Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold Sheet 10, December 23, 2022

Quick check A (consensus). Draw the input complex for 3-process consensus on $\{0,1\}$. Which 2-simplices may occur in the image of the task map?

Quick check B (prisoner's dilemma). Look up the *prisoner's dilemma*. Formalise this as a 2-person game. Does this game have pure Nash equilibria? Interpret!

Quick check C (mapping degrees for self-maps of the preference complex). Let X be the preference complex in the proof of Arrow's theorem and let $g\colon X \longrightarrow X$ be a simplicial map. Show that then $g = d \cdot \mathrm{id}_{H_1(X)}$ with $d \in \{-1, 0, 1\}$.

Exercise 1 (unanimity and dictators; 3 credits). Let $n \in \mathbb{N}_{\geq 2}$, let A be a finite set with $\#A \geq 3$, and let P be the set of total orders on A. Is the following statement true? Justify your answer with a suitable proof or counterexample.

> If $f\colon P^n \longrightarrow P$ has a dictator, then f satisfies unanimity.

Exercise 2 (the simplicial aggregation map; 3 credits). In the proof of Arrow's theorem, show that the map $F\colon V' \longrightarrow V$ obtained from the aggregation map $f\colon P^2 \longrightarrow P$ indeed defines a simplicial map $X' \longrightarrow X$.

Exercise 3 ("games"; 3 credits). We consider the two-person games described by the following payoff functions:

② \ ①	L	R
L	1 / 1	−1 / −1
R	−1 / −1	1 / 1

② \ ①	L	R
L	1 / −1	−1 / 1
R	−1 / 1	1 / −1

How can the left one be interpreted as "evasion on a narrow road" and the right one as "penalty kick"? Do these games have pure Nash equilibria? Justify your answer!

Exercise 4 (Nash equilibria; 3 credits). Let $G = (S_0, \dots, S_n, p_0, \dots, p_n)$ be an $(n+1)$-person game and let $\xi \in S(G)$. Show that ξ is a Nash equilibrium for G if and only if

$$\forall_{j \in \{0,\dots,n\}} \quad p_j(\xi) = \max_{\alpha \in S_j} p_j\bigl(\xi[j : \alpha]\bigr).$$

Bonus problem (social choice checking; 3 credits). Write a program that given $n \in \mathbb{N}$, a finite set A, the set P of total orders on A, and a map $f\colon P^n \longrightarrow P$ checks which of the properties "unanimity", "independence of irrelevant alternatives", "existence of a dictator" are satisfied by f. Document your code and apply your program to interesting examples.

Please turn over

Bonus problem (simplicial star; 3 credits). Construct a simplicial complex X that resembles a "nice" "star" shape; the simplicial complex X should have a simplicial automorphism group with more than 20 elements and $H_1(X)$ should be non-trivial. Illustrate your complex!

Bonus problem (Lefschetz fixed point theorem; 3 credits). Write a poem containing the statement and proof of the Lefschetz fixed point theorem.

Bonus problem (approximate agreement; 3 credits). Look up what the *approximate agreement* task is in distributed computing. Formalise this task via a suitable input complex, output complex, and task map.

Bonus problem (homology of preferences; 3 credits). For the simplicial complex X' in the proof of Arrow's theorem, use a computer program to compute $H_n(X'; \mathbb{F}_2)$ for all $n \in \mathbb{N}$. Document your code/solution!

Submission before January 13, 2023, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on January 12, 2023.