\times n}$

$$n+1 \leq \mathrm{cat}_{\mathrm{Am}}(F_2^{\times n}) \leq n+1$$

$b_n^{(2)}(F_2^{\times n}) = 1^n = 1 \neq 0$

Example: bounded cohomology:

$$H := H^k_{\Gamma,b}(\cdot\,; \mathbb{R}) \overset{}{\underset{T}{\Longrightarrow}} H^k_\Gamma(\cdot\,; \mathbb{R}) =: K$$

Wishlist for $H, K : \Gamma\text{-CW} \longrightarrow$ "Ab"

$T : H \Longrightarrow K$

- H is $\simeq_\Gamma$-inv.
- $H(\tilde{X} \to E\Gamma)$ is an iso
- $H(E\Gamma \to E_F\Gamma)$ is an iso
- $K(Y) \cong 0$ if $\dim Y \leq n-1$

$\rightsquigarrow$

$$\begin{tikzcd}
H(\tilde{X}) \arrow[r, equal] & H(\tilde{X}) \arrow[r, "T_{\tilde{X}}"] & K(\tilde{X}) \\
H(E\Gamma) \arrow[u, "H(c)", "\cong"'] & & \\
H(E_F\Gamma) \arrow[u, "\cong"] \arrow[r, "H(c)"'] \arrow[ur, phantom, "\circlearrowleft"] & H(N(\tilde{u})) \arrow[uu, "H(\tilde{u})"'] \arrow[r, "T_{N(\tilde{u})}"'] \arrow[uur, phantom, "\circlearrowleft"] & K(N(\tilde{u})) \cong 0 \arrow[uu, "K(\tilde{u})"']
\end{tikzcd}$$

$\rightsquigarrow$ $T_{\tilde{X}} : H(\tilde{X}) \to K(\tilde{X})$ is zero

Example:
- X path-conn. CW-complex
- $k \in \mathbb{N}_{\geq n}$
- $H := H^k_{\Gamma,b}$
- $K := H^k_\Gamma$ — comp. map
- $F := \mathrm{Amp}$ — is zero

$\rightsquigarrow$ comp map $H^k_b(X;\mathbb{R}) \to H^k(X;\mathbb{R})$

Concrete example:

If M is an occ hyp. n-mfd and $\Gamma := \pi_1(M)$, then

$$n+1 \leq \mathrm{cat}_{\mathrm{Am}}(M) \leq n+1$$

because $\|M\| > 0$ [Gromov, Thurston]

$\Updownarrow$

$$H_b^n(M;\mathbb{R}) \xrightarrow[\neq 0]{} H^n(M;\mathbb{R})$$