http://www.mathematik.uni-r.de/loeh/talk.pdf

# SIMPLICIAL VOLUME

I. DEFINITION   II. HOMOTOPY INVARIANCE & DEGREE ESTIMATE   III HYPERBOLIC CASE

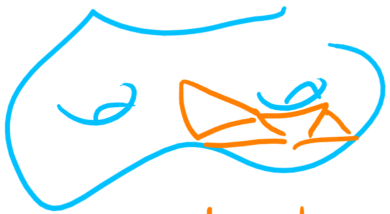
## I. DEFINITION

oriented closed connected

**Definition.** [Gromov] Let $M$ be an occ $n$-mfd. The <u>simplicial volume</u> of $M$ is

singular chain space

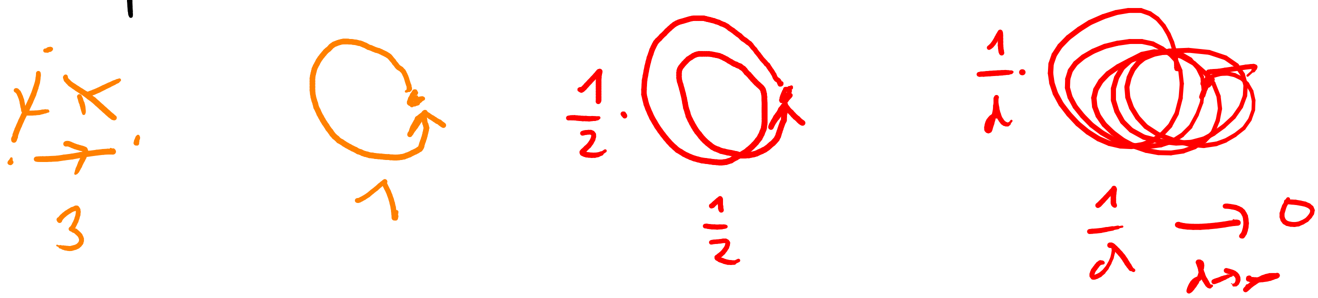
$$\|M\| := \inf\Big\{\sum_{j=1}^{m}|a_j| \;\Big|\; \sum_{j=1}^{m} a_j \sigma_j \in C_n(M;\mathbb{R})\ \ \mathbb{R}\text{-fundamental cycle}\Big\}$$



$\in \mathbb{R}_{\geq 0}$.

singular triangulations $\hat{=}$ singular cycles rep. the fundamental class of $M$

Simple example: $\|S^1\| = 0$.




## II HOMOTOPY INVARIANCE & DEGREE EST'S.

**Prop.** Let $f: M \to N$ be a cont. map between occ $n$-mfds. Then

$$|\deg f| \cdot \|N\| \leq \|M\|.$$

$$\rightsquigarrow \quad |\deg f| \leq \frac{\|M\|}{\|N\|}.$$

Proof. Let $c = \sum_{j=1}^{m} a_j \cdot \underbrace{\sigma_j}_{\Delta^n \to M} \in C_n(M;\mathbb{R})$ be an $\mathbb{R}$-fund. cycle of $M$.

$\rightsquigarrow$ $C_n(f;\mathbb{R})(c) = \sum_{j=1}^{m} a_j \cdot f \circ \sigma_j \in C_n(N;\mathbb{R})$

represents $(\deg f) \cdot [N]_{\mathbb{R}}$ (by def of $\deg f$)

$\rightsquigarrow$ $\overbrace{\| (\deg f) \cdot [N]_{\mathbb{R}} \|_1}^{|\deg f| \cdot \|N\|} \leq |C_n(f;\mathbb{R})(c)|_1$

$$\leq \sum_{j=1}^{m} |a_j|$$

$$= |c|_1$$

take inf over all $c$ $\rightsquigarrow$ $|\deg f| \cdot \|N\| \leq \|M\|$. $\square$

Example. If $M$ admits a self-map of degree $\neq 0, 1, -1$, then $\|M\| = 0$.

(e.g. $S^n$, tori, ...)

Corollary. Simplicial volume is a homotopy inv. of occ mfds. $\square$

Corollary. If $f : M \to N$ is a finite sheeted covering map of occ mfds, then

$$|\deg f| \cdot \|N\| = \|M\|.$$

Proof. $\leq$ : prop. above

$\geq$ : use transfer. $\square$

# III. THE HYPERBOLIC CASE

Theorem. [Gromov, Thurston]. If is a occ hyperbolic n-mfd, then

hyperbolic: Riemannian mfd whose Riem. universal covering is $\mathbb{H}^n$.

