http://www.mathematik.uni-r.de/loeh/talk2.pdf

SIMPLICIAL VOLUME & STABLE COMMUTATOR LENGTH

I. SIMPLICIAL VOLUME   II. FROM GROUPS TO MANIFOLDS   III USING SCL

I. SIMPLICIAL VOLUME

oriented, closed, connected

Definition. [Gromov] Let M be an occ n-mfd. Then the simplicial volume of M is

singular

$$\underbrace{\|M\|}_{\|[M]_{\mathbb{R}}\|_1} := \inf\Big\{\sum_{j=1}^m |a_j| \;\Big|\; \sum_{j=1}^m a_j \cdot \sigma_j \in C_n(M;\mathbb{R}) \text{ } \mathbb{R}\text{-fund. cycle}\Big\} \in \mathbb{R}_{\geq 0}.$$

Examples. • $\|S^1\| = 0$ $\frac{1}{d}$ · d

• If $\pi_1(M)$ is amenable (and $\dim M \neq 0$), then $\|M\| = 0$. [Gromov, Ivanov]

$\|\Sigma_g\| = 4\cdot(g-1)$ • If M is occ hyperbolic, then

$$\|M\| = \frac{\operatorname{vol} M}{v_n}. \quad \text{[Gromov, Thurston]}$$

countable set, closed under addition

• $\|\Sigma_g \times \Sigma_h\| = \frac{3}{2} \cdot \|\Sigma_g\| \cdot \|\Sigma_h\|$ [Bucher]

Question: What is $SV(d) := \{\|M\| \mid M \text{ occ } d\text{-mfd}\}$? (?)

Examples: • $SV(1) = \{0\}$

• $SV(2) = 4 \cdot \mathbb{N}$ [Soma, Gromov]

• $SV(3) = \mathbb{N}\left[\frac{\operatorname{vol} M}{v_3} \;\middle|\; M \text{ occ hyp } 3\text{-mfd}\right]$

Question. Does $SV(d)$ have a gap at 0 for $d \geq 4$? (?)

II Groups $\rightsquigarrow$ Manifolds

Theorem $\boxed{1}$ [Heuer, L] Let Γ be a fin. pres. group and let $\alpha \in H_2(\Gamma;\mathbb{R})$ be an integral homology class. Then there ex. an occ 4-mfd M with

$2 \overset{\times \Sigma_2}{\rightsquigarrow} 4$

$$\|M\| = 6 \cdot \|\alpha\|_1 .$$

Sketch of proof.

(1) The group: let $\tilde\Gamma := \underbrace{\Gamma \times \pi_1(\Sigma_2)}_{\text{fin pres group}}$.

and $\tilde\alpha := \underbrace{\alpha \times [\Sigma_{\cancel{g}\,2}]_{\mathbb{R}}}_{\text{integral homology class}} \in H_4(\tilde\Gamma;\mathbb{R})$.

Then $\overset{\text{gen. of [Bucher]}}{\rightsquigarrow}$

$$\|\tilde\alpha\|_1 = \frac{3}{2} \cdot \|\alpha\|_1 \cdot \|\Sigma_{\cancel{g}\,2}\|$$

$$= \frac{3}{2} \cdot \|\alpha\|_1 \cdot 4(2-1) = 6 \cdot \|\alpha\|_1 .$$

(2) $\rightsquigarrow$ manifold: There ex. an occ

[Thom] 4-mfd M and a cont. map $f : M \to B\tilde\Gamma$ with

$$H_4(f;\mathbb{R})\,[M]_{\mathbb{R}} = \tilde\alpha \in H_4(\tilde\Gamma;\mathbb{R})$$

+ surgery: can assume that f is π_1-iso

[Gromov] $\rightsquigarrow$ mapping thm

$$\|M\| = \|[M]_{\mathbb{R}}\|_1 = \|H_4(f;\mathbb{R})[M]_{\mathbb{R}}\|_1$$

$$= \|\tilde\alpha\|_1 = 6 \cdot \|\alpha\|_1 . \qquad \square$$

III. USING $\underbrace{\text{STABLE COMMUTATOR LENGTH}}_{\text{scl}}$

Theorem [2] [Heuer, L] Let Γ be a fin pres group with $H_2(\Gamma;\mathbb{R}) \cong 0$ and let $r \in [\Gamma,\Gamma]$ be of infinite order. Then there ex. a fin pres. group $\tilde{\Gamma}$ and an integral class $\alpha \in H_2(\tilde{\Gamma};\mathbb{R})$ with

$$\|\alpha\|_1 = 8 \cdot \operatorname{scl}_\Gamma r.$$

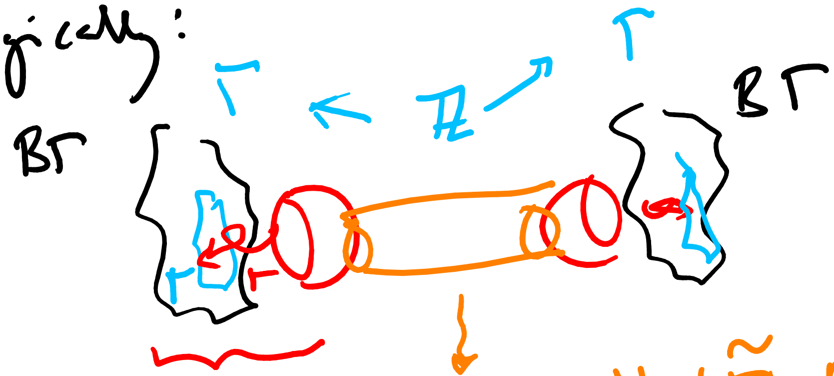
$$\operatorname{scl}_\Gamma r = \inf_{n \in \mathbb{N}_{>0}} \frac{\operatorname{cl}_\Gamma r^n}{n}, \quad \operatorname{cl}_\Gamma r := \min\{g \in \mathbb{N} \mid \exists_{a_1,\dots,a_g,b_1,\dots,b_g}\ r = [a_1,b_1]\cdots[a_g,b_g]\}$$

Sketch of proof:

"double"

Let $\tilde{\Gamma} := \Gamma *_{\mathbb{Z}} \Gamma$, $\quad r \leftarrow\!\shortmid \mathbb{Z} \mapsto r^{-1}$

Topologically:



$\rightsquigarrow \alpha \in H_2(\tilde{\Gamma};\mathbb{R})$

g commutators

$\rightsquigarrow 4g$ triangles

$\rightsquigarrow \|\alpha\|_1 = 8 \cdot \operatorname{scl}_\Gamma r.$ $\square$

Theorem $\boxed{1}$
Theorem $\boxed{2}$ $\Big\} \rightsquigarrow$ If Γ is a fin. pres. group with $H_2(\Gamma;\mathbb{R}) = 0$ and $r \in [\Gamma,\Gamma]$ of infinite order then: there ex. occ 4-mfd M with

$$\|M\| = 8 \cdot 6 \cdot \mathrm{scl}_\Gamma r$$
$$= 48 \cdot \mathrm{scl}_\Gamma r.$$

Corollary [Heuer L] We have:

- $\mathbb{Q}_{\geq 0} \subset SV(4)$

(use the universal central extension of Thompson's group T)

- $SV(4)$ contains arbitrarily small transcendental numbers

(use the Euler central extension of $SL_2(\mathbb{Z}[\frac{1}{2}])$)

+ . scl computations by Calegari et al

. $H^*_{(b)}$ computations by Ghys, Sergiescu, Burger, Monod.