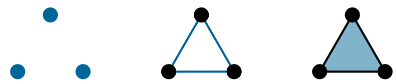
Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold Sheet 13, January 27, 2023

Quick check A (homogeneous matrix reduction via rows). Formulate a row-version of the homgeneous matrix reduction. Explain how to read off elementary divisors and graded module decompositions from the result.

Quick check B (persistent homology). Compute the barcode of persistent homology in degree 1 over a field of the following three-step filtration of $\Delta(2)$:



Exercise 1 (empty barcodes; 3 credits). Let K be a field and let (C^*, f^*) be a persistence K-chain complex of finite type with $C_k^n \not\cong 0$ for all $n, k \in \mathbb{N}$. Is the following statement true? Justify your answer with a suitable proof or counterexample.

> There exists a $k \in \mathbb{N}$ such that the barcode for persistent homology of (C^*, f^*) in degree k is non-empty.

Exercise 2 (persistent homology; 3 credits). Compute the barcode of persistent homology in degree 1 over a field of the following four-step filtration:

Exercise 3 (persistent homology, reverse engineering; 3 credits). Give a filtration of a simplicial complex with four vertices whose persistent homology in degree 1 over $\mathbb{Q}$ has the barcode $(1, 4), (2, 2), (2, 0)$. Justify your answer!

Exercise 4 (a modified homogeneous matrix reduction algorithm; 3 credits). Show that the following algorithm results in a reduced graded matrix. How can one read off a graded decomposition for $\bigoplus_{j=1}^r \Sigma^{n_j} K[T]/\operatorname{im} A$ from the resulting matrix ? Justify your answer!
Given a field K, numbers $r, s \in \mathbb{N}$, monotonically increasing sequences $n_1, \dots, n_r$, $m_1, \dots, m_s \in \mathbb{N}$, and an (n_*, m_*)-graded matrix $A \in M_{r \times s}(K[t])$, do:

- For each k from 1 up to s (in ascending order):
  Let $\ell := \operatorname{low}_A(k)$.
  If $\ell \neq 0$, then:
  - For each j from ℓ down to 1 (in descending order):
    If $\operatorname{low}_A(k) = j$ and there exists $k' \in \{1, \dots, k-1\}$ with $\operatorname{low}_A(k') = j$, then: Update the matrix A by subtracting $A_{jk}/A_{jk'}$-times the column k' from column k.
- Return the resulting matrix A.

Bonus problem (persistent homology, implementation; 3 credits). Use a publicly available persistent homology library in a programming language of your choice to solve Exercise 2 over $\mathbb{Q}$ or over $\mathbb{F}_2$. Document your code/solution!

Submission before February 3, 2023, 8:30, via GRIPS (in English or German)
The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on February 2, 2023.
This is the last regular exercise sheet.