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GRADIENT INVARIANTS OF ASPHERICAL MFDS WITH SMALL AMENABLE CATEGORY

(joint work with Marco Moraschini, Roman Sauer)

I GRADIENT INVS II THE DYNAMICAL VIEW

I GRADIENT INVARIANTS

I $\mathbb{R}$-valued invariant of groups (or spaces) $\leadsto$ assoc. gradient inv. $\hat{I}$

$$X \mapsto \hat{I}(X) := \liminf_{Y \subset X \text{ finite index}} \frac{I(Y)}{[X:Y]}$$

($Y \to X$ finite covering)

Examples:

- rank gradient:

$$rg := \widehat{rk}$$

minimal number of gen's needed

- Betti number gradient:

$$\hat{b}_k \qquad rk_{\mathbb{Z}} H_k(\cdot\,;\mathbb{Z})$$

('analytical' def.)

If Γ is fin pres, then $\hat{b}_k(\Gamma) = b_k^{(2)}(\Gamma)$

[Lück approx. thm.]

- homology torsion gradient:

$$\widehat{t}_k := \left(\log |\operatorname{tors} H_k(\cdot;\mathbb{Z})|\right)^\wedge$$

Corollary ~~Theorem~~ [LMS] Let M be an occ aspherical n-mfd that admits an amenable open cover of multiplicity at most n. Let $\pi_1(M)$ be residually finite. Then:

$\operatorname{rg} \pi_1(M) = 0$, and

$$\underbrace{\forall_{k\in\mathbb{N}} \quad \widehat{b}_k(M) = 0, \quad \widehat{t}_k(M) = 0.}_{\text{known before [Sauer]}}$$

$\Uparrow$ (*)

Theorem. [LMS] Let M be an occ aspherical n-mfd (with $n>0$) that admits an amenable open cover of mult $\leq n$. Let $\pi_1(M)$ be res. finite. Then

$$\|M\|^\infty_{\mathbb{Z}} = 0.$$

gradient inv. of $\|\cdot\|_{\mathbb{Z}}$

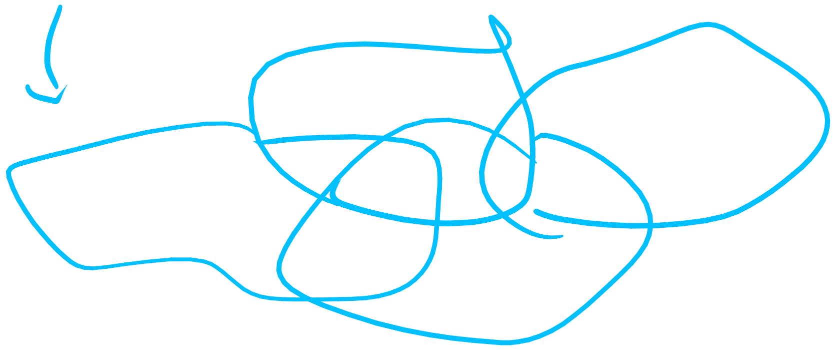
$\leadsto$ strong & unifying reason for the Cor.

About ⊛:

- rg: $\mathrm{rk}(\pi_1(M)) \le \|M\|_{\mathbb{Z}}$ + $\liminf$ ...
- $\hat{b}_k$: $b_k(M) \overset{PD}{\le} \|M\|_{\mathbb{Z}}$ [Gromov]

Attempt to prove the thm:

amenable



prove a vanishing result on the given amenable subob

+ assemble this

⚠ But: how should the finite index subgroups/finite coverings of the pieces interact and fit together?

<u>Idea:</u> Use another description of $\|M\|_{\mathbb{Z}}^{\infty}$.

