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Computing Simplicial Volumes

I A topological volume

II Values of simplicial volume

I **Theorem** (Mostow rigidity). Let $n \in \mathbb{N}_{\geq 3}$ and let M, N occ hyperbolic n-mfds. Then

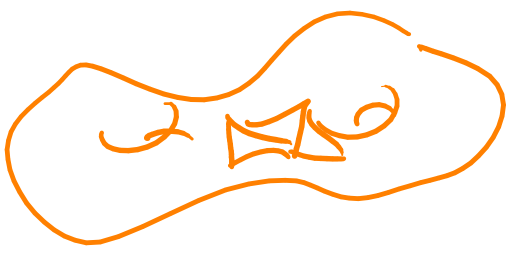
$$M \underset{\text{h-eq}}{\simeq} N \implies M \underset{\text{isometric}}{\cong} N.$$

[Gromov]

hyperbolic volume is a homotopy invariant

Definition (simplicial volume [Gromov]). The <u>simplicial volume</u> of an occ n-mfd M is

$$\|M\| := \|[M]_{\mathbb{R}}\|_1 := \inf\Big\{\sum_j |a_j| \;\Big|\; \sum_j \overset{\mathbb{R}}{a_j}\, \overset{\Delta^n \to M}{\sigma_j} \in C_n(M;\mathbb{R}) \text{ fund. cycle of } M\Big\} \in \mathbb{R}_{\geq 0}.$$



Observation: This is a homotopy invariant!

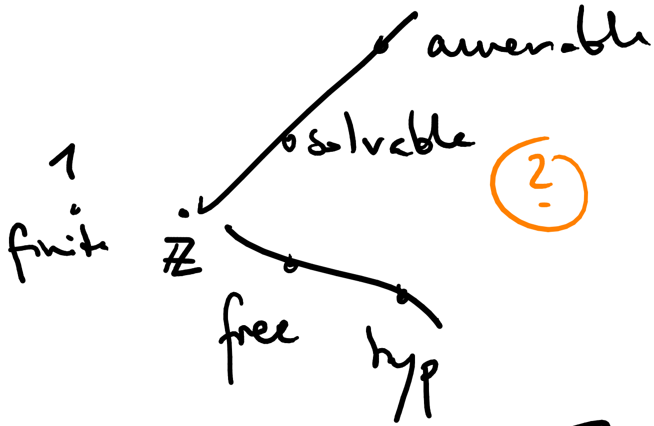
Theorem. [Gromov, Thurston] For all occ hyperbolic n-mfds M:

$$\|M\| = \frac{\mathrm{vol}(M)}{v_n}$$

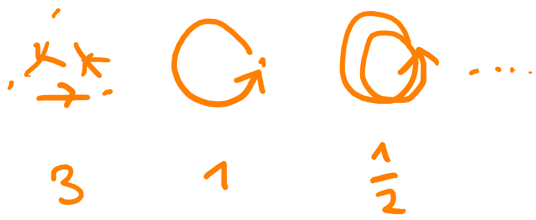
> 0 volume of ideal regular n-simplices in $\mathbb{H}^n$

Question: other values of simplicial volume?

Bridson's universe of groups:



Example: $\|S^1\| = 0$



Theorem [Gromov, Ivanov] If M is an occ mfd of dim > 0 and if $\pi_1(M)$ is amenable, then
$$\|M\| = 0.$$

(proof uses bounded cohomology)

$\rightsquigarrow$ If M is an occ mfd that admits a Ric > 0 metric, then $\|M\| = 0$.

There are some computations / estimates for
- connected sums / amenable gluings
- products

What does $SV(n) := \{\|M\| \mid M \text{ occ } n\text{-mfd}\}$ look like?

countable!

E.g.: $SV(2) = 4 \cdot \mathbb{N}$ $\quad \|\Sigma_2\| = 4$

$SV(3) = \ldots$ (geometrisation)

II

Theorem. [Heuer, L] Let $n \in \mathbb{N}_{\geq 4}$. Then

- $SV(n)$ has no gap at 0.
- $SV(4) \supset \mathbb{Q}_{\geq 0}$

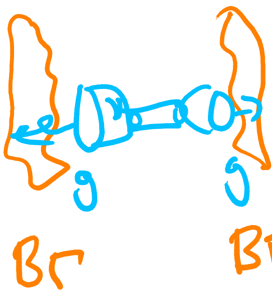
  $SV(4)$ contains arbitrarily small transcendental values.

Sketch of proof. (for $n=4$).

(A) Let Γ be a fin. pres. group with $H_2(\Gamma;\mathbb{R}) \cong 0$ and $g \in [\Gamma,\Gamma]$ of infinite order.

$\Delta = \Gamma *_{g \leftarrow \mathbb{Z} \rightarrow g} \Gamma$

Then: there ex. a fin. pres. group Δ and an integral class $\alpha \in H_2(\Delta;\mathbb{R})$ with



$$\|\alpha\|_1 = 8 \cdot \underbrace{\mathrm{scl}_\Gamma g}_{\inf_{k \in \mathbb{N}_{>0}} \frac{\mathrm{cl}_\Gamma g^k}{k}}$$

$4\,\mathrm{scl}_\Gamma g + 4 \cdot \mathrm{scl}_\Gamma g$

(B) In the situation of (A), there ex. an occ 4-mfd M with $\pi_1(M) \cong \Delta \times \pi_1(\Sigma_2)$ and

$$\|M\| \overset{\text{[Bucher–Karlsson]}}{=} \frac{3}{2} \cdot \underbrace{\|\alpha\|_1}_{= 8 \cdot \mathrm{scl}_\Gamma g} \cdot \underbrace{\|\Sigma_2\|}_{=4}$$

$$= 48 \cdot \mathrm{scl}_\Gamma g.$$

$M \xrightarrow{\pi_1\text{-iso}} B(\Delta \times \pi_1(\Sigma_2))$

$[M]_\mathbb{R} \longmapsto \alpha \times [\Sigma_2]_\mathbb{R}$

universal + central ext's

(C) Find such Γ and g! [Calegari et al.] $\square$

Theorem. [Heuer, L] Let $A \subset \mathbb{N}$ be recursively enumerable, but not recursive. Then

$$x_A := \sum_{n \in A} 2^{-n}$$

is <u>not</u> in $\bigcup_{n \in \mathbb{N}} SV(n)$. (and x_A is transcendental)

Proof. Definition. A real number x is <u>right-computable</u> if

$$\{ a \in \mathbb{Q} \mid x < a \}$$

is recursively enumerable.

Observation:

- x_A is <u>not</u> right-computable [Specker]
- For every occ mfd M, the number

  $$\|M\| = \inf \ldots = \inf_{d \in \mathbb{N}_{>0}} \frac{\| d \cdot [M]_{\mathbb{Z}} \|_{1,\mathbb{Z}}}{d}$$

  is right-computable. $\square$