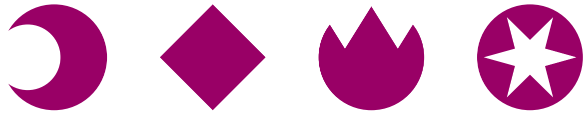
Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold Sheet 1, October 21, 2022

Quick check A (contractibility?). Which of the following subspaces of $\mathbb{R}^2$ are convex, star-shaped, contractible?



Quick check B (homotopies). Give explicit homotopies between the following maps; visualise your homotopies via suitable illustrations:

1. $[0,1] \longrightarrow S^1$, $x \longmapsto (\cos x, \sin x)$ and $[0,1] \longrightarrow S^1$, $x \longmapsto (\cos x, -\sin x)$;
2. id_{S^1} and $S^1 \longrightarrow S^1$, $(x_1, x_2) \longmapsto (-x_2, x_1)$;

Quick check C (motion planning on S^1). Sketch a reasonable motion planning for the state space S^1 (of course, this will not be continuous!).

Quick check D (deformations of images).

1. Read/explain the statement on hashing of images in the following tweet:
   https://twitter.com/marcan42/status/1428578906412437507
2. Read/learn more on hashing of images and technical and other issues connected with it:
   https://twitter.com/marcan42/status/1427896137696960513
3. Enjoy the result of the challenge:
   https://twitter.com/marcan42/status/1428758281476927488

Exercise 1 (contractibility? 3 credits). Is the following statement true? Justify your answer with a suitable proof or counterexample.

> If $n \in \mathbb{N}$ and $X \subset \mathbb{R}^n$ is non-empty, path-connected, and a finite union of star-shaped sets, then X is contractible.

Exercise 2 (spheres and stars; 3 credits). Let $n \in \mathbb{N}$. Show directly that the sphere S^n is *not* a star-shaped subset of $\mathbb{R}^{n+1}$.
Hints. You might need to resurrect some Euclidean geometry.

Exercise 3 (convex motion planning; 3 credits). Let $n \in \mathbb{N}$ and let $X \subset \mathbb{R}^n$ be non-empty and convex. Give an explicit continuous motion planning for X (and prove that it has the claimed property).

Exercise 4 (compact-open topology vs. uniform convergence; 3 credits). Let X be a compact space and let Y be a metric space. Show one of the two inclusions of the fact that the compact-open topology on $\mathrm{map}(X, Y)$ coincides with the topology of uniform convergence.

Bonus problem (real-life homotopy; 3 credits). Find three real-life situations that could be modelled by homotopies and explain your model.

Submission before October 28, 2022, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on October 27, 2022.