Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold — Sheet 11, January 13, 2023

Quick check A (simplicial homology: summary)**.** Recall the construction of simplicial homology, different strategies for the computation of simplicial homology, and applications of simplicial homology.

Quick check B (long-term persistence?)**.** Let $X \subset \mathbb{R}^N$ be a finite set, let $(\varepsilon_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathbb{R}_{>0}$ with $\lim_{n\to\infty} \varepsilon_n = \infty$. What can be said about $b_k^{i,j}(X, d_2, \varepsilon_*)$ for "large" j ?

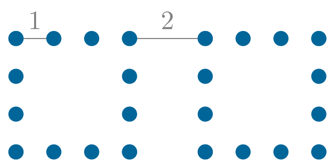
Quick check C (the graded ring $\mathbb{Z}[T]$)**.** Show that not every homogeneous ideal (i.e., generated by homogeneous elements) in $\mathbb{Z}[T]$ is principal. Here, we consider the grading of $\mathbb{Z}[T]$ given by the usual degree of polynomials.

Quick check D (elementary divisors)**.** Determine the elementary divisors of the $\mathbb{Z}$-module $\mathbb{Z}/(2) \oplus \mathbb{Z}/(3) \oplus \mathbb{Z}/(9) \oplus \mathbb{Z}/(81) \oplus \mathbb{Z}/(25) \oplus \mathbb{Z}/(25)$.

Exercise 1 (filtrations from functions; 3 credits)**.** Is the following statement true? Justify your answer with a suitable proof or counterexample.

> If X is a finite simplicial complex and $f\colon X \longrightarrow \mathbb{N}$ is a map, then the preimages sequence $(f^{-1}(\{0,\dots,n\}))_{n\in\mathbb{N}}$ are a filtration of X.

Exercise 2 (persistent Betti numbers; 3 credits)**.** Let X be the following subset of $\mathbb{R}^2$:



Let $\varepsilon_* := (0.1, 1.1, 2.1, 100, 101, 102, \dots)$. Compute the persistent Betti numbers $b_1^{i,j}(X, d_2, \varepsilon_*; \mathbb{Q})$ for all $(i,j) \in \{(1,1),(1,2),(1,3)\}$.

Exercise 3 (homogeneous elements; 3 credits)**.** Let R be a graded principal ideal domain. Show one of the following:

1. If $f, g, h \in R$ with $f = g \cdot h$ and $f \neq 0$ is homogeneous, then g and h are homogeneous.
2. If M is a graded R-module and $x \in M$ is homogeneous, then the annihilator $\operatorname{Ann}(x) := \{f \in R \mid f \cdot x = 0\}$ is a principal ideal that is generated by a homogeneous element of R.

Exercise 4 (sensor networks; 3 credits)**.** Give an example of a sensor network that does *not* satisfy the sufficient coverage condition, but such that the network still covers the whole fenced region.

Bonus problem (zigzag persistence; 3 credits)**.** What is zigzag persistence? What is the structure theorem for zigzag persistence and on which theory is it based? As always: Cite all sources!

Submission before January 20, 2023, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on January 19, 2023.