Applied Algebraic Topology: Exercises

Quick check A (Haus/Rakete des Nikolaus). Which of the following doodles can be drawn in a single stroke (without re-drawing lines)? Can you do it in such a way that the path closes up?

Quick check B (Eulerian cycles in infinite graphs?). Let X be an infinite connected graph, all of whose vertices have finite even degree. Does X then necessarily contain a partial Eulerian cycle of non-zero length?

Quick check C (real-life simplicial complexes). Construct (fragments of) the simplicial complexes described as follows:

- Food ingredients that taste well together.

- Pitches that sound consonantly.

- Football players that could play well on the same team.

Exercise 1 (regular polyhedra; 3 credits). Is the following statement true? Justify your answer with a suitable proof or counterexample.

> If P is a regular polyhedron, then the graph of vertices/edges of P does *not* admit an Eulerian cycle.

Exercise 2 (subgraphs and connectedness; 3 credits). Let X be a finite connected graph. Let $X' = (V', E')$ be a subgraph of X with $V' \neq \emptyset$ and

$$\forall_{v \in V'} \quad \deg_{X'} v = \deg_X v.$$

Show that $X' = X$. Illustrate your proof with suitable pictures!

Exercise 3 (Eulerian paths; 3 credits). State and prove a characterisation for the existence of Eulerian paths in finite connected graphs! Argue efficiently!

Exercise 4 (independence! 3 credits). Let K be a field and let V be a K-vector space. Then we consider the simplicial complex

$$I(V) := \{\sigma \in P_{\text{fin}}(V) \mid \sigma \text{ is linearly independent over } K\}.$$

In the following, let $X := I(\mathbb{F}_2{}^3)$. Justify your answers!

1. What is $\dim X$?

2. How many edges does X have?

3. Do there exist vertices $x, y, z \in X(0)$ with $\{x, y\}, \{x, z\}, \{y, z\} \in X(1)$ and $\{x, y, z\} \notin X(2)$?

Bonus problem (de Bruijn graphs in DNA reconstruction; 3 credits). Give a formal definition of directed graphs. How are de Bruijn graphs in DNA reconstruction defined? How can Eulerian paths in de Bruijn graphs help in DNA reconstruction? Do not forget to cite your sources!

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on November 3, 2022.