Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold Sheet 14, February 3, 2023

Quick check A (persistent homology: summary). Recall the definition of persistent homology, the definition of barcodes, strategies for the computation, and important properties.

Quick check B (bottleneck distance). Compute the bottleneck distance between the following two weighted barcodes:

$$(0,42),(0,2023),(2023,0) \quad \text{and} \quad (2023,2022)$$

Quick check C (Gromov–Hausdorff distance). Compute the Gromov–Hausdorff distance between the following two subsets of $\mathbb{R}^2$ (with the Euclidean metric):

$$\bigl\{(0,0),(1,0),(1,1),(0,1)\bigr\} \quad \text{and} \quad \bigl\{(0,0),(1.1,0),(1,1),(0,1)\bigr\}$$

Exercise 1 (stability of Betti numbers; 3 credits). Let $X, Y \subset \mathbb{R}^2$ be finite sets (with the Euclidean metric). Is the following statement true? Justify your answer with a suitable proof or counterexample.

$$\text{Then } \bigl|b_1(R_1(X,d_2);\mathbb{Q}) - b_1(R_1(Y,d_2);\mathbb{Q})\bigr| \leq 2 \cdot d_{\mathrm{GH}}\bigl((X,d_2),(Y,d_2)\bigr).$$

Exercise 2 (bottleneck distance, properties; 3 credits). Show that the bottleneck distance defines a pseudo-metric on the set of all weighted barcodes.

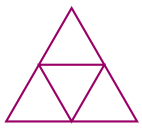
Exercise 3 (Gromov–Hausdorff distance, non-degeneracy; 3 credits). Solve one of the following:

1. Show that the Gromov–Hausdorff distance satisfies the triangle inequality.
2. Let (X,d), (X',d') be finite metric spaces with $d_{\mathrm{GH}}((X,d),(X',d')) = 0$. Show that (X,d) and (X',d') are isometric.

   Hints. This is less obvious than it looks.

Exercise 4 (persistent homology of point clouds; 3 credits). Sketch an implementation plan for the computation of barcodes of persistent homology in degree 1 for point clouds in $\mathbb{R}^2$, taking the following into account: Which input is needed and how could it be represented? How could the output be represented? Which intermediate steps are necessary? How could one solve these intermediate steps?

Bonus problem (persistent triforce; 3 credits). Use a persistent homology library of your choice to generate 42 random points on the lines of



and to compute the barcodes of associated Rips filtrations (with respect to reasonable radii) in degree 1. Document your code and the results! Are the results consistent with your expectations?

Optional submission before February 10, 2023, 8:30, via GRIPS

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on February 9, 2023.
All credits on this exercise sheet count as bonus credits.