http://www.mathematik.uni-r.de/loeh/talk.pdf 18_05_2021

Stable Integral Simplicial Volume of 3-Manifolds

I Simplicial volume II Residually finite view III Dim 3

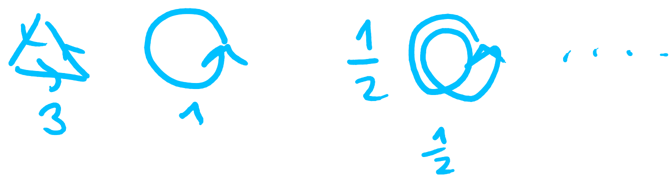
(I) Definition [Gromov] (simplicial volume). If M is an occ n-mfd, then

singular chain complex

$$\|M\|_{\mathbb{R}} := \inf\{|c|_1 \mid c \in C_n(M;\mathbb{R}) \text{ fund. cycle of } M\} \in \mathbb{R}_{\geq 0}$$

$$\Big|\sum_j a_j \cdot \sigma_j\Big|_1 := \sum_j |a_j|$$

Examples: • $\|S^1\|_{\mathbb{R}} = 0$



gen: If $\pi_1(M)$ is amenable, then $\|M\|_{\mathbb{R}} = 0$ and $\dim M \neq 0$ [Gromov, Ivanov]

• $\|\Sigma_g\|_{\mathbb{R}} = 4\cdot g - 4$ if $g \geq 1$ [Gromov]

gen: If M is hyperbolic, then

homotopy invariant → $$\|M\|_{\mathbb{R}} = \frac{\operatorname{vol}(M)}{v_n}.$$ [Gromov, Thurston]

(II) Def. (stable integral simplicial vol.) If M is an oc mfd, then

$$\|M\|_{\mathbb{Z}}^{\infty} := \inf\Big\{\frac{\|N\|_{\mathbb{Z}}}{|\deg p|} \;\Big|\; p : N \to M \text{ finite covering}\Big\}$$

Questions: ① For which aspherical oc mfds M (with res. fin. π_1) do we have
$$\|M\|_{\mathbb{R}} = \|M\|_{\mathbb{Z}}^{\infty} \; ?$$

② Why is ① interesting?

Observations: • If $\pi_1(M) \cong 1$, then $\|M\|_{\mathbb{Z}}^{\infty} = \|M\|_{\mathbb{Z}} \geq 1 \neq 0 = \|M\|_{\mathbb{R}}$

$\leadsto$ focus on the aspherical case

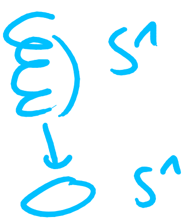
• For all $k \in \mathbb{N}$: $b_k(M) \overset{PD}{\leq} \|M\|_{\mathbb{Z}}$. [Gromov]

$\leadsto$ $|\chi(M)| \leq (\dim M + 1) \cdot \|M\|_{\mathbb{Z}}^{\infty}$

$\leadsto$: $\|M\|_{\mathbb{Z}}^{\infty} = 0 \implies \chi(M) = 0$.

Question [Gromov] If M is an oc aspherical mfd, then
$$\|M\|_{\mathbb{R}} = 0 \overset{?}{\implies} \chi(M) = 0.$$

Elementary examples:



$\|\Sigma_g\|_{\mathbb{Z}} \leq 4g-2$

• $\|S^1\|_{\mathbb{Z}}^{\infty} = 0 = \|S^1\|_{\mathbb{R}}$

• For all $g \in \mathbb{N}_{\geq 1}$: $\underset{\substack{\| \\ 4g-4}}{\|\Sigma_g\|_{\mathbb{R}}} \;\underset{=}{\cancel{\leq}}\; \|\Sigma_g\|_{\mathbb{Z}}^{\infty} = 4g-4$

Problem: How to compute further examples?

Idea: look at other gradient invariants (of groups)

fin pres., res fin.