(IV) THE DYNAMICAL VIEW

[Gromov, Schmidt]

Definition. (integral foliated simplicial volume)
Let M be an occ n-mfd with $\Gamma := \pi_1(M)$,
let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a pmp action of Γ
on a std. Borel prob. space. Then

$$\|M\|^\alpha := \|M\|_{1, L^\infty(X, \mathbb{Z})}$$

twisted coeff's by α

$$:= \inf \Big\{ \sum_j \int_X |f_j| \, d\mu \;\Big|\; \sum_j f_j \otimes \sigma_j \text{ is a fund. cycle of } M \text{ in } L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}\Gamma} C_n(\tilde{M}; \mathbb{Z}) \Big\}.$$

"flexible" "rigid"

The <u>integral foliated simpl. vol.</u> of M is

$$\|M\|_{\mathcal{F}, \mathbb{Z}} := \inf_\alpha \|M\|^\alpha.$$

profinite completion of $\pi_1(M)$

Theorem. [L-Pagliantini] If M is an occ mfd with res. fin. $\pi_1(M)$. Then: $\|M\|^\infty_{\mathbb{Z}} = \|M\|^{\widehat{\pi_1(M)}}$

Theorem [LMS] Let M be an aspherical occ n-mfd $(n > 0)$ that admits an amenable open cover of mult $\leq n$. Let $\alpha : \pi_1(M) \curvearrowright (X,\mu)$ be ess. free. Then:

$$\|M\|^{\alpha} = 0.$$

$(\leadsto \|M\|_{\mathcal{F},\mathbb{Z}} = 0)$

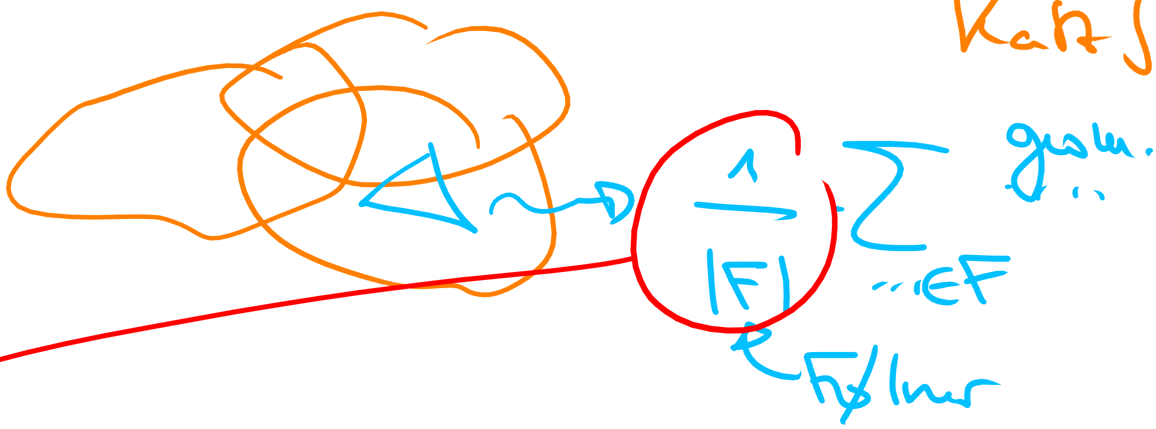
If $\pi_1(M)$ is res. fin, then $\pi_1(M) \curvearrowright \widehat{\pi_1(M)}$ is ess free

$$\Downarrow$$

$$\|M\|^{\infty}_{\mathbb{Z}} = 0.$$

Idea of proof:

- Use a version of the "amenable reduction lemma" [Gromov, Alpert-Katz]



- replace this by "the" Rokhlin lemma ($\leadsto$ "division" on the level of (X,μ)) $\square$

Basic Rokhlin lemma: let $\mathbb{Z} \curvearrowright (X,\mu)$ $\;(\mathbb{Z} = \langle t \rangle)$ pmp

ess free, let $\varepsilon \in \mathbb{R}_{>0}$, let $N \in \mathbb{N}$.

Then: there ex. a meas. $A \subset X$

s.t.

$$\forall_{\substack{j,i \in \{0,\dots,N\} \\ j \neq i}} \quad t^j \cdot A \cap t^i \cdot A = \emptyset$$

and $$\mu\Big(X \setminus \bigcup_{j=0}^{N} t^j A\Big) < \varepsilon,$$