$$\|M\| = \frac{\mathrm{vol}\, M}{v_n} > 0$$

vol M: hyperbolic volume

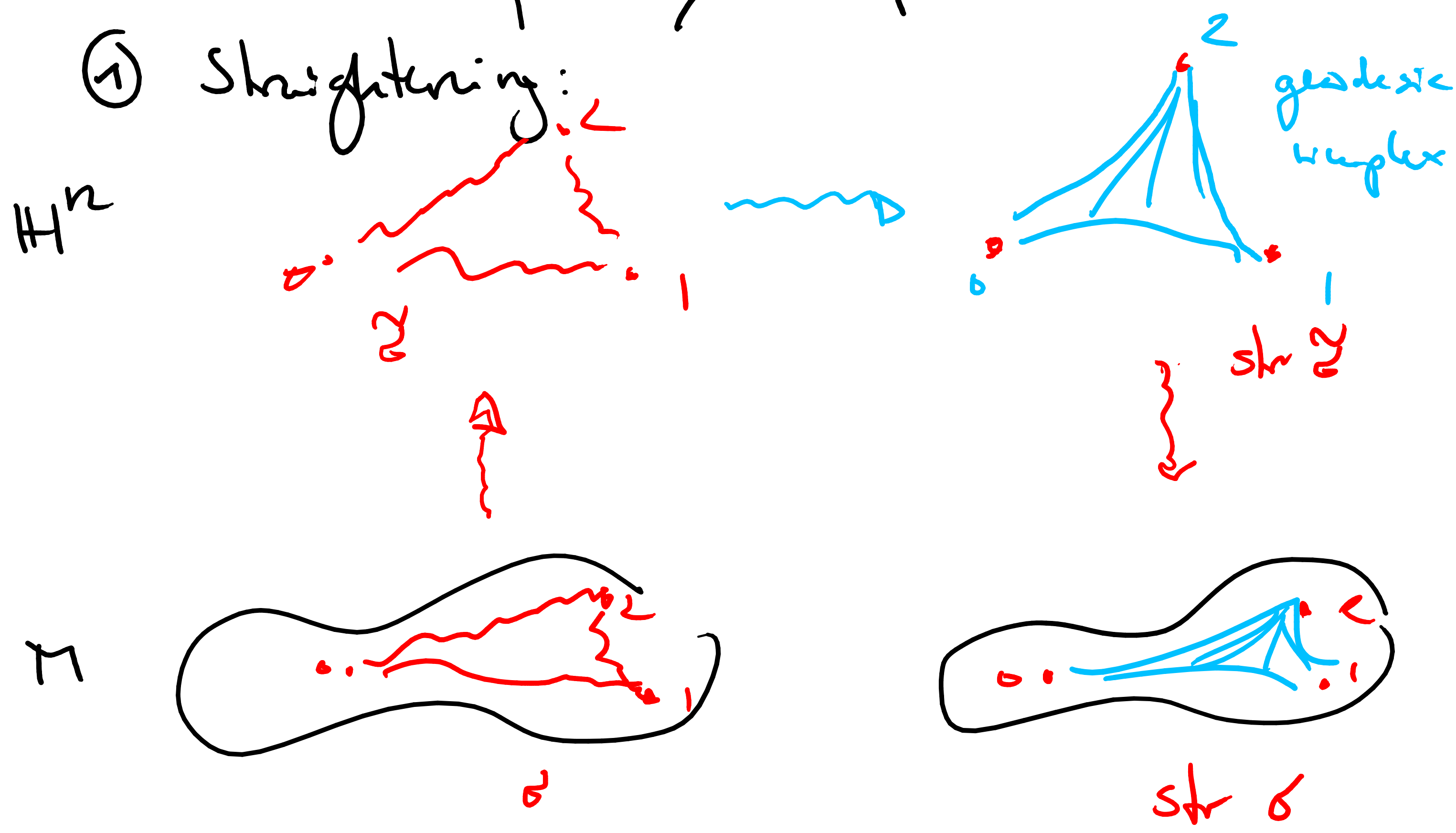
$v_n$: sup volumes of geodesic n-simplices in $\mathbb{H}^n$.

homotopy invariance of $\|\cdot\|$ $\rightsquigarrow$ vol. of occ hyperbolic mfds is homotopy invariant (!)

(RIGIDITY!)

Sketch of proof. $(\geq)$ let $c = \sum_{j=1}^{k} a_j \cdot \sigma_j \in C_n(M;\mathbb{R})$ be an $\mathbb{R}$-fund. cycle of M.

① Straightening:



$\leadsto [\mathrm{str}\, c] = [c] = [M]_{\mathbb{R}}$ in $H_n(M;\mathbb{R})$.

② Integration:

$$\mathrm{vol}\, M = \int_{\mathrm{str}\, c} \mathrm{vol}_M = \sum_{j=1}^{m} a_j \cdot \int_{\mathrm{str}\, \sigma_j} \mathrm{vol}_M$$

$$= \sum_{j=1}^{m} a_j \cdot \underbrace{\int_{\mathrm{str}\, \tilde{\sigma}_j} \mathrm{vol}_{\mathbb{H}^n}}_{\text{signed volume of image of } \mathrm{str}\, \tilde{\sigma}_j.}$$

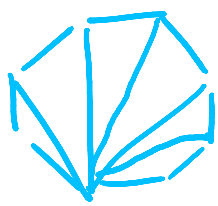
$\leadsto |\cdot| \le v_n$

$$\leadsto \mathrm{vol}\, M \le \underbrace{\sum_{j=1}^{m} |a_j|}_{= |c|_1} \cdot v_n$$

take inf over all $c$ $\leadsto$ $\mathrm{vol}\, M \le \|M\| \cdot v_n$.

$\ge$: (for the case $\dim M = 2$) $\leadsto M \cong \Sigma_g$ with $g \ge 2$

① $\|\Sigma_g\| \le 4g - 2$



$4g$ gon

② pass to finite coverings:

For each $d \ge 1$, there ex. a fin. covering

$$\Sigma_h \xrightarrow[d \text{ sheets}]{} \Sigma_g$$

$$\overset{\chi}{\rightsquigarrow} \quad d\cdot(g-1) = h-1$$

$$\rightsquigarrow \quad h = d(g-1)+1$$

degree estimate

$$\rightsquigarrow \|\Sigma_g\| \le \frac{\|\Sigma_{d(g-1)+1}\|}{d}$$

$$\overset{(1)}{\le} \frac{4\cdot d(g-1)+4-2}{d}$$

$$= 4\cdot(g-1) + \underset{d\to\infty}{\frac{2}{d} \longrightarrow 0}$$

inf over d

$$\rightsquigarrow \|\Sigma_g\| \le 4\cdot(g-1) = \frac{\operatorname{vol}\Sigma_g}{v_2}. \quad \square$$