Betti numbers $b_k(\Gamma)$ $\leadsto$ $\lim\limits_{\substack{\Lambda \triangleleft \Gamma \\ \text{fin index}}} \dfrac{b_k(\Lambda)}{[\Gamma : \Lambda]}$ $= b_k^{(2)}(\Gamma) = b_k^{(2)}(\mathcal{R}_{\Gamma \curvearrowright (X,\mu)})$

[Lück] [Gaboriau]

[Abért, Nikolov] profinite completion

rank $d(\Gamma)$ $\leadsto$ $\inf\limits_{\substack{\Lambda \triangleleft \Gamma \\ \text{fin. index}}} \dfrac{d(\Lambda) - 1}{[\Gamma : \Lambda]}$ $=: \operatorname{rg}(\Gamma) = \operatorname{cost}(\Gamma \curvearrowright \hat{\Gamma}) - 1$

min # of gen's

rank gradient

$\leadsto$ can use ergodic theory!

(Rmk: $b_k^{(2)}(M) \leq \|M\|^\infty_{\mathbb{Z}}$, $\operatorname{rg} \pi_1(M) \leq \|M\|^\infty_{\mathbb{Z}}$)

[Gromov] [L]

Def. [Gromov, Schmidt] (integral foliated simpl. vol.)
Let M be an occ mfd and let $\alpha : \pi_1(M) \curvearrowright (X,\mu)$ be a p.m.p. action on a (standard Borel) prob. space. Then

$$\|M\|^\alpha := \|M\|_{L^\infty(X,\mu;\mathbb{Z})}$$

twisted coeff's!

flexible, rigid

and

$$\|M\|_{\mathcal{F},\mathbb{Z}} := \inf \{ \|M\|^\alpha \mid \alpha : \pi_1(M) \curvearrowright (X,\mu) \} \in \mathbb{R}_{\geq 0}.$$

Rmk: $\int_{\ldots}$

$$\|M\|_{\mathbb{R}} \leq \|M\|_{\mathbb{F},\mathbb{Z}} \leq \|M\|_{\mathbb{Z}}^{\infty} \leq \|M\|_{\mathbb{Z}}$$

Theorem. [L-Pagliantini, Frigerio-L-Pagliantini-Sauer]
If M is an occ mfd, then

with res fin π_1

$$\|M\|_{\mathbb{Z}}^{\infty} = \|M\|^{\pi_1(M) \curvearrowright \widehat{\pi_1(M)}}.$$

Theorem. [L-Pagliantini, FLPS, Fauser-L-Moraschini-Quintanilha]

Let M be an occ aspherical 3-mfd. Then

$$\|M\|_{\mathbb{Z}}^{\infty} = \|M\|_{\mathbb{R}} = \frac{\text{hypvol}(M)}{v_3}.$$

⚠ · In general, one cannot drop asphericity

· In general, one cannot drop dim 3:
<u>Not</u> true for hyp 4-mfds
[Francaviglia, Frigerio, Martelli]

Sketch proof.

Ⓐ The closed hyperbolic case: occ hyp 3-mfds

① geometric input: $\exists (M_n)_{n \in \mathbb{N}}$ $\lim_{n\to\infty} \frac{\|M_n\|_{\mathbb{Z}}}{\|M_n\|_{\mathbb{R}}} = 1$

② proportionality principle: If M, N are occ hyp n-mflds, then

$$\frac{\|M\|_{\mathcal{F},\mathbb{Z}}}{\mathrm{vol}(M)} \leq \frac{\|N\|^{\infty}_{\mathbb{Z}}}{\mathrm{vol}(N)}$$

(uses ergodic theory).

③ more ergodic theory: If $\dim M = 3$, then

$$\|M\|_{\mathcal{F},\mathbb{Z}} \underset{\substack{\uparrow \\ \pi_1(M) \\ \text{has EMD}^*}}{=} \|M\|^{\pi_1(M) \curvearrowright \widehat{\pi_1(M)}} = \|M\|^{\infty}_{\mathbb{Z}}$$

$\leadsto$ $\|M\|^{\infty}_{\mathbb{Z}} = \|M\|_{\mathbb{R}}$ if M is an occ hyp 3-mfd.

Ⓑ General case:

① JSJ-decomp. ⟶ cut along 2-tori ⟶

- hyperbolic pieces ⟵ Ⓐ
- "Seifert fibred" pieces ⟵ 0

② (Sub) Additivity for gluings along tori again using ergodic theory + specifics on profinite completions of 3-mfd groups. □

[Bucher]: $\|\Sigma_g \times \Sigma_h\|_{\mathbb{R}} = \frac{3}{2} \cdot \|\Sigma_g\|_{\mathbb{R}} \cdot \|\Sigma_h\|_{\mathbb{R}}$.