http://www.mathematik.uni-r.de/loeh/talk.pdf

The Spectrum of Simplicial Volume I: Foundations

joint with Nicolaus Heuer

I Simplicial Volume   II Bounded Cohomology

III From Groups to Manifolds

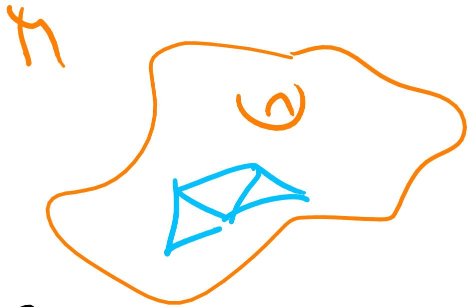
I

or., closed, conn.

Def. [Gromov] Let M be an occ n-mfd. The <u>simplicial volume</u> of M is

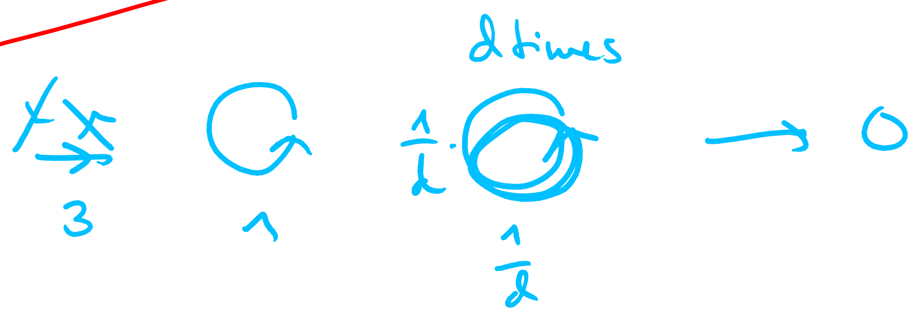
$$\|M\| := \inf\Big\{ \sum_j |a_j| \;\Big|\; \sum_j a_j \sigma_j \in C_n(M;\mathbb{R}) \ \mathbb{R}\text{-fund. cycle of } M \Big\}, \quad a_j \in \mathbb{R},\ \sigma_j : \Delta^n \to M$$

$\in \mathbb{R}_{\geq 0}$ homotopy invariant!



Basic examples:

- $\|S^1\| = 0$



More generally: If $\pi_1(M)$ is amenable, then $\|M\| = 0$. [Gromov]

- If M is hyperbolic, then [Gromov, Thurston]

$$\|M\| = \frac{\mathrm{vol}(M)}{v_n} > 0$$

$\geq$ straightening

$\leq$ smearing

rigidity!

More generally: loc. sym. sp. [Lafont, Schmidt]

- $\|M\| \cdot \|N\| \le \|M \times N\| \le \text{const} \cdot \|M\| \cdot \|N\|$
  e.g. [Bucher]
  $$\|\Sigma_g \times \Sigma_h\| = \frac{3}{2} \cdot \underbrace{\|\Sigma_g\|}_{4\cdot(g-1)} \cdot \|\Sigma_h\|$$
- In dim ≥ 3: $\|M \# N\| = \|M\| + \|N\|$
  [Gromov]

via H_b^*

More gen: gluings along amenable π_1-inj. boundaries.

- In general: hard to compute [Weinberger]

Question: For $n \in \mathbb{N}$, what is

$$SV(n) := \{\|M\| \mid M \text{ occ } n\text{-mfd}\} \subset \mathbb{R}_{\geq 0} \quad ?$$

countable

- Does $SV(n)$ have a gap at 0?
- What about transcendental values?

II BOUNDED COHOMOLOGY H_b^*

Def. Let X be top. space. We def:

- The <u>bounded cochain complex</u> of X
  $$C_b^*(X;\mathbb{R}) := \{f : \text{map}(\Delta^*, X) \to \mathbb{R} \mid |f|_\infty < \infty\}$$
  subcomplex of $C^*(X;\mathbb{R})$
- The <u>bounded cohomology</u> of X

does <u>not</u> satisfy excision!

$$H_b^*(X;\mathbb{R}) := H^*(C_b^*(X;\mathbb{R}))$$

$\varphi \mapsto \|\varphi\|_\infty := \inf\{|f|_\infty \mid f \in C_b^*(X;\mathbb{R}), \delta f = 0, [f] = \varphi\}$

In gen: neither injective, nor surjective

The comparison map

$$c_X : H_b^*(X;\mathbb{R}) \longrightarrow H^*(X;\mathbb{R})$$
$$[f] \longmapsto [f]$$

$\rightsquigarrow$ can evaluate H_b^* on H_*

Prop. (duality principle [Gromov]) Let X be a top. space, let $\alpha \in H_*(X;\mathbb{R})$. Then

$$\|\alpha\|_1 = \sup\left\{ \frac{1}{\|\varphi\|_\infty} \;\middle|\; \varphi \in H_b^*(X;\mathbb{R}),\ \langle \varphi, \alpha \rangle = 1 \right\}.$$

$\|M\| = \|[M]_\mathbb{R}\|_1$

Proof. $\geq$: $|\langle \varphi, \alpha \rangle| \leq \|\varphi\|_\infty \cdot \|\alpha\|_1$

$\leq$: Hahn-Banach. $\square$

Consequence: Let $f : X \to Y$ be continuous. If $H_b^*(f;\mathbb{R}) : H_b^*(Y;\mathbb{R}) \to H_b^*(X;\mathbb{R})$ is an isometric iso, wrt $\|\cdot\|_\infty$

then $H_*(f;\mathbb{R}) : H_*(X;\mathbb{R}) \to H_*(Y;\mathbb{R})$ is isometric. wrt $\|\cdot\|_1$

Theorem. (mapping theorem [Gromov, Ivanov]). Let $f : X \to Y$ be a cont. map between path-connected spaces with:

$\pi_1(f)$ is surjective and $\ker \pi_1(f)$ is amenable.

Then: $H_b^*(f;\mathbb{R}) : H_b^*(Y;\mathbb{R}) \to H_b^*(X;\mathbb{R})$ is an isometric iso.

Proof. • amenability ⟹ invariant mean of $\ker \pi_1(f)$ and of π_k with $k \geq 2$

• iterated transfer construction. $\square$

III From Groups to Manifolds

Theorem (normed Thom realisation [Crowley–L]). Let Γ be a fin. pres. group and let $\alpha \in H_4(B\Gamma;\mathbb{R})$ be an integral class. Then there ex. an occ 4-mfd M with $\pi_1 M \cong \Gamma$

$$\|M\| = \|\alpha\|_1 .$$

Proof. • Thom realisation:

⟿ there ex. an occ 4-mfd M and a cont. map $f\colon M \to B\Gamma$ with

$$H_4(f;\mathbb{R})[M]_{\mathbb{R}} = \alpha .$$

• surgery

⟿ may assume that $\pi_1(f)\colon \pi_1(M) \to \Gamma$ is an iso.

• mapping thm

$$⟿ \|M\| = \|H_*(f;\mathbb{R})[M]_{\mathbb{R}}\|_1 = \|\alpha\|_1 .$$

$\square$

$$M \xrightarrow[\text{map } f]{\text{class.}} B\pi_1 M \quad ⟿ \quad \|M\| = \|H_*(f;\mathbb{R})[M]_{\mathbb{R}}\|_1$$