Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold Sheet 12, January 20, 2023

Quick check A (barcodes)**.** Draw a "random" (or not) finite set of points in $\mathbb{R}^2$. What barcodes (over $\mathbb{Q}$) you would expect for a given sequence of radii?

Quick check B (graded modules)**.** Let K be a field and let M be a finitely generated graded torsion $K[T]$-module. What can be said about $(T+1) \cdot M$?

Quick check C (barcodes and disjoint unions)**.** How can the barcode of persistent homology of the disjoint union of two filtered finite simplicial complexes be computed from the individual barcodes?

Exercise 1 (persistent Betti numbers; 3 credits)**.** Let K be a field and let (C^*, f^*) be a persistence K-chain complex of finite type. Let $i, j \in \mathbb{N}$ with $i \leq j$ and let $k \in \mathbb{N}$. Is the following statement true? Justify your answer with a suitable proof or counterexample.

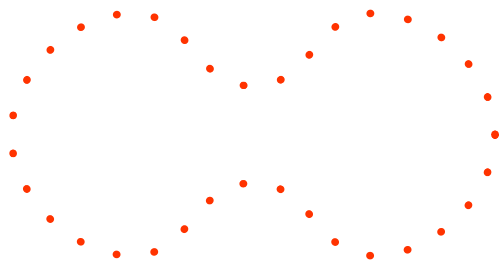
If $b_k^{(i,j)}(C^*, f^*) = 0$, then $b_k^{(i+2022,j)}(C^*, f^*) = 0$.

Exercise 2 (graded structure; 3 credits)**.** Let $\varphi\colon \bigoplus_{j=1}^{6} \Sigma^j \mathbb{F}_2[T] \longrightarrow \bigoplus_{j=1}^{5} \Sigma^j \mathbb{F}_2[T] =: F$ be the homomorphism of graded $\mathbb{F}_2[T]$-modules given by the matrix

$$\begin{pmatrix} 1 & T & T^2 & T^3 & T^4 & 0 \\ 0 & 0 & T & 0 & T^3 & T^4 \\ 0 & 0 & 0 & 0 & T^2 & T^3 \\ 0 & 0 & 0 & 1 & 0 & T^2 \\ 0 & 0 & 0 & 0 & 1 & T \end{pmatrix}.$$

Determine a graded decomposition of $F/\operatorname{im}\varphi$ as in the structure theorem.

Exercise 3 (persistent eight; 3 credits)**.** We consider the following finite subset of $\mathbb{R}^2$ and $\varepsilon_* := (n \cdot \varepsilon')_{n \in \mathbb{N}}$, where ε' is roughly the minimal distance between any of the two points. What do you expect for the barcode of the Rips complexes $R_{\varepsilon_*}(X, d_2)$ (connected by inclusion) with $\mathbb{Q}$-coefficients? Justify your answer!



Exercise 4 (persistent Betti numbers from barcodes; 3 credits)**.** How can the persistent Betti numbers be computed from the barcodes of persistent homology? Formulate a precise statement and provide a proof.

Bonus problem (applications of persistent homology; 3 credits)**.** Find three research papers that describe (different) applications of persistent homology outside of mathematics and appeared after 2017.

Submission before January 27, 2023, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on January 26, 2023.