## **Applied Algebraic Topology: Exercises**

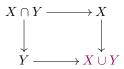
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Sheet 3, November 4, 2022

**Quick check A** (constant maps). Let X and Y be simplicial complexes and let  $y \in V(Y)$ . Show that then the constant map  $V(X) \longrightarrow V(Y)$  with value y indeed is a simplicial map. What is there to prove anyway?!

**Quick check B** (pushouts and products). Recall/Learn about the universal property of *pushouts* and *products* in categories.

**Quick check C** (unions/intersections lead to pushouts). Let X and Y be simplicial complexes. Show that the following diagram of inclusions of simplicial complexes is a pushout diagram in SC:



**Quick check D** (simplicial maps and connectedness). Let X and Y be simplicial complexes and let  $f: X \longrightarrow Y$  be a simplicial map. Show that f(X) is a subcomplex of Y and that f(X) is connected if X is connected. Compare this connectedness result/proof with the topological case!

**Exercise 1** (simplicial complexes with three vertices; 3 credits). Is the following statement true? Justify your answer with a suitable proof or counterexample.

There exist exactly five isomorphism classes of simplicial complexes with exactly three vertices.

**Exercise 2** (simplicial products of standard simplices; 3 credits). Let  $n, m \in \mathbb{N}$ . What is  $\Delta(n) \boxtimes \Delta(m)$ ? Give a concrete description and prove your claim!

**Exercise 3** (connectedness from covers; 3 credits). Let X be a simplicial complex and let  $(U_i)_{i \in I}$  be a family of connected subcomplexes of X with  $\bigcup_{i \in I} V(U_i) = V(X)$ . Let N be the graph  $(I, \{\{i, j\} \mid i, j \in I, i \neq j, V(U_i) \cap V(U_j) \neq \emptyset\})$ . Show that X is connected if N is connected. Illustrate!

**Exercise 4** (transitivity of contiguity? 3 credits). Let X and Y be simplicial complexes. Two maps  $f, g: X \longrightarrow Y$  are *contiguous* if the following holds:

$$\forall_{\sigma \in X} \quad f(\sigma) \cup g(\sigma) \in Y.$$

Show that contiguity in general is *not* transitive on the set  $\operatorname{map}_{\Delta}(X, Y)$ .

**Bonus problem** (adjoints of the vertex functor; 3 credits). Let  $V: SC \longrightarrow Set$  be the functor mapping simplicial complexes to the underlying set of vertices and mapping simplicial maps to the underlying map between the sets of vertices. Show that this functor V has both a left adjoint and a right adjoint.

*Hints.* There are two generic ways to convert sets into simplicial complexes with the given set as set of vertices. These ways are the desired left/right adjoints of V.

Submission before November 11, 2022, 8:30, via GRIPS (in English or German) The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on November 10, 2022.