Applied Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold — Sheet 5, November 18, 2022

Quick check A (barycentric counting)**.** Let $n \in \mathbb{N}$. How many vertices does the iterated subdivision $\operatorname{sd}^2 \Delta(2)$ have?

Quick check B (simplicial approximation and compositions)**.** Show that compositions of simplicial approximations are simplicial approximations of the composition.

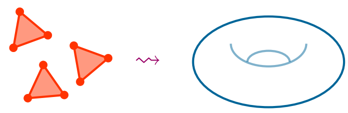
Quick check C (approximating triple wrapping)**.** Give an $N \in \mathbb{N}$ and a simplicial approximation $\operatorname{sd}^N S(1) \longrightarrow S(1)$ of "the" map $|S(1)| \longrightarrow |S(1)|$ wrapping three times around the circle.

Quick check D (real-world triangulations)**.** Give a real-world example, where triangulations are used to approximate geometric objects.

Exercise 1 (counting homotopy classes? 3 credits)**.** Is the following statement true? Justify your answer with a suitable proof or counterexample.

> If X is a countable simplicial complex, then $[|X|, |X|]$ is (at most) countable.

Exercise 2 (triangulating the torus; 3 credits)**.** Give an example of a simplicial complex X such that $|X|$ is homeomorphic to the torus $S^1 \times S^1$.
Hints. Give a precise specification of X. It is not necessary to give a formal proof that $|X|$ is homeomorphic to $S^1 \times S^1$; it is sufficient to give pictures and explanations that make it plausible.



Exercise 3 (simplicial approximation and barycentric subdivision; 3 credits)**.** Let X be a simplicial complex. Show that then the barycentric subdivision homeomorphism $\beta_X \colon |\operatorname{sd} X| \longrightarrow |X|$ admits a simplicial approximation.
Hints. Choose a total ordering on $V(X)$ (you may use this without proof), then go for the minimum.

Exercise 4 (the nerve map; 3 credits)**.** Let Z be a paracompact topological space, let $U = (U_i)_{i \in I}$ be an open cover of Z, and let $\varphi = (\varphi_i)_{i \in I}$ be a paritition of unity on Z that is subordinate to U. Show that the *nerve map*

$$\begin{aligned} \nu_\varphi \colon Z &\longrightarrow \bigl|N(U)\bigr| \\ \zeta &\longmapsto \sum_{i \in I} \varphi_i(\zeta) \cdot e_i \end{aligned}$$

is well-defined and continous. Moreover, show that if $\nu_{\varphi'}$ is another partition of unity on Z subordinate to U, then $\nu_\varphi \simeq \nu_{\varphi'}$.

Bonus problem (measurability of Čech-realising a given space; 3 credits)**.** Let Z be a topological space, let $n, N \in \mathbb{N}$, and let $\varepsilon \in \mathbb{R}_{>0}$. Show that the subset

$$\bigl\{ x \in (\mathbb{R}^N)^n \bigm| \bigl|\check{C}_\varepsilon(\{x_1, \dots, x_n\}, \mathbb{R}^N, d_2)\bigr| \simeq Z \bigr\}$$

of $(\mathbb{R}^N)^n$ is measurable (with respect to the Borel σ-algebra).

Submission before November 25, 2022, 8:30, via GRIPS (in English or German)

The Quick checks are not to be submitted and will not be graded; they will be solved and discussed in the exercise class on November 24, 2022.