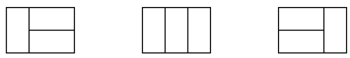
Exercises for *The ζ-function in combinatorics*

Ringvorlesung GK CCC June 16, 2015 (C. Löh)

Exercise 1 (a classic on generating functions)**.** For $n \in \mathbb{N}$ let t_n denote the number of ways in which a $(2 \times n)$-rectangle can be filled by (2×1)-bricks. For example, $t_3 = 3$:



Let $T \in \mathbb{Z}[\![x]\!]$ be the ordinary generating function of t.

1. Show combinatorially that T satisfies the following relation in $\mathbb{Z}[\![x]\!]$:
$$T = 1 + x \cdot T + x^2 \cdot T.$$
2. Find a closed form for t be rewriting T using the recursion formula above for T and a corresponding partial fraction decomposition.
3. How does this relate to the Fibonacci numbers?

Exercise 2 (exponential generating functions)**.** Algebraic exponential generating functions (EGF) are defined by

$$\begin{aligned} \mathrm{EGF}\colon \mathbb{Z}^{\mathbb{N}} &\longrightarrow \mathbb{Q}[\![x]\!] \\ (a_n)_{n \in \mathbb{N}} &\longmapsto \sum_{n=0}^{\infty} \frac{a_n}{n!} \cdot x^n. \end{aligned}$$

1. What is the convolution law for exponential generating functions?
2. For $n \in \mathbb{N}$, a *derangement of* $\{1, \dots, n\}$ is a permutation of $\{1, \dots, n\}$ without fixed points. Let D_n be the number of such derangements. Show combinatorially that
$$n! = \sum_{k=0}^{n} \binom{n}{k} \cdot D_{n-k}$$
holds and then use exponential gerating functions to deduce that
$$\frac{D_n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \cdot \frac{1}{n!}.$$

Exercise 3 (Möbius inversion formula)**.** Prove the Möbius inversion formula using the Euler product decomposition for the classical ζ-function.

Exercise 4 (DGF of the Euler φ-function)**.** Let φ be the Euler φ-function. Show that the (analytic) Dirichlet generating function corresponding to φ is given by (on a suitable domain in $\mathbb{C}$)

$$z \longmapsto \frac{\zeta(z-1)}{\zeta(z)}.$$