Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold Sheet 3, November 4, 2022

Quick check A (constant maps). Let X and Y be simplicial complexes and let $y \in V(Y)$. Show that then the constant map $V(X) \longrightarrow V(Y)$ with value y indeed is a simplicial map. What is there to prove anyway?!

Quick check B (pushouts and products). Recall/Learn about the universal property of *pushouts* and *products* in categories.

Quick check C (unions/intersections lead to pushouts). Let X and Y be simplicial complexes. Show that the following diagram of inclusions of simplicial complexes is a pushout diagram in SC:

$$\begin{array}{ccc} X \cap Y & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & X \cup Y \end{array}$$

Quick check D (simplicial maps and connectedness). Let X and Y be simplicial complexes and let $f \colon X \longrightarrow Y$ be a simplicial map. Show that $f(X)$ is a subcomplex of Y and that $f(X)$ is connected if X is connected. Compare this connectedness result/proof with the topological case!

Exercise 1 (simplicial complexes with three vertices; 3 credits). Is the following statement true? Justify your answer with a suitable proof or counterexample.

> There exist exactly five isomorphism classes of simplicial complexes with exactly three vertices.

Exercise 2 (simplicial products of standard simplices; 3 credits). Let $n, m \in \mathbb{N}$. What is $\Delta(n) \boxtimes \Delta(m)$? Give a concrete description and prove your claim!

Exercise 3 (connectedness from covers; 3 credits). Let X be a simplicial complex and let $(U_i)_{i \in I}$ be a family of connected subcomplexes of X with $\bigcup_{i \in I} V(U_i) = V(X)$. Let N be the graph $(I, \{\{i,j\} \mid i,j \in I,\ i \neq j,\ V(U_i) \cap V(U_j) \neq \emptyset\})$. Show that X is connected if N is connected. Illustrate!

Exercise 4 (transitivity of contiguity? 3 credits). Let X and Y be simplicial complexes. Two maps $f, g \colon X \longrightarrow Y$ are *contiguous* if the following holds:

$$\forall_{\sigma \in X} \quad f(\sigma) \cup g(\sigma) \in Y.$$

Show that contiguity in general is *not* transitive on the set $\operatorname{map}_\Delta(X, Y)$.

Bonus problem (adjoints of the vertex functor; 3 credits). Let $V \colon \mathsf{SC} \longrightarrow \mathsf{Set}$ be the functor mapping simplicial complexes to the underlying set of vertices and mapping simplicial maps to the underlying map between the sets of vertices. Show that this functor V has both a left adjoint and a right adjoint.

Hints. There are two generic ways to convert sets into simplicial complexes with the given set as set of vertices. These ways are the desired left/right adjoints of V.

Submission before November 11, 2022, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on November 10, 2022.